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Bounds and Computational Results for Exponential Sums Related to Cusp Forms

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Abstract. The aim of this paper is to present some computer data suggesting the correct size of bounds for exponential sums of Fourier coefficients of holomorphic cusp forms.

1 Introduction

Holomorphic cusp forms can be represented as Fourier series

$$F(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{\kappa-1}{2}} e(nz)\,,$$

where $\Im z > 0$, $e(x) = e^{2\pi i x}$, and the numbers $a(n)$ are called normalized Fourier coefficients and κ is the weight of the form; see e.g. [2] or [8] for an account of the theory holomorphic modular forms.

Exponential sums of Fourier coefficients of holomorphic cusp forms have been researched for a long time already. In 1929 Wilton [10] proved the estimate

$$\sum_{n \le M} a(n) e(n\alpha) \ll \sqrt{M} \log M\,,$$

where α is an arbitrary real number. This remained the best known bound for nearly sixty years until year 1987 Jutila [7] proved the estimate $\ll \sqrt{M}$. Jutila's bound is the best possible as can been seen by using the Rankin-Selberg mean value result (see e.g. [9]) and Parseval's formula:

$$\int_0^1 \left| \sum_{n \le M} a(n) e(n\alpha) \right|^2 d\alpha = \sum_{n \le M} |a(n)|^2 \asymp M\,. \qquad (1)$$

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Key Words and Phrases: cusp forms, exponential sums, Ramanujan tau function, analytic computations

With short sums

$$\sum_{M \le n \le M+\Delta} a(n)e(n\alpha)\,,$$

where Δ is considerably smaller than M, the situation is very different: only when $\Delta \gg M^{7/10}$ a sharp bound is know. In this paper, we will concentrate on the known bounds in the case when $\Delta \ll M$, and on computer evidence, particularly in the case with $\Delta = M^{1/2}$, as evidence in this point seems to reveal information about other points, too.

The constants implied by symbols $\ll$, $\asymp$, and the O notation, depend only on ε. Also, ε is an unspecified very small number, not necessarily same in its all appearances. The Ω-notation is to be read as follows: $f = \Omega(g)$ iff $f = o(g)$ does not hold.

The Fourier coefficients used in the computation are the normalized values of the Ramanujan τ function (that is, $\tau(n)n^{-5.5}$), the Fourier coefficients of the non-trivial holomorphic cusp form of weight 12. We will denote these by $a(n)$. Also, as we are interested in the upper bound estimates, it is sufficient to consider sums with $\alpha \in \left[0, \frac{1}{2}\right]$.

2 About the Behavior of Exponential Sums

Let us first look at the real part of the sum

$$\sum_{M \le n \le M+\Delta} a(n)e(n\alpha)\,.$$

In the picture, the following choices are made: $M = 10^6$, $\Delta = 10^4$ and $\alpha \in [0.0005, 0.0015]$ with steps of 10^{-7}. α is on the x-axis and the size of the real part is on the y-axis.

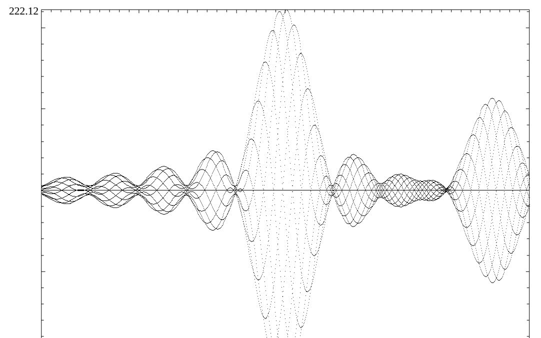

Figure 1

The behavior may seem a bit curious at the first glance as the graph looks nearly symmetric. However, this is easily explained since

$$\sum_{M \le n \le M+\Delta} a(n)\,e\left(n\left(\alpha + \frac{1}{2M}\right)\right) = -\sum_{0 \le m \le \Delta} a(M+m)\,e\left((M+m)\,\alpha + \frac{m}{2M}\right),$$

and hence

$$\left|\sum_{M\le n\le M+\Delta} a(n)\, e\left(n\left(\alpha+\frac{1}{2M}\right)\right) + \sum_{M\le n\le M+\Delta} a(n)e(n\alpha)\right|$$

$$= \left|\sum_{0\le m\le \Delta} a(M+m)e((M+m)\alpha)\left(1-e\left(\frac{m}{2M}\right)\right)\right|$$

$$\le \left|\frac{\pi i}{M}\sum_{0\le m\le\Delta} a(M+m)e(m\alpha)m\right| + O\Bigg(M^{-2+\varepsilon}\sum_{0\le m\le\Delta} m^2\Bigg).$$

The second term yields $\ll \Delta^3 M^{-2+\varepsilon}$, which is small and the first term can be seen to be considerably smaller than the original sum.

The sum oscillates very rapidly, returning close to back to the starting point M^{-1} later. This is, of course, a property of the exponential function. Close by, the behavior looks like the following:

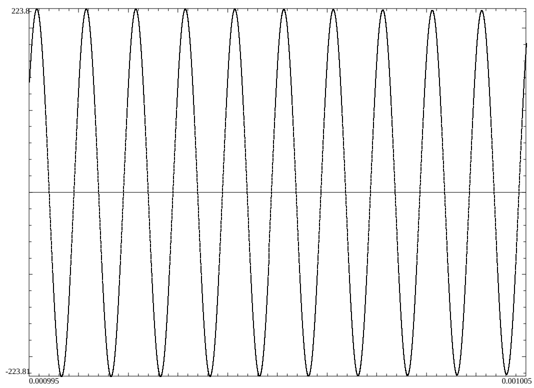

Figure 2

Interestingly, when we look at similar graphs of various sums, we notice that with exponential sums of Fourier coefficients of holomorphic cusp forms the behavior looks somewhat regular while with the Moebius function in the place of the Fourier coefficients, the graph looks very random. With the divisor function (which is always positive), the decay is very rapid on this interval. We don't know the reason for the neat behavior of the exponential sums related to cusp forms as opposed to the behavior of the exponential sums related to the Moebius function.

When we look at absolute values of exponential sums, it is worth noting that

$$\left|e(-M(\alpha+\beta))\sum_{M\le n\le M+\Delta} a(n)e(n(\alpha+\beta)) - e(-M\alpha)\sum_{M\le n\le M+\Delta} a(n)e(n\alpha)\right|$$

$$= \left|\sum_{0\le m\le\Delta} a(m+M)\left(e(m(\alpha+\beta))-e(m\alpha)\right)\right| \ll M^{\varepsilon}\Delta^2\beta\,.$$

Of course, we could have used partial summation to estimate but this is sufficient for our purposes. This yields a very simple criterion for judging the correctness of computational bounds with respect to the density of the used α's.

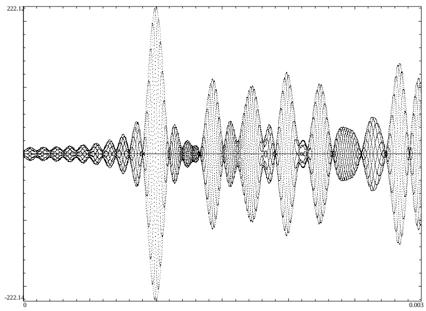

Figure 3 $a(n)$

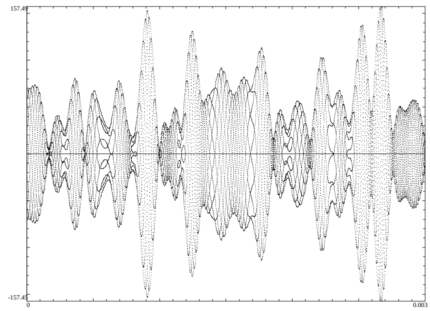

Figure 4 $\mu(n)$

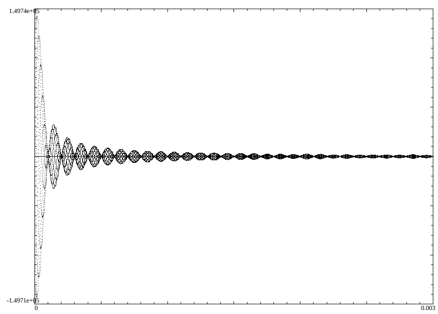

Figure 5 $d(n)$

3 Theoretical Bounds and Computational Evidence

Let us start with stating the best known upper bounds for short ($\Delta \ll M$) sums. For very short sums $\Delta \ll M^{2/5}$ only the trivial estimate ΔM^{ε} is known. On the other hand, when $\Delta \gg M^{3/4}$, the best general upperbound is $M^{1/2}$ ([7] and [4]). Between these, the situation gets a lot more interesting.

Theorem 1. *Let* $M^{2/5} \ll \Delta \ll M^{3/4}$, *then*

$$\sum_{M \le n \le M+\Delta} a(n) e(n\alpha) \ll M^{-1/4}\Delta + M^{1/3+\varepsilon}\Delta^{1/6}.$$

When $\Delta \gg M^{7/10+\varepsilon}$, *the bound is sharp.*

The proof of this can be found in [5]. This improves the interval of Δ on which a sharp estimate is known from $\Delta \gg M^{23/32+\varepsilon}$ to $\Delta \gg M^{7/10+\varepsilon}$. However, looking at the computer data at the point $\Delta \asymp \sqrt{M}$, it seems that $M^{-1/4}\Delta$ is the correct bound for $\Delta \gg M^{1/2+\varepsilon}$. We studied sums of type

$$\sum_{M \le n \le M+\sqrt{M}} a(n) e(n\alpha)$$

where $M = 25 \cdot 10^6 + b \cdot 5000$, where $b \in \{0, 1, \ldots, 1000\}$ and $\alpha = \frac{a}{2 \cdot 10^6}$, where $a = \{1, \ldots, 10^6\}$. We notice that the difference between the bounds we obtain and between the theoretical upper bounds (which correspond to the case when the maximal value is obtain at some point which is not of form $\frac{a}{2 \cdot 10^6}$) is at most around 15 and certainly less than 30 (look at the calculations at the end of the previous section). However, even $30 < \frac{1}{2}M^{1/4}$, and therefore, on this interval our computations treat cases sufficiently close to the worst case. As

$$\left|\sum a(n)e(n\alpha)\right| = \left|\sum a(n)e(n(1-\alpha))\right| ,$$

it is sufficient to consider only α's on the interval $[0, 0.5]$.

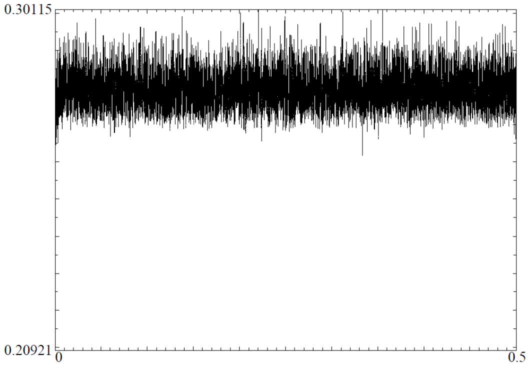

Figure 6 Maximal absolute value of sum for different $\alpha = a/(2 \cdot 10^6)$, $a \in \{1, \ldots, 10^6\}$ over $M = 25 \cdot 10^6 + b \cdot 5000$, where $b \in \{0, 1, \ldots, 1000\}$ and $\Delta = \sqrt{M}$ (α on the x-axis and absolute value of the maximal size of the sums on the y-axis)

Here it is worth noting that as the biggest value of M considered is $M = 3 \cdot 10^7$, for instance

$$M^{0.30115} < 2.5 \cdot M^{1/4} .$$

Therefore, even though the values are not below $M^{0.25}$, they are only a very small constant away from that. This is a serious problem with computational data: in theoretical results the constants do not matter but with computations we would need M to be larger than 10^{30} to ensure that for instance a constant factor 2 would give an error less than 0.01 in the exponent.

Figure 7 gives an idea about the behavior of an exponential sum of squareroot length:

From the Figure 6 we see that 0.30115 is the maximum of all those maxima listed in that graph. Although 0.3 is exceeded as a maximum, the average of the maximal values presented in Figure 6 is about 0.287643, while for the maximal values presented in Figure 7 average is only 0.280293.

On the other hand, when considering the maximal values over all $\alpha = a/(2 \cdot 10^6)$, $a \in \{1, \ldots, 10^6\}$, for $M \in \{25 \cdot 10^6, 25 \cdot 10^6 + 5000, \ldots, 30 \cdot 10^6\}$, we can determine

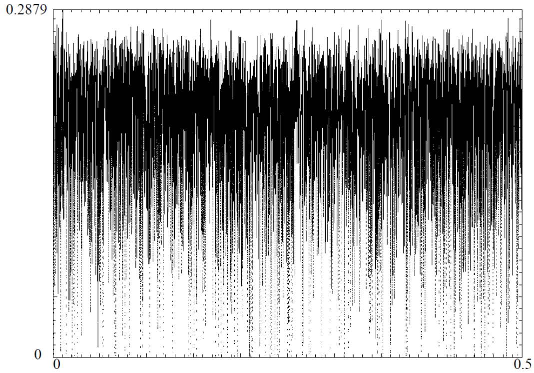

Figure 7 Absolute value of short sum with $\alpha = a/(2\cdot 10^6), a \in \{1, \ldots, 10^6\}$ for $M = 3\cdot 10^7$ and $\Delta = \sqrt{M}$ (α on the x-axis and absolute value of the size of the sum on the y-axis)

that there are actually only four separate values of M for which maximum is greater than 0.3:

M	maximum over α
27445000	0.3003303
26665000	0.3004903
27515000	0.3008670
26615000	0.3011450

In order to present the distribution of maximums for various M in a compact table, we can use the first four digits (first three decimals) as the class identifiers illustrating the fact that distribution is centered around classes 0.286 and 0.287.

class	0.278	0.279	0.280	0.281	0.282	0.283	0.284	0.285
items in class	5	6	5	29	45	87	90	90
class	0.286	0.287	0.288	0.289	0.290	0.291	0.292	0.293
items in class	105	106	96	91	64	57	47	26
class	0.294	0.295	0.296	0.297	0.298	0.299	0.300	0.301
items in class	17	7	10	9	3	2	3	1

The following theorem was proved in [4], and in a bit more general form in [3].

Theorem 2. *Let $M^{1/2+\delta} < \Delta \leq \lambda M^{3/4}$, where $0 < \lambda < 1$ is a constant. Let w be a smooth weight function on the interval $[M, M+\Delta]$ which equals 1 on the interval $[a, b] \subset [M, M+\Delta]$ where*

$$a - M = M + \Delta - b = \Delta^{1-\delta}$$

with δ a sufficiently small fixed positive real number. Assume further that $\alpha = M^{-\frac{1}{2}}$. Then

$$\left| \sum_{M \le n \le M+\Delta} a(n)w(n)e(\alpha n) \right| \asymp \Delta M^{-1/4}.$$

This gives sharp bounds, and hence also lower bounds for smoothed sums on the interval $\Delta \gg M^{1/2+\varepsilon}$.

Also, the first author has proved the following Ω result for sums without the exponential term:

Theorem 3. *Let $c > 0$ be an arbitrary real number. Then*

$$\sum_{M \le n \le M+c\sqrt{M}} a(n) = \Omega\left(M^{1/4}\right).$$

This extends the Ω result by Ivic [6] to the missing point:

Theorem 4. *Let $\Delta \ll M^{1/2-\varepsilon}$. Then*

$$\sum_{M \le n \le M+\Delta} a(n) = \Omega\left(\sqrt{\Delta}\right).$$

The picture of the current best bounds looks like the following:

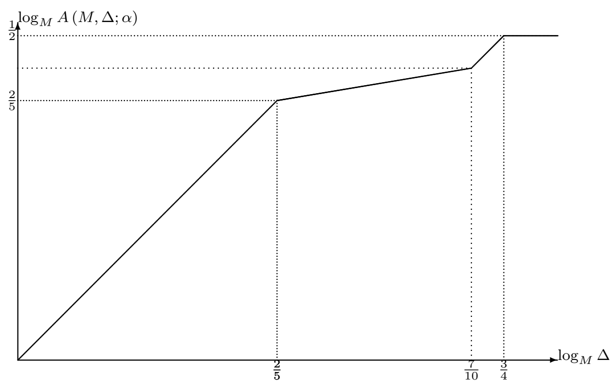

Figure 8

The graph with the added computational data of the maximum values for each $1 \le \Delta \le M^{3/4}$ for $M = 10^6$ is shown on the Figure 9.

It is worth noting that since $M^{1/6} = 10$, for small values of Δ (small in the sense of as a power of M) the computational data does not necessarily give any information. Also, values of α are much less dense than what might be necessarily sensible but this figure is more like a curiosity as Figure 6 seems to reveal the real behavior of maximal values of exponential sums with $M^{1/2} \ll \Delta \ll M^{3/4}$.

However, these two graphs together show quite clearly how much there seems to be space between the theoretical estimates and real ones. Although, there is again the problem with computations being only possible with real numbers (which can

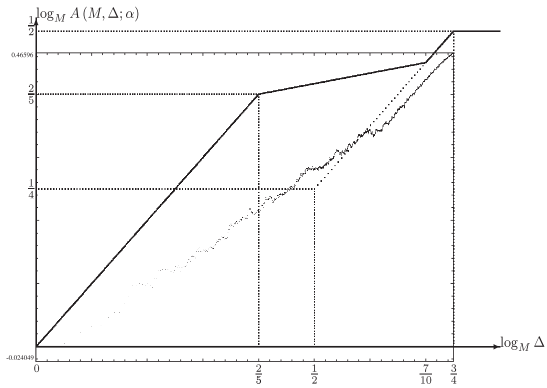

Figure 9 Maximum values of short sums for $M = 10^6$ with $1 \le \Delta \le M^{3/4}$ over $\alpha = a/(2 \cdot 10^3), a \in \mathbb{Z}$

therefore not go to infinity): the constants which appear in front of estimates do not affect the theoretical logarithmic upper bounds but they are visible in the computational bounds.

The exact formulation of the Rankin-Selberg mean value theorem reads as following:

Theorem 5.

$$\sum_{n \le M} a(n) = HM + O\left(M^{3/5}\right),$$

where H is a constant.

From this we obtain using the Parseval formula

$$\int_0^{\alpha} \left| \sum_{M \le n \le M+\Delta} a(n) e(n\alpha) \right|^2 d\alpha \asymp \Delta,$$

when $\Delta \gg M^{3/5+\varepsilon}$. Similarly as in (1) on this interval, the estimates are hence $\sqrt{\Delta}$ in average. There is a general belief that the error term in the Rankin-Selberg mean value theorem is too big. However, there is also the Ω-result by Ivic [6] stating that the error term has to be at least $M^{3/8}$. However, it has been conjectured by Ivic in the same article to be $O\left(M^{3/8+\varepsilon}\right)$. In particular, this would imply that for $\Delta \gg M^{1/2+\varepsilon}$ the average estimate would be $\sqrt{\Delta}$. Taking into account the fact that the upper bound at $\Delta \asymp \sqrt{M}$ seems to be $\sqrt{\Delta}$, this would mean that this bound would have to be attained in a set of points of positive measure.

4 Shortly About Computations and Graphs

The computations were done in two phases by using 32-bit version of PARI/GP [1] software on pc platform. In the preliminary phase, values of $\tau(x)$ were computed for $x \leq 3 \cdot 10^7$. By using these precomputed values of τ, the computations for various short sums of type $\sum_M^{M+\Delta} a(n)e(n\alpha)$ were performed.

In the first phase, computation of values for the Ramanujan τ function is based on the following three well-known equations:

$$\tau(rs) = \tau(r)\tau(s)\,,$$
$$\tau(p^k) = \tau(p)\tau(p^{k-1}) - p^{11}\tau(p^{k-2})\,, \quad k \geq 2\,,$$
$$\tau(p) = \sum_{\substack{k\in\mathbb{Z} \\ 1\leq \frac{k(k+1)}{2} < p}} (-1)^{k+1}(2k+1)\Big(p-1-9\frac{m(m+1)}{2}\Big)\tau\Big(p-\frac{m(m+1)}{2}\Big)\,,$$

where p is a prime and r and s are relative primes. Resulting algorithm is a fairly straightforward one for which the last equation of the above three equations clearly takes the most of the computational operations. Finally, the overall amount of both multiplications and additions for generating a table of values of $\tau(k)$ for $k \leq n$ is $o(n\sqrt{n})$.

In the computation of the absolute value of sum $\sum_{n=M}^{M+\Delta} a(n)e(n\alpha)$, the equality

$$\left|\sum_{n=M}^{M+\Delta} a(n)e(n\alpha)\right| = |e((M+\Delta)\alpha)| \left|\sum_{n=M}^{M+\Delta} a(n)e((n-M-\Delta)\alpha)\right|\,,$$

simplifies computation by reducing the amount of computational operations to one multiplication and one addition per each term of the sum. Basically, the algorithms used in computations are based on a recursive assignment of type $r <- r \cdot e + c_i$, where e is a constant and c_i are values of $a(n)$. Computing the real and the imaginary parts of the sum this way requires one additional multiplication with $e((M+\Delta)\alpha)$. Furthermore, when computing sums for different values of M, α and Δ the exponential function is needed once for each value of α and, depending on whether value of sum is presented in logarithmic or linear scale, the logarithmic function is needed once for each M and each absolute value of sum needed to collect necessary data. Apart of these operations, a fixed number of additional multiplications and additions are needed for some values which are computed before actual computations. There are also some ways to decrease the execution time, like not repeating computations unnecessarily and keeping in mind the physical properties of the used computation platform.

In order to guarantee that final computed values of sums have relative error less than 10^{-6}, the normalized values $a(n)$ of the Fourier coefficients of the Ramanujan τ function was computed with sufficient precision. For example, when computing data for the graphs presented in Figures 9 and 6 the application of multiplications and additions in a repeated fashion introduces an inaccuracy to a value of a calculated sum (for Figure 6 the total amount of computational operations was over 10^{12}). Also, using values of a greater precision (as using short steps of α in the graph representing the values of the real part for some short sums) increases a need for greater accuracy in computations.

The PARI/GP software while being fast for numerical computations with great accuracy is a bit difficult to use for presenting computed data as graphs. This is because the software presents the accurate computed data in a graphical plot as in a screen. Therefore, there exists an effect similar to the drawing a non-axis line to a screen, i.e. two plotted points in graph could have the same x-axis position while they do not have the same actual value corresponding x-axis position.

There is also one other interesting effect in graphical presentations of numerical data which appears in the first plot. In the case of PARI/GP it is a bit worse because of the above effect in graphical plots. A plot seems to be a collection of several sine-like functions while, in fact, there is only one sine-like function. When gathering computational data about functions the values of x for which value $f(x)$ is computed usually form uniformly distributed sequence. And sampling a function like sine in that way can create an interference effect.

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