Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Adrijan Varbanov Borisov; Margarita Georgieva Spirova

On the measurability of sets of pairs of intersecting nonisotropic straight lines of type beta in the simply isotropic space

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 48 (2009), No. 1, 7--16

Persistent URL: http://dml.cz/dmlcz/137502

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **48** (2009) 7–16

On the Measurability of Sets of Pairs of Intersecting Nonisotropic Straight Lines of Type Beta in the Simply Isotropic Space

Adrijan Varbanov BORISOV [1], Margarita Georgieva SPIROVA [2]

[1] *Department of Mathematics, South-West University,*
"Neofit Rilski" 66, Ivan Mihailov Str., 2700 Blagoevgrad, Bulgaria
e-mail: adribor@aix.swu.bg

[2] *Fakultät für Mathematik, TU Chemnitz*
D-09107 Chemnitz, Germany
e-mail: margarita.spirova@mathematik.tu-chemnitz.de

(Received October 24, 2008)

Abstract

The measurable sets of pairs of intersecting non-isotropic straight lines of type β and the corresponding densities with respect to the group of general similitudes and some its subgroups are described. Also some Crofton-type formulas are presented.

Key words: Simply isotropic space, density, measurability.

2000 Mathematics Subject Classification: 53C65

1 Introduction

The *simply isotropic space* $I_3^{(1)}$ (see [8]) is defined as a projective space $\mathbb{P}_3(\mathbb{R})$ in which the absolute consists of a plane ω (the *absolute plane*) and two complex conjugate straight lines f_1, f_2 (the *absolute lines*) within ω. In homogeneous coordinates (x_0, x_1, x_2, x_3) we can choose the plane $x_0 = 0$ as the plane ω, the line $x_0 = 0$, $x_1 + ix_2 = 0$ as the line f_1, and the line $x_0 = 0$, $x_1 - ix_2 = 0$ as the line f_2. Then the intersecting point F of f_1 and f_2, which is called an *absolute point*, has coordinates $(0, 0, 0, 1)$. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of *general simply* *isotropic similitudes.* In affine coordinates (x, y, z) with respect to the affine coordinate system $(O, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$, any similitude of G_8 can be written in the form ([8, p. 3])

$$\begin{aligned} \overline{x} &= c_1 + c_7(x\cos\varphi - y\sin\varphi), \\ \overline{y} &= c_2 + c_7(x\sin\varphi + y\cos\varphi), \\ \overline{z} &= c_3 + c_4x + c_5y + c_6z, \end{aligned} \tag{1}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$, and φ are real parameters and $c_7 > 0$.

A plane in ${I_3}^{(1)}$ is said to be *non-isotropic* if its infinite line is not incident with the absolute point F; otherwise the plane is called *isotropic.*

A straight line in ${I_3}^{(1)}$ is said to be *(completely) isotropic* if its infinite point coincides with the absolute point F; otherwise the straight line is said to be *non-isotropic* ([8, p. 5]).

Let G_1 and G_2 be two non-isotropic straight lines and let us denote by U_1 and U_2 their infinite points, respectively. The straight lines G_1 and G_2 are said to be *of type* β if the points U_1, U_2, and F are collinear; otherwise the straight lines are said to be *of type* α ([8, p. 45]).

We will consider also the following subgroups of G_8:

I. $B_7 \subset G_8 \Longleftrightarrow c_7 = 1$. This is the group of simply isotropic similitudes of the δ*-distance* ([8, p. 5]).

II. $S_7 \subset G_8 \Longleftrightarrow c_6 = 1$. This is the group of simply isotropic similitudes of the s*-distance* ([8, p. 6]).

III. $W_7 \subset G_8 \Longleftrightarrow c_6 = c_7$. This is the group of simply isotropic *angular* similitudes ([8, p. 18]).

IV. $G_7 \subset G_8 \Longleftrightarrow \varphi = 0$. This is the group of simply isotropic *boundary* similitudes ([8, p. 8]).

V. $V_7 \subset G_8 \Longleftrightarrow c_6c_7^2 = 1$. This is the group of simply isotropic *volume preserving* similitudes ([8, p. 8]).

VI. $G_6 = G_7 \cap V_7$. This is the group of simply isotropic *volume preserving boundary* similitudes ([8, p. 8]).

VII. $B_6 = B_7 \cap G_7$. This is the group of *modular boundary* motions ([8, p. 9]).

VIII. $B_5 = B_7 \cap S_7 \cap G_7$. This is the group of *unimodular boundary* motions ([8, p. 9]).

Basic references on the geometry of the simply isotropic space ${I_3}^{(1)}$ are Sachs' book [8] and Strubecker's papers [8], [11] and [12].

Using some basic concepts from integral geometry in the sense of R. Deltheil [3], M. I. Stoka [10], G. I. Drinfel'd, and A. V. Lucenko [4], [5], [6], we study the measurability of sets of pairs of intersecting nonisotropic straight lines of type β with respect to G_8 and indicated above subgroups. Analogous problems about sets of pairs of intersecting non-isotropic straight lines of type α in ${I_3}^{(1)}$ have been treated in [2].

2 Measurability with respect to $\mathbf{G_8}$

Let (G_1, G_2) be a pair of intersecting non-isotropic straight lines of type β. Let G_i have Plücker coordinates (p^i_j), $i = 1, 2$, $j = 1, \ldots, 6$, which satisfy the relations ([8, p. 38])

$$p^i_1 p^i_4 + p^i_2 p^i_5 + p^i_3 p^i_6 = 0, \quad i = 1, 2. \tag{2}$$

Since G_1 and G_2 are intersecting non-isotropic lines of type β, we have

$$p^1_1 p^2_4 + p^1_2 p^2_5 + p^1_3 p^2_4 + p^1_4 p^2_1 + p^1_5 p^2_2 + p^1_6 p^2_3 = 0, \qquad p^1_3 - p^2_3 \neq 0, \tag{3}$$

$$|p^i_1| + |p^i_2| \neq 0, \quad i = 1, 2, \tag{4}$$

$$p^1_1 p^2_2 - p^1_2 p^2_1 = 0. \tag{5}$$

Having in mind (4), we can assume, without loss of generality, that $p^i_1 = 1$. From (2), p^i_4 can be expressed by the remaining Plücker coordinates of G_i, and in view if (3) and (5), p^2_2 and p^2_6 also can be expressed by p^1_2, p^1_3, p^1_5, p^1_6, p^2_3 and p^2_5. Thus the pair (G_1, G_2) can be determined by p^1_2, p^1_3, p^1_5, p^1_6, p^2_3, p^2_5.

Remark 2.1 We note that if G_i, $i = 1, 2$, are represented in the usual way by the equations

$$G_1\colon \begin{cases} x = a_1(z - r) + p \\ y = b_1(z - r) + q \end{cases}, \qquad G_2\colon \begin{cases} x = a_2(z - r) + p \\ y = \dfrac{a_2}{a_1} b_1(z - r) + q \end{cases}, \tag{6}$$

where $P(p, q, r) = G_1 \cap G_2$ and $a_1 \neq 0$, $a_2 \neq 0$, then

$$\begin{gathered} p^1_2 = \frac{b_1}{a_1}, \quad p^1_3 = \frac{1}{a_1}, \quad p^1_5 = r - \frac{p}{a_1}, \quad p^1_6 = p\frac{b_1}{a_1} - q, \\ p^2_3 = \frac{1}{a_2}, \quad p^2_5 = r - \frac{p}{a_2}. \end{gathered} \tag{7}$$

Under the action of (1) the pair $(G_1, G_2)(p^1_2, p^1_3, p^1_5, p^1_6, p^2_3, p^2_5)$ is transformed into the pair $(\overline{G}_1, \overline{G}_2)(\overline{p}^1_2, \overline{p}^1_3, \overline{p}^1_5, \overline{p}^1_6, \overline{p}^2_3, \overline{p}^2_5)$. Thus we have

$$\begin{aligned} \overline{p}^1_2 &= Kc_7(\sin\varphi + p^1_2 \cos\varphi), \\ \overline{p}^1_3 &= K(c_4 + c_5 p^1_2 + c_6 p^1_3), \\ \overline{p}^1_5 &= K\{(c_3 - c_5 p^1_6 + c_6 p^1_5) c_7 \cos\varphi \\ &\quad - [c_3 + c_4 p^1_6 + c_6(p^1_2 p^1_5 + p^1_3 p^1_6)] c_7 \sin\varphi - c_1(c_4 + c_5 + c_6 p^1_3)\}, \\ \overline{p}^1_6 &= Kc_7[(c_1 p^1_2 - c_2)\cos\varphi + (c_1 + c_2 p^1_2)\sin\varphi + c_7 p^1_6], \\ \overline{p}^2_3 &= K(c_4 + c_5 p^1_2 + c_6 p^2_3), \\ \overline{p}^2_5 &= K\{(c_3 - c_5 p^1_6 + c_6 p^2_5) c_7 \cos\varphi \\ &\quad - [c_3 + c_4 p^1_6 + c_6(p^1_2 p^2_5 + p^2_3 p^1_6)] c_7 \sin\varphi - c_1(c_4 + c_5 + c_6 p^2_3)\}, \end{aligned} \tag{8}$$

where $K = [c_7(\cos\varphi - p_2^1 \sin\varphi)]^{-1}$, $i = 1, 2$. The transformations (8) form the associated group $\overline{G_8}$ of G_8 ([10, p. 34]). The group $\overline{G_8}$ is isomorphic to G_8 and the density with respect to G_8 of the pairs (G_1, G_2) if it exists, coincides with the density with respect to $\overline{G_8}$ of the set of parameters $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$.

The associated group $\overline{G_8}$ has the infinitesimal operators

$$X_1 = p_3^1 \frac{\partial}{\partial p_5^1} - p_2^1 \frac{\partial}{\partial p_6^1} - p_3^2 \frac{\partial}{\partial p_5^2}, \quad X_2 = \frac{\partial}{\partial p_6^1}, \qquad X_3 = \frac{\partial}{\partial p_5^1} + \frac{\partial}{\partial p_5^2},$$

$$X_4 = \frac{\partial}{\partial p_3^1} + \frac{\partial}{\partial p_3^2}, \quad X_5 = p_2^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_5^1} + p_2^1 \frac{\partial}{\partial p_3^2} - p_6^1 \frac{\partial}{\partial p_5^2},$$

$$X_6 = p_3^1 \frac{\partial}{\partial p_3^1} + p_5^1 \frac{\partial}{\partial p_5^1} + p_3^2 \frac{\partial}{\partial p_3^2} + p_5^2 \frac{\partial}{\partial p_5^2}, \quad X_7 = p_3^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_6^1} + p_3^2 \frac{\partial}{\partial p_3^2},$$

$$X_8 = [1 + (p_2^1)^2] \frac{\partial}{\partial p_2^1} + p_2^1 p_3^1 \frac{\partial}{\partial p_3^1} - p_3^1 p_6^1 \frac{\partial}{\partial p_5^1} + p_2^1 p_6^1 \frac{\partial}{\partial p_6^1} + p_2^1 p_3^2 \frac{\partial}{\partial p_3^2} - g_6^1 p_3^2 \frac{\partial}{\partial p_5^2}, \tag{9}$$

and it acts transitively on the set of parameters $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$. The infinitesimal operators X_1, X_2, X_3, X_4, X_7, and X_8 are arcwise unconnected and

$$X_6 = \frac{p_5^2 - p_5^1}{p_3^2 - p_3^1} X_1 + p_6^1 X_2 + \frac{p_3^1 p_5^2 - p_5^1 p_3^2}{p_3^2 - p_3^1} X_3 + X_7.$$

Since

$$X_1 (\frac{p_5^2 - p_5^1}{p_3^2 - p_3^1}) + X_2 (p_6^1) + X_3 (\frac{p_3^1 p_5^2 - p_5^1 p_3^2}{p_3^2 - p_3^1}) + X_7(1) = 3 \neq 0,$$

we can establish the following

Theorem 2.1 *The set of pairs of intersecting non-isotropic straight lines is not measurable with respect to the group G_8, and it has no measurable subsets.*

3 Measurability with respect to S_7

The associated group $\overline{S_7}$ of the group S_7 has the infinitesimal operators X_1, X_2, X_3, X_4, X_5, X_7, and X_8 from (9), and it acts transitively on the set of parameters $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$. The integral invariant function

$$f = f(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$$

satisfying the so-called system of R. Deltheil (see [3, p. 28]; [10, p. 11])

$$X_1(f) = 0, \quad X_2(f) = 0, \quad X_3(f) = 0, \quad X_4(f) = 0, \quad X_5(f) = 0$$
$$X_7(f) + f = 0, \quad X_8(f) + 5 p_2^1 f = 0$$

has the form

$$f = \frac{h}{(p_3^1 - p_3^2)[1 + (p_2^1)^2]^2},$$

where $h = const.$

Thus we state the following

Theorem 3.1 *The set of pairs* $(G_1, G_2)(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$ *is measurable with respect to the group* S_7 *and has the density*

$$d(G_1, G_2) = \frac{1}{|p_3^2 - p_3^1|[1 + (p_2^1)^2]^2} \, dp_2^1 \wedge dp_3^1 \wedge dp_5^1 \wedge dp_6^1 \wedge dp_3^2 \wedge dp_5^2. \tag{10}$$

Differentiating (7) and substituting into (10) we obtain other expression for the density:

Corollary 3.1 *The density (10) for the pairs* (G_1, G_2) *represented by (6) can be written in the form*

$$d(G_1, G_2) = \left|\frac{a_1}{a_2^2(a_1^2 + b_1^2)^2}\right| da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr. \tag{11}$$

4 Some Crofton-type formulas with respect to S_7

Let us consider the isotropic plane ι, which is determined by the lines G_1 and G_2. The plane ι has the equation

$$\iota \colon b_1 x - a_1 y + a_1 q - b_1 p = 0.$$

If $\widetilde{P}$ is the orthogonal projection of P into Oxy, consider the affine coordinate system $(\widetilde{P}\overrightarrow{e_1}'\overrightarrow{e_2}')$ in the isotropic plane ι, where $\overrightarrow{e_1}' = (a_1, b_1, 1)$, $\overrightarrow{e_2}' = \overrightarrow{e_3}$. It should be noticed, that if $\widetilde{G} = \iota \cap Oxy$ then $\overrightarrow{e_1}' \| \widetilde{G}$. Let $J^1 = Oxz \cap \iota$ and $J^2 = Oyz \cap \iota$. Obviously

$$J^1 \colon x = p - \frac{a_1}{b_1} q, \; y = 0, \qquad J^2 \colon y = q - \frac{b_1}{a_1} p, \; x = 0,$$

and J^1, J^2 have the equations

$$J^1 \colon x = -\frac{q}{b_1}, \qquad J^2 \colon x = -\frac{p}{a_1}$$

with respect to $(\widetilde{P}\overrightarrow{e_1}'\overrightarrow{e_2}')$.

Then the density $d(J^1, J^2)$ for the pairs (J^1, J^2) with respect to the group H_4^1, which is the restriction of S_7 into ι, is (see [1, p. 201])

$$d(J^1, J^2) = \left(\frac{p}{a_1} - \frac{q}{b_1}\right)^2 d\frac{p}{a_1} \wedge d\frac{q}{b_1}.$$

Recall that ([8, p. 45])

$$s = \frac{a_1 - a_2}{a_2\sqrt{a_1^2 + b_1^2}} \tag{12}$$

is the angle from G_1 to G_2, we find

$$d(J^1, J^2) \wedge dP \wedge ds = \frac{(pb_1 - qa_1)pq}{a_1^3 b_1^4 a_2^2 \sqrt{a_1^2 + b_1^2}} \, da_1 \wedge db_1 \wedge dp \wedge dq \wedge dr \wedge da_2.$$

Comparing with (11), we get

$$d(G_1, G_2) = \left| \frac{a_1^4 b_1^4}{pq(pb_1 - qa_1)(a_1^2 + b_1^2)^{\frac{3}{2}}} \right| d(J^1, J^2) \wedge ds \wedge dP. \tag{13}$$

Let φ_i, $i = 1, 2$, b the angle between G_i and Oxy. Then ([8, p. 48])

$$\varphi_1 = \frac{1}{\sqrt{a_1^2 + b_1^2}}, \qquad \varphi_2 = \frac{a_1}{a_2\sqrt{a_1^2 + b_1^2}}, \tag{14}$$

and (13) becomes

$$d(G_1, G_2) = \left| \frac{a_1^4 b_1^4 \varphi_1^3}{pq(pb_1 - qa_1)} \right| d(J^1, J^2) \wedge ds \wedge dP. \tag{15}$$

By differentiation of (14) and by exterior multiplication by (12), we obtain

$$\begin{aligned} d(G_1, G_2) &= \left| \frac{a_1^4 b_1^4}{pq(pb_1 - qa_1)(a_1^2 + b_1^2)^{\frac{3}{2}}} \right| d(J^1, J^2) \wedge d\varphi_2 \wedge dP \\ &= \left| \frac{a_1^4 b_1^4 \varphi_1^3}{pq(pb_1 - qa_1)} \right| d(J^1, J^2) \wedge d\varphi_2 \wedge dP. \end{aligned} \tag{16}$$

If $\widetilde{\varphi}$ is the isotropic distance from J^1 to J^2, then ([7, p. 19])

$$\widetilde{\varphi} = -\frac{p}{a_1} + \frac{q}{b_1}. \tag{17}$$

Putting (17) into (15) and (16), we find

$$d(G_1, G_2) = \left| \frac{a_1^3 b_1^3 \varphi_1^3}{pq\widetilde{\varphi}} \right| d(J^1, J^2) \wedge ds \wedge dP = \left| \frac{a_1^3 b_1^3 \varphi_1^3}{pq\widetilde{\varphi}} \right| d(J^1, J^2) \wedge d\varphi_2 \wedge dP. \tag{18}$$

Let G_i^1 and G_i^2 be now the projections of G_i into Oxz and Oyz obtained in a parallel way to Oy and Ox, respectively. Then

$$\begin{aligned} G_i^1 &: \quad z = \frac{1}{a_i}x + r - \frac{p}{a_i}, \; y = 0, \; i = 1, 2, \\ G_1^2 &: \quad z = \frac{1}{b_1}y + r - \frac{q}{b_1}, \; x = 0, \\ G_2^2 &: \quad z = \frac{a_1}{a_2 b_1}y + r - \frac{a_1}{a_2 b_1}q, \; x = 0. \end{aligned}$$

Furthermore,

$$d(G_1^1, G_2^1) = \left| \frac{1}{a_1 a_2 (a_2 - a_1)} \right| da_1 \wedge da_2 \wedge dp \wedge dr \tag{19}$$

is the density for the pairs (G_1^1, G_2^1) in the isotropic plane Oxz with respect ${}^1H_4^1$ which is the restriction of S_7 into Oxz and

$$d(G_1^2, G_2^2) = \left| \frac{1}{b_1^2 a_2 (a_2 - a_1)} \right| (a_1 db_1 \wedge da_2 - a_2 db_1 \wedge da_1) \wedge dq \wedge dr$$

is the density for the pairs (G_1^2, G_2^2) in the isotropic plane Oyz with respect $^2H_4^1$ which is the restriction of S_7 into Oyz (see [1, p. 177]).

By exterior multiplication of (G_1^1, G_2^1) and $ds \wedge dq$, we get

$$d(G_1, G_2) = \left|\frac{a_1^2 \varphi_1}{b_1}\right| d(G_1^1, G_2^1) \wedge ds \wedge dq, \tag{20}$$

and by exterior multiplication of (19) and $d\varphi_1 \wedge dq$:

$$d(G_1, G_2) = \left|\frac{a_1^2 s}{b_1}\right| d(G_1^1, G_2^1) \wedge d\varphi_1 \wedge dq. \tag{21}$$

If, instead of using $d\varphi_1 \wedge dq$, we multiply by $d\varphi_2 \wedge dq$, we obtain

$$d(G_1, G_2) = \left|\frac{a_1 a_2 s}{b_1}\right| d(G_1^1, G_2^1) \wedge d\varphi_2 \wedge dq. \tag{22}$$

Analogously, we can derive the following formulas:

$$\begin{aligned} d(G_1, G_2) &= \left|\frac{a_1^2 b_1^2 \varphi_1}{a_2^3}\right| d(G_1^2, G_2^2) \wedge ds \wedge dp \\ &= \left|\frac{b_1^2 s}{a_1}\right| d(G_1^2, G_2^2) \wedge d\varphi_1 \wedge dp \\ &= \left|\frac{a_2 b_1^2 s}{a_1^2}\right| d(G_1^2, G_2^2) \wedge d\varphi_2 \wedge dp. \end{aligned} \tag{23}$$

In summary, the following theorem holds.

Theorem 4.1 *The density for the set of pairs (G_1, G_2) of intersecting non-isotropic straight lines of type β, determined by (6), with respect to the group S_7 satisfies the relations (15), (16), (18), (20), (21), (22), and (23).*

5 Measurability with respect to G_6

Now, the corresponding associated group $\overline{G_6}$ has the infinitesimal operators

$$\begin{aligned} &Y_1 = p_3^1 \frac{\partial}{\partial p_5^1} - p_2^1 \frac{\partial}{\partial p_6^1} + p_3^2 \frac{\partial}{\partial p_5^2}, \quad Y_2 = \frac{\partial}{\partial p_6^1}, \\ &Y_3 = \frac{\partial}{\partial p_5^1} + \frac{\partial}{\partial p_5^2}, \quad Y_4 = p_2^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_5^1} + p_2^1 \frac{\partial}{\partial p_3^2} - p_6^1 \frac{\partial}{\partial p_5^2}, \\ &Y_7 = 3p_3^1 \frac{\partial}{\partial p_3^1} + 2p_5^1 \frac{\partial}{\partial p_5^1} - p_6^1 \frac{\partial}{\partial p_6^1} + 3p_3^2 \frac{\partial}{\partial p_3^2} + 2p_5^2 \frac{\partial}{\partial p_5^2}, \quad Y_8 = \frac{\partial}{\partial p_1^1} + \frac{\partial}{\partial p_3^2}. \end{aligned}$$

The group $\overline{G_6}$ acts intransitively on the set of points $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$ and therefore the set of pairs (G_1, G_2) has not invariant density with respect to G_6. The system

$$Y_1(f) = 0, \; Y_2(f) = 0, \; Y_3(f) = 0, \; Y_4(f) = 0, \; Y_7(f) = 0, \; Y_8(f) = 0$$

has the solution

$$f = p_2^1,$$

and it is an absolute invariant of G_6. Consider the subset of pairs (G_1, G_2) satisfying the condition

$$p_2^1 = h, \tag{24}$$

where $h = const$. The group $\overline{G_6}$ induces on this subset the group G_6^* with the infinitesimal operators

$$Z_1 = p_3^1 \frac{\partial}{\partial p_5^1} - h \frac{\partial}{\partial p_6^1} + p_3^2 \frac{\partial}{\partial p_5^2}, \quad Z_2 = \frac{\partial}{\partial p_6^1},$$
$$Z_3 = \frac{\partial}{\partial p_5^1} + \frac{\partial}{\partial p_5^2}, \quad Z_4 = p_2^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_5^1} + p_2^1 \frac{\partial}{\partial p_3^2} - p_6^1 \frac{\partial}{\partial p_5^2},$$
$$Z_7 = 3p_3^1 \frac{\partial}{\partial p_3^1} + 2p_5^1 \frac{\partial}{\partial p_5^1} - p_6^1 \frac{\partial}{\partial p_6^1} + 3p_3^2 \frac{\partial}{\partial p_3^2} + 2p_5^2 \frac{\partial}{\partial p_5^2}, \quad Z_8 = \frac{\partial}{\partial p_1^1} + \frac{\partial}{\partial p_3^2}.$$

The integral invariant function $f = f(p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$, which satisfies the Deltheil system

$$Z_1(f) = 0,\ Z_2(f) = 0,\ Z_3(f) = 0,\ Z_4(f) = 0,\ Z_7(f) - 9f = 0,\ Z_8(f) = 0,$$

has the form

$$f = \frac{c}{(p_3^1 - p_3^2)^3},$$

where $c = \text{const}$.

Thus we state the following

Theorem 5.1 *The set of pairs $(G_1, G_2)(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$ of intersecting non-isotropic lines of type β is not measurable with respect to G_6, but it has the measurable subset*

$$p_2^1 = h, \quad h = const,$$

with the density

$$d(G_1, G_2) = \frac{1}{|p_3^2 - p_3^1|^3}\, dp_3^1 \wedge dp_5^1 \wedge dp_6^1 \wedge dp_3^2 \wedge dp_5^2. \tag{25}$$

Differentiating (7), (24), and replacing into (25), we establish

Corollary 5.1 *The set of pairs (G_1, G_2) of intersecting non-isotropic lines of type β, determined by (6), is not measurable with respect to the group G_6, but it has the measurable subset*

$$\frac{b_1}{a_1} = h, \quad h = const,$$

with the density

$$d(G_1, G_2) = \frac{1}{(a_1 - a_2)^2}\, da_1 \wedge da_2 \wedge dp \wedge dq \wedge dr.$$

6 Measurability with respect to $\mathbf{B_7}$, $\mathbf{W_7}$, $\mathbf{G_7}$, $\mathbf{V_7}$, $\mathbf{B_6}$, and $\mathbf{B_5}$

By arguments similar to those used in the sections 2, 3, and 5, we investigated the measurability with respect to all the remaining groups. We have the following results:

Theorem 6.1 *The set of pairs (G_1, G_2) of intersecting non-isotropic straight lines of type β, determined by (6), is measurable with respect to the group*

(i) B_7 and it has the density

$$d(G_1, G_2) = \left| \frac{a_1 a_2}{(a_1 - a_2)^3 \sqrt{a_1^2 + b_1^2}} \right| da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr;$$

(ii) V_7 and it has the density

$$d(G_1, G_2) = \frac{|a_1|}{(a_1 - a_2)^2 (a_1^2 + b_1^2)} da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr.$$

Theorem 6.2 *With respect to the groups W_7 and S_7 the set of pairs (G_1, G_2) of intersecting non-isotropic lines of type β is not measurable and it has no measurable subsets.*

Theorem 6.3 *The set of pairs (G_1, G_2) of intersecting non-isotropic straight lines of type β, determined by (6), is not measurable with respect to the group*

(i) B_6, but it has the measurable subset

$$\frac{b_1}{a_1} = h, \quad h = const,$$

with the density

$$d(G_1, G_2) = \left| \frac{a_1 a_2}{(a_1 - a_2)^3} \right| da_1 \wedge da_2 \wedge dp \wedge dq \wedge dr;$$

(ii) B_5, but it has the measurable subset

$$\frac{b_1}{a_1} = h_1, \quad \frac{1}{a_1} - \frac{1}{a_2} = h_2, \quad h_1, h_2 = const,$$

with the density

$$d(G_1, G_2) = \left| \frac{a_2}{a_1(a_1 - a_2)} \right| da_1 \wedge da_2 \wedge dp \wedge dq \wedge dr.$$

References

[1] Borisov, A.: Integral geometry in the Galilean plane. *Research work qualifying for a degree full-professor, Sofia,* 1998.

[2] Borisov, A. V., Spirova, M. G.: *Crofton-type formulas relating sets of pairs of intersecting nonisotropic straight lines in the simply isotropic space.* Tensor **67** (2006), 243–253.

[3] Deltheil, R.: *Sur la théorie des probabilité géométriques.* Thése Ann. Fac. Sc. Univ. Toulouse **11** (1919), 1–65.

[4] Drinfel'd, G. I.: *On the measure of the Lie groups.* Zap. Mat. Otdel. Fiz. Mat. Fak. Kharkov. Mat. Obsc. **21** (1949), 47–57 (in Russian).

[5] Drinfel'd, G. I., Lucenko, A. V.: *On the measure of sets of geometric elements.* Vest. Kharkov. Univ. **31**, 3 (1964), 34–41 (in Russian).

[6] Lucenko, A. V.: *On the measure of sets of geometric elements and thear subsets.* Ukrain. Geom. Sb. **1**, 3 (1965), 39–57 (in Russian).

[7] Sachs, H.: Ebene isotrope Geometrie. *Friedr. Vieweg Sohn, Braunschweig,* 1987.

[8] Sachs, H.: Isotrope Geometrie des Raumes. *Friedr. Vieweg and Sohn, Braunschweig–Wiesbaden,* 1990.

[9] Santaló, L. A.: Integral Geometry and Geometric Probability. *Addison-Wesley, London,* 1976.

[10] Stoka, M. I.: Geometrie Integrala. *Ed. Acad. RPR, Bucuresti,* 1967.

[11] Strubecker, K.: *Differentialgeometrie des isotropen Raumes I.* Sitzungsber. Österr. Akad. Wiss. Wien **150** (1941), 1–53.

[12] Strubecker, K.: *Differentialgeometrie des isotropen Raumes II, III, IV, V.* Math. Z. **47** (1942), 743–777; **48** (1942), 369–427; **50** (1944), 1–92; **52** (1949), 525–573.