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Abstract. Several authors gave various factorizations of the Fibonacci and Lucas numbers. The relations are derived with the help of connections between determinants of tridiagonal matrices and the Fibonacci and Lucas numbers using the Chebyshev polynomials. In this paper some results on factorizations of the Fibonacci–like numbers U_n and their squares are given. We find the factorizations using the circulant matrices, their determinants and eigenvalues.

1. Introduction

There are several well–known factorizations of the Fibonacci or Lucas numbers and some specific linear subsequences of them. In [1] Cahill et al. studied certain families of tridiagonal matrices and their correspondence to these sequences. In [2] the same authors derived these complex factorizations:

$$F_n = \prod_{k=1}^{n-1} \left(1 - 2i\cos\frac{k\pi}{n}\right),\ n \geq 2\ ,$$

and

$$L_n = \prod_{k=1}^{n} \left(1 - 2i\cos\frac{(2k-1)\pi}{2n}\right),\ n \geq 1\ .$$

They proved them by considering in what way these numbers can be connected to Chebyshev polynomials by determinants of sequences of suitable tridiagonal matrices.

In [3] Cahill and Narayan extended the previous results to construct families of tridiagonal matrices whose determinants generate an arbitrary linear subsequence F_{an+b} or L_{an+b}, where a, n are positive integers and b is a nonnegative integer.

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They chose a specific linear subsequence of the Fibonacci numbers and used it to derive the factorization

$$F_{2mn} = F_{2m} \prod_{k=1}^{n-1} \left(L_{2m} - 2\cos\frac{k\pi}{n} \right),$$

which was a generalization of the factorization

$$F_{2n} = \prod_{k=1}^{n-1} \left(3 - 2\cos\frac{k\pi}{n} \right)$$

presented in [2].

We have chosen a new way how to find out the factorization of squares of one type of the generalized Fibonacci numbers. Our method is based on the using of suitable circulant matrices and expressing the determinants of them by their eigenvalues.

Throughout the paper we adopt the conventions that the sum and the product over an empty set is 0 and 1, respectively.

2. Preliminary results

Lemma 1. ([2], Lemma 1) *Let $\{H(n), n = 1, 2, \dots\}$ be a sequence of tridiagonal matrices of the form:*

$$H(1) = \begin{pmatrix} h_{1,1} \end{pmatrix}, \qquad H(2) = \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}$$

and for $n \geq 3$

$$H(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & \\ & h_{3,2} & h_{3,3} & h_{3,4} & & \\ & & h_{4,3} & h_{4,4} & \ddots & \\ & & & \ddots & \ddots & h_{n-1,n} \\ & & & & h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

Then the successive determinants of $H(n)$ are given by the following recurrence formula

$$\begin{aligned} |H(1)| &= h_{1,1}\ , \\ |H(2)| &= h_{1,1}h_{2,2} - h_{1,2}h_{2,1}\ , \\ |H(n)| &= h_{n,n}\,|H(n-1)| - h_{n-1,n}\,h_{n,n-1}\,|H(n-2)| \ \textit{for } n \geq 3\ . \end{aligned} \tag{1}$$

In the notation of Horadam [5], we define the sequence of numbers $W_n = W_n(a, b\,; p, q)$, with arbitrary integer parameters a, b, p, q, so that

$$W_n = p\,W_{n-1} - q\,W_{n-2}\ , \quad n \geq 2\ , \qquad 2$$

where $W_0 = a$, $W_1 = b$.

The n-th terms of the Fibonacci and Lucas sequences are

$$F_n = W_n(0,1;1,-1) \ , \qquad L_n = W_n(2,1;1,-1) \ .$$

More generally, we name the Fibonacci–type sequence $U_n = W_n(0,1;p,q)$ and the Lucas–type sequence $V_n = W_n(2,p;p,q)$. The Binet formulas for U_n and V_n have forms similar to the formulas for F_n, L_n

$$U_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \ , \quad V_n = \gamma^n + \delta^n \ ,$$

where $\gamma = \frac{p+\sqrt{p^2-4q}}{2}$ and $\delta = \frac{p-\sqrt{p^2-4q}}{2}$ are the roots (mutually distinct) of the quadratic equation $x^2 - px + q = 0$. It means that the following relations hold for the numbers γ, δ:

$$\gamma + \delta = p \ , \quad \gamma - \delta = \sqrt{p^2 - 4q} \ , \quad \gamma\delta = q \ , \quad \gamma^2 + \delta^2 = p^2 - 2q \ .$$

The properties of circulant matrices are well known and widely used. A circulant matrix $C(n) = (c_k)_{k=1}^n$ of type $n \times n$ has such form where each row is a cyclic shift of the row above it. Its structure can also be characterized by noting that the (i,j) entry $C_{i,j}$ of $C(n)$ is given by

$$C_{i,j} = c_{(j-i) \pmod n)+1} \ ,$$

which identifies $C(n)$ as a special type of Toeplitz matrix.

For example Gradshteyn and Ryzhik expanded determinants of circulant matrices and gave eigenvalues of them.

Lemma 2. ([4], pp. 1111 – 1112) *Let c_k, $k = 1, \dots, n$, be complex numbers. Then*

$$\begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ & & \cdots & \\ c_2 & c_3 & \cdots & c_1 \end{vmatrix} = \prod_{k=1}^{n} (c_1 + c_2\varepsilon_k + c_3\varepsilon_k^2 + \cdots + c_n\varepsilon_k^{n-1}) \ , \qquad 3$$

where ε_k, $k = 1, \dots, n$, are the n-th roots of unity. The eigenvalues λ_k of the corresponding $n \times n$ circulant matrix are

$$\lambda_k = c_1 + c_2\varepsilon_k + c_3\varepsilon_k^2 + \cdots + c_n\varepsilon_k^{n-1} \ . \qquad 4$$

Let us denote by $B(n)$ the $n \times n$ tridiagonal matrix

$$B(n) = \begin{pmatrix} b & c & & & & \\ a & b & c & & & \\ & a & b & c & & \\ & & a & b & \ddots & \\ & & & \ddots & \ddots & c \\ & & & & a & b \end{pmatrix} \ ,$$

where a, b, c are any complex numbers. Further let $A(n)$ be the $n \times n$ circulant matrix obtained from $B(n)$ by adding only two "corner" entries a, c.

$$A(n) = \begin{pmatrix} b & c & & & & a \\ a & b & c & & & \\ & a & b & c & & \\ & & a & b & \ddots & \\ & & & \ddots & \ddots & c \\ c & & & & a & b \end{pmatrix} .$$

Lemma 3 *Let $n > 1$ be any integer. For the determinant of $A(n)$ the recursive relation*

$$|A(n+1)| = b\,|B(n)| - 2ac\,|B(n-1)| + (-1)^n(c^{n+1} + a^{n+1}) \qquad 5$$

holds.

Proof The determinant $|A(n+1)|$ can be expanded with respect to the first row

$$|A(n+1)| = b\,|B(n)| - c \begin{vmatrix} a & c & & & 0 \\ 0 & b & c & & \\ & a & b & \ddots & \\ & & \ddots & \ddots & c \\ c & & & a & b \end{vmatrix} + (-1)^n a \begin{vmatrix} a & b & c & & \\ 0 & a & b & \ddots & \\ & 0 & a & \ddots & c \\ & & \ddots & \ddots & b \\ & & & 0 & a \end{vmatrix} .$$

Expanding the last two determinants with respect to the first column or to the last row, respectively, we have

$$|A(n+1)| = b|B(n)| - c \left(a|B(n-1)| - (-1)^n c \begin{vmatrix} c & 0 & & & & \\ b & c & 0 & & & \\ a & b & c & 0 & & \\ & a & b & c & \ddots & \\ & & \ddots & \ddots & \ddots & 0 \\ & & & a & b & c \end{vmatrix} \right)$$

$$+ (-1)^n a \left(a \begin{vmatrix} a & b & c & & & \\ 0 & a & b & c & & \\ & 0 & a & b & \ddots & \\ & & 0 & a & \ddots & c \\ & & & \ddots & \ddots & b \\ & & & & 0 & a \end{vmatrix} + (-1)^{n+1} c\,|B(n-1)| \right) =$$

$$= b\,|B(n)| - ac\,|B(n-1)| + (-1)^n c^{n+1} + (-1)^n a^{n+1} - ac\,|B(n-1)| =$$

$$= b\,|B(n)| - 2ac\,|B(n-1)| + (-1)^n(c^{n+1} + a^{n+1}) . \quad (1)$$

Lemma 4 *Let n be any nonnegative integer. Then*

$$(i) \qquad U_{n+4} - q^2 U_n = pV_{n+2}\ , \tag{2}$$

$$(ii) \qquad V_{2n} - 2q^n = (p^2 - 4q)\, U_n^2\ . \tag{3}$$

Proof We can prove these identities by the Binet formulas for U_n and V_n.

(i)

$$U_{n+4} - q^2 U_n = \frac{\gamma^{n+4} - \delta^{n+4}}{\gamma - \delta} - \gamma^2\delta^2 \frac{\gamma^n - \delta^n}{\gamma - \delta} \tag{4}$$

$$= \frac{1}{\gamma - \delta}(\gamma^{n+4} - \delta^{n+4} - \gamma^{n+2}\delta^2 + \gamma^2\delta^{n+2}) \tag{5}$$

$$= \frac{1}{\gamma - \delta}(\gamma^2 - \delta^2)(\gamma^{n+2} + \delta^{n+2}) = p\, V_{n+2}\ , \tag{6}$$

(ii)

$$V_{2n} - 2q^n = \gamma^{2n} + \delta^{2n} - 2(\gamma\delta)^n = (\gamma^n - \delta^n)^2 \tag{7}$$

$$= (\gamma - \delta)^2 \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right)^2 = (p^2 - 4q)\, U_n^2\ . \tag{8}$$

3. The main result

Our main result is concentrated into the following theorem.

Theorem 1 *The factorization of squares of the Fibonacci–like numbers is given as follow*

$$U_n^2 = \prod_{k=1}^{n-1} \left(p^2 - 2q - 2q\cos\frac{2k\pi}{n}\right)\ ,\ n \geq 1\ .$$

Proof First, consider the $n \times n$ tridiagonal matrix $B(n)$ given by

$$B(n) = \begin{pmatrix} p^2 - 2q & -q & & & & \\ -q & p^2 - 2q & -q & & & \\ & -q & p^2 - 2q & -q & & \\ & & -q & p^2 - 2q & \ddots & \\ & & & \ddots & \ddots & -q \\ & & & & -q & p^2 - 2q \end{pmatrix}.$$

We will show by induction that the determinant $|B(n)| = \frac{1}{p}U_{2n+2}$. It is easy to see that $|B(1)| = p^2 - 2q = \frac{1}{p}U_4$ and $|B(2)| = p^4 - 4p^2q + 3q^2 = \frac{1}{p}U_6$.

Using Lemma 1 we can write for $n > 2$

$$|B(n)| = (p^2 - 2q)\,|B(n-1)| - q^2\,|B(n-2)|$$

and further as $U_n = pU_{n-1} - qU_{n-2}$ or $U_{n-2} = \frac{p}{q}U_{n-1} - \frac{1}{q}U_n$

$$\begin{aligned}|B(n)| &= (p^2 - 2q)\,\frac{1}{p}\,U_{2n} - q^2\frac{1}{p}\,U_{2n-2} \\ &= \left(p - 2\,\frac{q}{p}\right) U_{2n} - \frac{q^2}{p}\,\left(\frac{p}{q}\,U_{2n-1} - \frac{1}{q}\,U_{2n}\right) \\ &= \left(p - \frac{q}{p}\right) U_{2n} - q\left(\frac{p}{q}\,U_{2n} - \frac{1}{q}\,U_{2n+1}\right) \\ &= U_{2n+1} - \frac{q}{p}\,U_{2n} = \frac{1}{p}\,U_{2n+2}\ . \end{aligned} \tag{6}$$

Consider now a circulant matrix $A(n) = (a_{i,j})$ of type $n \times n$ which has the same entries as $B(n)$ only $a_{1,n} = a_{n,1} = -q$. With respect to Lemma 3 we can express for $n > 2$ the determinant of $A(n)$ in the following form

$$|A(n)| = (p^2 - 2q)\,|B(n-1)| - 2q^2\,|B(n-2)| - 2q^n\ .$$

Using relation (6) and Lemma 4 we can write

$$\begin{aligned}|A(n)| &= (p - 2\,\frac{q}{p})\,U_{2n} - 2\,\frac{q^2}{p}\,U_{2n-2} - 2q^n \\ &= \frac{1}{p}\,U_{2n+2} - \frac{q^2}{p}\,U_{2n-2} - 2q^n = V_{2n} - 2q^n = (p^2 - 4q)\,U_n^2\ . \end{aligned} \tag{7}$$

But we can calculate the determinant of the circulant matrix $A(n)$ in an alternative way using Lemma 2. Then

$$|A(n)| = \prod_{k=1}^{n} (p^2 - 2q - q\varepsilon_k - q\,\varepsilon_k^{n-1})\ ,$$

where ε_k, $k = 1, 2, \ldots, n$, are the n-th roots of unity. Obviously,

$$\varepsilon_k = \cos\frac{2k\pi}{n} + i\,\sin\frac{2k\pi}{n},\ \ k = 1, 2, \ldots, n$$

and

$$\begin{aligned}\varepsilon_k^{n-1} &= \cos\frac{(n-1)2k\pi}{n} + i\,\sin\frac{(n-1)2k\pi}{n} \\ &= \cos\left(2k\pi - \frac{2k\pi}{n}\right) + i\,\sin\left(2k\pi - \frac{2k\pi}{n}\right) = \cos\frac{2k\pi}{n} - i\,\sin\frac{2k\pi}{n}\ . \end{aligned} \tag{9}$$

Therefore

$$|A(n)| = \prod_{k=1}^{n} \left(p^2 - 2q - 2q\cos\frac{2k\pi}{n}\right) \tag{8}$$

and combining (7) and (8) we have

$$U_n^2 = \frac{1}{p^2 - 4q} \prod_{k=1}^{n} \left(p^2 - 2q - 2q\cos\frac{2k\pi}{n}\right)\ .$$

As for $k = n$ the corresponding factor is $p^2 - 4q$ the proved relation follows. □

Theorem 2 *The factorization of the Fibonacci–like numbers has the form*

$$U_n = \prod_{k=1}^{n-1} \left(p - 2\sqrt{q} \cos \frac{k\pi}{n} \right) \ , \ n \geq 1 \ .$$

Proof From Theorem 1 we have $U_n^2 = \prod_{k=1}^{n-1} \left(p^2 - 2q - 2q \cos \frac{2k\pi}{n} \right)$. Using the well–known formulas for cosines we can write successively

$$\begin{aligned} U_n^2 &= \prod_{k=1}^{n-1} \left(p^2 - 4q \frac{1 + \cos \frac{2k\pi}{n}}{2} \right) = \prod_{k=1}^{n-1} \left(p^2 - 4q \cos^2 \frac{k\pi}{n} \right) \\ &= \prod_{k=1}^{n-1} \left(p - 2\sqrt{q} \cos \frac{k\pi}{n} \right) \left(p + 2\sqrt{q} \cos \frac{k\pi}{n} \right) \\ &= \prod_{k=1}^{n-1} \left(p - 2\sqrt{q} \cos \frac{k\pi}{n} \right) \left(p - 2\sqrt{q} \cos \frac{(n-k)\pi}{n} \right) \\ &= \prod_{k=1}^{n-1} \left(p - 2\sqrt{q} \cos \frac{k\pi}{n} \right)^2 \end{aligned}$$

and the factorization of U_n follows as the coefficient of the highest power of p on the both sides is equal to 1. □

4. Concluding remarks

Special cases of the sequence $\{W_n\}$ which interest us in the number theory are above all the following ones. Their factorizations are derived from Theorem 1:
the Fibonacci sequence $\{F_n\}$:

$$F_n^2 = \prod_{k=1}^{n-1} \left(3 + 2 \cos \frac{2k\pi}{n} \right) \ , \quad n \geq 1 \ ,$$

the Pell sequence $\{P_n\} = \{W_n(0, 1; 2, -1)\}$:

$$P_n^2 = \prod_{k=1}^{n-1} \left(6 + 2 \cos \frac{2k\pi}{n} \right) = 2^{n-1} \prod_{k=1}^{n-1} \left(3 + \cos \frac{2k\pi}{n} \right) \ , \quad n \geq 1 \ ,$$

the Fermat sequence $\{f_n\} = \{W_n(0, 1\,; 3, 2)\}$ (its terms are also known as the Mersenne numbers $M_n = 2^n - 1$):

$$f_n^2 = \prod_{k=1}^{n-1} \left(5 - 4 \cos \frac{2k\pi}{n} \right) \ , \quad n \geq 1 \ .$$

Some open questions arise if we want to use circulants to factorizations of the numbers related to the generalized Fibonacci numbers. For example, is it possible to find out suitable circulant matrices for factorizations of the Lucas–like numbers or their squares?

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