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# THE DESIGN OF THE CENTURY

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ABSTRACT. We construct a 2-chromatic Steiner system $S(2,4,100)$ in which every block contains three points of one colour and one point of the other colour. The existence of such a design has been open for over 25 years.



## 1. The background

A *Steiner system* $S(t,k,v)$ is an ordered pair $(V,\mathscr{B})$ where $V$ is a set of cardinality $v$, the *base set*, and $\mathscr{B}$ is a collection of $k$-subsets of $V$, the *blocks*, which collectively have the property that every $t$-element subset of $V$ is contained in precisely one block. Elements of $V$ are called *points*. In this paper we are principally concerned with the case in which $t = 2$ and $k = 4$. Steiner systems $S(2,4,v)$ exist if and only if $v \equiv 1$ or $4 \pmod{12}$, [4]; such values of $v$ are called *admissible*. Given a Steiner system $S(2,4,v)$, we may ask whether it is possible to colour each point of the base set $V$ with one of two colours, say red or blue, so that no block is monochromatic. A Steiner system $S(2,4,v)$ having this property is said to be 2-*chromatic* or to have a *blocking set*. It was shown in [5] that 2-chromatic $S(2,4,v)$s exist for all admissible $v$ with the possible exception of three values, $v = 37$, 40 and 73. Existence for these three values was established in [3]. Perhaps we should also remark here that it is known that for all $v \geq 25$ there exists a Steiner system $S(2,4,v)$ which is not 2-chromatic, [8].

In a 2-chromatic $S(2,4,v)$ let $c$ and $v - c$ be the cardinalities of the red and blue colour classes, respectively. If $b_1$, $b_2$ and $b_3$ are the numbers of blocks with

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colour patterns $RRRB$, $RRBB$ and $RBBB$, respectively, then by counting pairs we have:

$$\begin{aligned} 3b_1 + b_2 &= \frac{c\,(c-1)}{2}, \\ b_2 + 3b_3 &= \frac{(v-c)(v-c-1)}{2}, \\ 3b_1 + 4b_2 + 3b_3 &= c\,(v-c). \end{aligned}$$

Solving the equations for $b_2$ gives $b_2 = (4vc - 4c^2 + v - v^2)/4$, which is non-negative for

$$\frac{v-\sqrt{v}}{2} \le c \le \frac{v+\sqrt{v}}{2}.$$

Furthermore, in the extreme cases where $\{c, v-c\} = \{(v-\sqrt{v}\,)/2, (v+\sqrt{v}\,)/2\}$ it follows that $b_2 = 0$; i.e. every block contains three points of one colour and one of the other colour. Moreover, the monochromatic triples of each colour appearing in the blocks form Steiner systems $S\big(2,3,(v-\sqrt{v}\,)/2\big)$ and $S\big(2,3,(v+\sqrt{v}\,)/2\big)$. An $S(2,3,w)$ is usually called a *Steiner triple system* and denoted by STS$(w)$; they exist if and only if $w \equiv 1$ or $3 \pmod 6$, [6]. A modern account of Kirkman's work is given in [1]. From the preceding discussion, it is easy to deduce that a 2-chromatic $S(2,4,v)$ having all blocks containing three points of one colour and one of the other colour can exist only if $v$ is of the form $(12s+2)^2$ or $(12s+10)^2$, $s \ge 0$.

The smallest non-trivial case is therefore $v = 100$, and has become known as "the Design of the Century". Its existence, and possible construction, has been a problem in Design Theory for over 25 years. An early reference is [7]. In this paper we construct the design. We make no claim for uniqueness and, indeed, we think it highly unlikely.

## 2. The method

The cardinalities of the two colour classes are 55, the red points, and 45, the blue points. Denote the former by $A_0, A_1, \dots, A_{54}$ and the latter by $\infty, B_0, B_1, \dots, B_{43}$. We will seek an $S(2,4,100)$ having an automorphism $\sigma$ of order 11 defined by

$$\sigma : A_i \mapsto A_{i+5 \pmod{55}}, \quad B_j \mapsto B_{j+4 \pmod{44}}, \quad \infty \mapsto \infty.$$

Our method is based on a simple backtrack algorithm with four distinct stages.

*Stage 1.* Select systems STS(55) and STS(45), both having automorphism $\sigma$, on the red and blue points respectively. The latter is an example of a 4-rotational STS$(v)$; such systems exist for $v \equiv 1, 9, 13$ or $21 \pmod{24}$, [2].

*Stage 2.* The blue system has 30 orbits under the automorphism. We partition these into five classes of six orbits, and label each class with a different point from the set $\{A_0, A_1, A_2, A_3, A_4\}$. Within each class we then assign the label to one block of each of the six orbits in such a way that the blocks to which the label is assigned form a partial parallel class; i.e. the blocks are pairwise disjoint. The assignment of red points to the other blocks of blue points is completely determined by $\sigma$. It is clear that this assignment ensures that there are no repeated pairs of a blue point with a red point.

*Stage 3.* The red system has 45 orbits under the automorphism. We next deal with the blue point $\infty$. In the course of performing stage 2 of the algorithm the point $\infty$ will have been paired with two of the five subsets $\{A_{i+j} : j = 0, 5, 10, \ldots, 50\}$, $i = 0, 1, 2, 3, 4$. We assign $\infty$ to all blocks of a single orbit whose red points cover the remaining three subsets.

*Stage 4.* This leaves 44 orbits of the red system. As in stage 2 we partition these into four classes of 11 orbits and label each class with a different point from the set $\{B_0, B_1, B_2, B_3\}$. Within each class, we then assign the label, say $X$, to one block of each of the 11 orbits in such a way that the blocks to which $X$ is assigned form a partial parallel class, say $\mathscr{P}$. We attempt to do this while satisfying the further constraint that none of the 22 red points with which $X$ has already been paired in stage 2 occur in $\mathscr{P}$. This latter is, of course, a very severe constraint. Again, the assignment of the blue points to the other blocks of red points is completely determined by $\sigma$.

Finally, we make a brief remark about our implementation of the algorithm. Stages 3 and 4 execute very quickly on a modern computer system and we always ran the backtracking to completion. However, for each particular choice of systems STS(55) and STS(45), we did not run the backtracking of stage 2 to completion, preferring instead to return to stage 1 after a certain period of time and select new systems.

## 3. The design

Listed below are 75 blocks which, under the mapping $\sigma$, give "the Design of the Century". As described in the last section, the construction of the design involved significant computing. However, it is perfectly feasible, although perhaps a little tedious, to check the design by hand, and the dedicated reader is invited to do this.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B0 | B1 | B9 | A0 | B4 | B6 | B23 | A0 | B8 | B11 | B13 | A0 |
| B12 | B20 | B10 | A0 | B36 | B5 | B7 | A0 | B2 | B3 | B38 | A0 |
| B0 | B4 | B33 | A1 | B40 | B3 | B6 | A1 | B8 | B24 | B5 | A1 |
| B16 | B7 | B11 | A1 | B1 | B2 | B21 | A1 | B9 | B13 | B42 | A1 |
| B0 | B6 | B24 | A2 | B8 | B19 | B40 | A2 | B4 | B31 | B3 | A2 |
| B9 | B14 | B43 | A2 | B1 | B7 | B29 | A2 | B5 | B26 | B2 | A2 |
| B0 | B14 | B21 | A3 | B4 | B35 | B41 | A3 | B1 | B10 | B31 | A3 |
| B5 | B23 | $\infty$ | A3 | B2 | B7 | B18 | A3 | B22 | B34 | B3 | A3 |
| B28 | B1 | B14 | A4 | B4 | B22 | B26 | A4 | B0 | B38 | $\infty$ | A4 |
| B29 | B39 | B2 | A4 | B37 | B5 | B19 | A4 | B3 | B11 | B35 | A4 |
| A25 | A29 | A19 | B0 | A20 | A32 | A5 | B0 | A35 | A48 | A18 | B0 |
| A15 | A39 | A11 | B0 | A41 | A43 | A8 | B0 | A21 | A42 | A13 | B0 |
| A31 | A14 | A28 | B0 | A16 | A9 | A12 | B0 | A17 | A23 | A49 | B0 |
| A37 | A44 | A33 | B0 | A22 | A34 | A38 | B0 | | | | |
| A35 | A36 | A13 | B1 | A10 | A17 | A18 | B1 | A25 | A44 | A11 | B1 |
| A20 | A43 | A19 | B1 | A5 | A37 | A39 | B1 | A15 | A6 | A12 | B1 |
| A30 | A23 | A28 | B1 | A26 | A29 | A34 | B1 | A21 | A38 | A48 | B1 |
| A27 | A42 | A9 | B1 | A32 | A49 | A7 | B1 | | | | |
| A5 | A7 | A21 | B2 | A20 | A25 | A46 | B2 | A35 | A41 | A11 | B2 |
| A40 | A54 | A15 | B2 | A30 | A47 | A19 | B2 | A36 | A37 | A23 | B2 |
| A26 | A39 | A24 | B2 | A16 | A32 | A53 | B2 | A31 | A12 | A22 | B2 |
| A17 | A8 | A9 | B2 | A13 | A28 | A49 | B2 | | | | |
| A40 | A43 | A32 | B3 | A35 | A44 | A15 | B3 | A10 | A20 | A38 | B3 |
| A5 | A27 | A47 | B3 | A25 | A6 | A13 | B3 | A21 | A26 | A36 | B3 |
| A31 | A42 | A11 | B3 | A16 | A28 | A34 | B3 | A41 | A9 | A29 | B3 |
| A12 | A17 | A48 | B3 | A8 | A24 | A54 | B3 | A0 | A11 | A37 | $\infty$ |

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