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THE ALGEBRAIC CLOSURE OF A p-ADIC NUMBER FIELD IS A COMPLETE TOPOLOGICAL FIELD

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ABSTRACT. The algebraic closure of a p-adic field is not a complete field with the p-adic topology. We define another field topology on this algebraic closure so that it is a complete field. This new topology is finer than the p-adic topology and is not provided by any absolute value. Our topological field is a complete, not locally bounded and not first countable field extension of the p-adic number field, which answers a question of Mutylin.

1. Introduction

A *topological ring* $(R, \mathcal{T})$ is a ring R provided with a topology $\mathcal{T}$ such that the algebraic operations $(x,y) \mapsto x \pm y$ and $(x,y) \mapsto xy$ are continuous. A *topological field* $(K, \mathcal{T})$ is a field K equipped with a ring topology $\mathcal{T}$ such that the inversion $x \mapsto x^{-1}$ is also continuous. For an introduction to topological fields, the books [3], [15], [17] are recommended.

We consider the field of p-adic numbers $\mathbb{Q}_p$ and its algebraic closure $\overline{\mathbb{Q}}_p$. There is a unique extension of the p-adic absolute value $|\ |_p$ from $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p$. The field $(\overline{\mathbb{Q}}_p, |\ |_p)$ is not complete, its completion $\mathbb{C}_p$ is an algebraically closed and complete field with an absolute value extended from $\mathbb{Q}_p$. This field $\mathbb{C}_p$ is called the p-adic analog of the field of complex numbers. The cardinality of the three fields, $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$, is $2^{\aleph_0}$. See, for instance, the books of p-adic analysis [2], [4], [6], [11], [14], [16].

In this paper, we propose a change in the above scheme. Instead of performing the completion of $(\overline{\mathbb{Q}}_p, |\ |_p)$, we introduce a field topology $\mathcal{T}_\mu$ on $\overline{\mathbb{Q}}_p$ such that $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is a complete topological field. Our field topology $\mathcal{T}_\mu$ is finer

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than the p-adic topology on $\overline{\mathbb{Q}}_p$. Nevertheless, for each finite field extension $K/\mathbb{Q}_p$, with $K \subset \overline{\mathbb{Q}}_p$, the subspace topology that K inherits from $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is just the p-adic topology on K. Our topology $\mathcal{T}_\mu$ does not satisfy the first axiom of countability, and therefore does not correspond to any absolute value. The topological field $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is a complete, not locally bounded and not first countable field extension of the p-adic number field, which answers a question of Mutylin [12; Table 2]. In the last section we make some comments about the possibility of defining analytic functions on $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$.

Poonen [13] constructs a p-adic version of a Mal'cev-Neumann field of series in which the elements are formal series of the form $\sum_{g\in S} \alpha_g p^g$, where S is a well-ordered subset of $\mathbb{Q}$ and the α_g's are residue class representatives. This field contains $\mathbb{C}_p$ strictly; both fields share similar properties. This construction is inspired by [7]. Besides, the closed subfields of $(\mathbb{C}_p, |\ |_p)$ are studied in [1], [5].

Given an element $\alpha \in \overline{\mathbb{Q}}_p$, we denote by $\deg(\alpha)$ the degree of α over the p-adic field $\mathbb{Q}_p$. We denote by $v_p(\alpha)$ the p-adic valuation which corresponds to the unique extension of the p-adic absolute value to $\overline{\mathbb{Q}}_p$ (and also to $\mathbb{C}_p$), that is, $|\alpha|_p = p^{-v_p(\alpha)}$. We recall that the value group of $\overline{\mathbb{Q}}_p$ is $\mathbb{Q}$. We denote the open and closed disks in $\overline{\mathbb{Q}}_p$ by

$$B(0,r) = \{\alpha \in \overline{\mathbb{Q}}_p :\ |\alpha|_p < r\}, \qquad \overline{B}(0,r) = \{\alpha \in \overline{\mathbb{Q}}_p :\ |\alpha|_p \le r\}.$$

We recall that for a family $\{U_i\}_{i\in I}$ of subsets of a commutative ring R to be a fundamental system of neighbourhoods of zero for a Hausdorff ring topology $\mathcal{T}$ on R, it suffices that the following properties hold.

(1) For all $i \in I$, $0 \in U_i$, $U_i = -U_i$.
(2) For all $i, j \in I$ there exists $k \in I$ such that $U_k \subseteq U_i \cap U_j$.
(3) For all $i \in I$ there exists $k \in I$ such that $U_k + U_k \subseteq U_i$.
(4) For all $i \in I$ there exists $k \in I$ such that $U_k U_k \subseteq U_i$.
(5) For all $i \in I$ and $x \in R$ there exists $k \in I$ such that $xU_k \subseteq U_i$.
(6) $\bigcap_{i\in I} U_i = \{0\}$.

If, in addition, R is a field, then $\mathcal{T}$ is a field topology if $\{U_i\}_{i\in I}$ also satisfies the following condition.

(7) For all $i \in I$ there exists $k \in I$ such that $(1+U_k)^{-1} \subseteq 1 + U_i$.

See [15; p. 4] or [17; p. 3], for instance.

2. The field topology on $\overline{\mathbb{Q}}_p$

In this section we define a field topology on $\overline{\mathbb{Q}}_p$ (the algebraic closure of the p-adic field $\mathbb{Q}_p$) and show some of its properties. Throughout this article we will denote by $\mathcal{F}$ the set of strictly increasing functions $f\colon \mathbb{N} \to \mathbb{N}$. The family $\mathcal{F}$ is a directed set with the partial order $f \geq g$ if $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. We recall a result about the intermediate fields of the extension $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ which will have important consequences in the sequel.

LEMMA 1. ([14; p. 132]) *For any integer $n \geq 1$, there are only finitely many extensions of $\mathbb{Q}_p$ of degree n in $\overline{\mathbb{Q}}_p$. Thus, $\overline{\mathbb{Q}}_p$ is the union of a countable number of finite field extensions of $\mathbb{Q}_p$.*

Applying the previous lemma, we conclude that, for each $n \in \mathbb{N} \cup \{0\}$, there exists a finite field extension

$$K_n/\mathbb{Q}_p\,, \qquad K_n \subset \overline{\mathbb{Q}}_p\,, \tag{8}$$

such that every $\alpha \in \overline{\mathbb{Q}}_p$ with $\deg(\alpha) \leq n$ belongs to K_n (see also [2; p. 74]). We also assume that $K_n \subsetneq K_{n+1}$ for all n and $K_1 = \mathbb{Q}_p$. We define

$$\lambda(n) = [K_n : \mathbb{Q}_p]\,; \tag{9}$$

notice that $\lambda \in \mathcal{F}$. Certainly, for $n > 1$, there are $\beta \in K_n$ such that $\deg(\beta) > n$. It is clear that

$$\overline{\mathbb{Q}}_p = \bigcup_{n\in\mathbb{N}} K_n\,.$$

We introduce some subsets of $\overline{\mathbb{Q}}_p$. For $t, n \in \mathbb{N}$ we define

$$B\lfloor t, n\rfloor = \{\alpha \in K_n :\ v_p(\alpha) \geq t\} = \overline{B}(0, p^{-t}) \cap K_n\,.$$

Each of these subsets is a compact additive subgroup of K_n (provided with the p-adic topology). If $t_1 \geq t_2$ and $n_1 \leq n_2$, we have the inclusion $B\lfloor t_1, n_1\rfloor \subseteq B\lfloor t_2, n_2\rfloor$. In particular, we set

$$A_n = B\lfloor 1, n\rfloor = \overline{B}(0, p^{-1}) \cap K_n\,. \tag{10}$$

It is clear that $A_n \subset A_{n+1}$ and $A_n + A_m = A_s$, where $s = \max\{n, m\}$.

For each $f \in \mathcal{F}$, we define the following subset, which is an additive subgroup of $\overline{\mathbb{Q}}_p$.

$$W_f = \bigcup_{m=1}^{\infty} \left(\sum_{s=1}^{m} B\lfloor f(s), s\rfloor \right). \tag{11}$$

THEOREM 2. *The family $\{W_f\}_{f\in\mathcal{F}}$ is a neighbourhood base at zero for a Hausdorff field topology on $\overline{\mathbb{Q}}_p$ finer than the p-adic topology. We denote this topology by $\mathcal{T}_\mu$.*

P r o o f . We shall check that the family $\{W_f\}_{f\in\mathcal{F}}$ satisfies properties (1)–(7). Properties (1) and (2) are immediate. Since each W_f is an additive subgroup of $\overline{\mathbb{Q}}_p$, the property (3) is satisfied. We verify property (4). Let us see that $W_f W_f \subseteq W_f$ for all $f \in \mathcal{F}$. It suffices to show that, if $n \geq s$, then

$$B\lfloor f(s), s\rfloor\, B\lfloor f(n), n\rfloor \subseteq B\lfloor f(n), n\rfloor .$$

Let $\alpha_s \in B\lfloor f(s), s\rfloor$ and $\alpha_n \in B\lfloor f(n), n\rfloor$. We have that $\alpha_s\alpha_n \in K_n$, and $v_p(\alpha_s\alpha_n) = v_p(\alpha_s) + v_p(\alpha_n) \geq f(s) + f(n) \geq f(n)$. Therefore $\alpha_s\alpha_n \in B\lfloor f(n), n\rfloor$.

We check property (5). Given W_f and $\beta \in K_s \subset \overline{\mathbb{Q}}_p$, we define $g \in \mathcal{F}$ as $g(n) = f(n+s) + m$, where $m \in \mathbb{N}\cup\{0\}$ satisfies $-m < v_p(\beta)$. Let $\alpha = \sum_{n=1}^{k} \alpha_n \in W_g$, where $\alpha_n \in B\lfloor g(n), n\rfloor$. We have that $\beta\alpha_n \in K_t$, where $t = \max\{n, s\}$. Besides,

$$v_p(\beta\alpha_n) \geq -m + v_p(\alpha_n) \geq -m + g(n) = f(n+s) \geq f(t) .$$

Therefore $\beta\alpha_n \in B\lfloor f(t), t\rfloor$, and so $\beta\alpha \in W_f$. We have proven that $\beta W_g \subseteq W_f$.

Now we verify property (7) by showing that $(1 + W_f)^{-1} \subseteq 1 + W_f$ for all $f \in \mathcal{F}$. Let $\alpha = \sum_{n=1}^{t} \alpha_n \in W_f$, where $\alpha_n \in B\lfloor f(n), n\rfloor$. In order to construct the inverse of $1+\alpha$ we write

$$\left(1 + \sum_{n=1}^{t} \alpha_n\right)\left(1 + \sum_{n=1}^{t} \beta_n\right) = 1 ,$$

where the β_n are defined inductively according to the following rule:

$$0 = \alpha_n + \beta_n + \sum_{\max\{i,j\}=n} \alpha_i\beta_j , \qquad n = 1, \dots, t .$$

That is,

$$\beta_1 = \frac{-\alpha_1}{1+\alpha_1} ,$$

$$\beta_2 = \frac{-\alpha_2(1+\beta_1)}{1+\alpha_1+\alpha_2} ,$$

$$\vdots$$

$$\beta_n = \frac{-\alpha_n\left(1 + \sum_{i=1}^{n-1} \beta_i\right)}{1 + \sum_{i=1}^{n} \alpha_i} .$$

It is easy to check inductively that $v_p(\beta_n) = v_p(\alpha_n) \geq f(n)$ and $\beta_n \in K_n$. Hence $\beta = \sum_{n=1}^{t} \beta_n \in W_f$.

Finally, we check that the topology $\mathcal{T}_\mu$ is finer than the p-adic topology on $\overline{\mathbb{Q}}_p$, which implies property (6). Given an open ball $B(0, p^{-s})$ of center 0 and radius p^{-s}, we choose $f \in \mathcal{F}$ such that $f(1) > s$, it is clear that $W_f \subseteq B(0, p^{-s})$. □

Later, we shall see that $\mathcal{T}_\mu$ is strictly finer than the p-adic topology. Now we show some immediate consequences from the definition of the topology $\mathcal{T}_\mu$. Since each W_f in the basis $\{W_f\}_{f\in\mathcal{F}}$ is an additive subgroup, a series $\sum_{n=1}^{\infty} \alpha_n$ converges in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ if and only if $\alpha_n \to 0$. Let $\alpha \in \overline{\mathbb{Q}}_p$ such that $\deg(\alpha) \leq m$ and $v_p(\alpha) \geq f(m)$ for some $f \in \mathcal{F}$ and $m \in \mathbb{N}$, it is immediate that $\alpha \in W_f$.

Lemma 3. *Let $K \subset \overline{\mathbb{Q}}_p$ be a finite field extension of $\mathbb{Q}_p$, then the p-adic topology on K coincides with the subspace topology inherited from $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$.*

P r o o f. For each open ball $B(0, p^{-s})$, we consider $f \in \mathcal{F}$ such that $f(1) > s$; it is clear that $W_f \cap K \subseteq B(0, p^{-s})$. On the other hand, let $[K : \mathbb{Q}_p] = m$ be the degree of the field extension. For each neighbourhood W_f, we take the ball $B(0, p^{-f(m)}) \subset K$. Every element $\alpha \in B(0, p^{-f(m)}) \subset K$ satisfies that $\deg(\alpha) \leq m$ and $v_p(\alpha) > f(m)$, and so $\alpha \in B\lfloor f(m), m\rfloor \cap K \subset W_f \cap K$. That is, $B(0, p^{-f(m)}) \subseteq W_f \cap K$. □

An immediate consequence is that all the sets $B\lfloor t, n\rfloor$ and A_n are compact in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$.

Lemma 4. *The topological space $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is separable and σ-compact.*

P r o o f. Each finite field extension $K/\mathbb{Q}_p$ is separable and σ-compact with the p-adic topology. By Lemma 3, this topology coincides in K with the subspace topology $\mathcal{T}_\mu|_K$. Therefore $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is a countable union of subspaces which are separable and σ-compact, and so $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ satisfies both conditions. □

We introduce another family of subsets of $\overline{\mathbb{Q}}_p$ which also constitutes a neighbourhood base at zero. They give a more clear vision of the underlying idea in the topology $\mathcal{T}_\mu$. For each $f \in \mathcal{F}$, we define

$$Z_f = \Big\{ \sum_{n=1}^{t} \alpha_n : \ t \in \mathbb{N}, \ \ f(\deg(\alpha_n)) \leq v_p(\alpha_n) \Big\} \subset \overline{\mathbb{Q}}_p . \tag{12}$$

Notice that the subscript in the above sum $\sum \alpha_n$ does not play any role, it only matters that the sum is finite. Observe that each Z_f is an additive subgroup of $\overline{\mathbb{Q}}_p$.

LEMMA 5. *The family $\{Z_f\}_{f\in\mathcal{F}}$ is another fundamental system of zero neighbourhoods for the topological field $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$.*

Proof. First, we show that $Z_f \subseteq W_f$ for each $f \in \mathcal{F}$. Given $\alpha = \sum_{n=1}^{t} \alpha_n \in Z_f$ written according to (12), we sum the terms which have the same degree:

$$\beta_s = \sum_{\deg(\alpha_n)=s} \alpha_n \in K_s .$$

We have that $v_p(\beta_s) \geq \min\{v_p(\alpha_n) : \deg(\alpha_n) = s\} \geq f(s)$. Therefore $\beta_s \in B\lfloor f(s), s\rfloor$, and so

$$\alpha = \sum_{s=1}^{m} \beta_s \in \bigcup_{m=1}^{\infty} \left(\sum_{s=1}^{m} B\lfloor f(s), s\rfloor \right) = W_f .$$

Second, we prove that $W_{f\circ\lambda} \subseteq Z_f$ for each $f \in \mathcal{F}$, where λ is the function defined in (9). Since Z_f is an additive subgroup of $\overline{\mathbb{Q}}_p$, it suffices to prove that $B\lfloor f(\lambda(s)), s\rfloor \subseteq Z_f$ for every $s \in \mathbb{N}$. Now, if $\beta \in B\lfloor f(\lambda(s)), s\rfloor$, then $\deg(\beta) \leq \lambda(s)$ and $v_p(\beta) \geq f(\lambda(s)) \geq f(\deg(\beta))$, that is, $\beta \in Z_f$. □

Since both Z_f and W_f are additive subgroups and neighbourhoods of zero, then both are open (and closed) subgroups of $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$.

We are going to prove that $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is complete. We follow a development analogous to that in [8; §8], which is highly inspired in [18], [19]. First, we study the convergent sequences in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$, we need the compact sets A_n, defined in (10).

LEMMA 6. *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence converging to zero in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$. Then there exists $l \in \mathbb{N}$ such that $h_n \in A_l$ for all n but a finite number, that is, there exists a bound $m \in \mathbb{N}$ such that $\deg(h_n) \leq m$ for all n.*

Proof. We reason by the way of contradiction. We may assume that $h_n \in \overline{B}(0, p^{-1})$ and $h_n \notin A_n$ for all n, after passing to subsequences if required. We shall construct a neighbourhood of zero W_f not containing any value of the sequence $(h_n)_{n\in\mathbb{N}}$, which is an absurd. We define $H_n = \{h_i\}_{i\in\mathbb{N}} \cap A_n$, which is a finite set for every n. Since $\overline{B}(0, p^{-1}) = \bigcup_{n\in\mathbb{N}} A_n$, it is clear that $\{h_i\}_{i\in\mathbb{N}} = \bigcup_{n\in\mathbb{N}} H_n$. There exists $t_1 > 1$ such that $H_1 \cap B\lfloor t_1, 1\rfloor = \emptyset$. If

$H_2 \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor m, 2\rfloor\big) \neq \emptyset$ for all $m > t_1$, then, since H_2 is finite set, there exists $\alpha \in H_2 \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor m, 2\rfloor\big)$ for all $m > t_1$. This means that $\alpha = \beta_m + \gamma_m$, where $\beta_m \in B\lfloor t_1, 1\rfloor$ and $\gamma_m \in B\lfloor m, 2\rfloor$ for all $m > t_1$. Since $\gamma_m \to 0$, then $\beta_m \to \alpha$. The set $B\lfloor t_1, 1\rfloor$ is closed, thus $\alpha \in B\lfloor t_1, 1\rfloor \subset A_1$, and so $\alpha \in H_1$; but this contradicts the choice of t_1. Hence there exists $t_2 > t_1$ satisfying $H_2 \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor t_2, 2\rfloor\big) = \emptyset$. Continuing in the same manner, at the sth step we find $t_s > t_{s-1}$ such that

$$H_s \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor t_2, 2\rfloor + \ldots + B\lfloor t_s, s\rfloor\big) = \emptyset .$$

We choose $f \in \mathcal{F}$ such that $f(n) = t_n$ for all $n \in \mathbb{N}$. Taking into account (11), we conclude that $\Big(\bigcup_{s\in\mathbb{N}} H_s\Big) \cap W_f = \emptyset$, that is, $\{h_n\}_{n\in\mathbb{N}} \cap W_f = \emptyset$, which is an absurd. □

Let us give some immediate consequences: if a sequence $(h_n)_{n\in\mathbb{N}}$ converges to any value in $\big(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu\big)$, then there exists a bound $m \in \mathbb{N}$ such that $\deg(h_n) \leq m$ for all n. If $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then there also exists a bound m such that $\deg(g_n) \leq m$ for all n, and consequently $g_n \in K_m$ for all n. By Lemma 3, K_m is complete with the subspace topology inherited from $\big(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu\big)$, hence the Cauchy sequence $(g_n)_{n\in\mathbb{N}}$ has limit. We have proven that $\big(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu\big)$ is sequentially complete.

COROLLARY 7. *The field topology $\mathcal{T}_\mu$ on $\overline{\mathbb{Q}}_p$ is strictly finer than the p-adic topology.*

Proof. We have seen that $\mathcal{T}_\mu$ is finer than the p-adic topology. Consider a sequence of elements $\alpha_n \in \overline{\mathbb{Q}}_p$ such that $v_p(\alpha_n) \geq 0$ and $\deg(\alpha_n) \geq n$ for all $n \in \mathbb{N}$. Then we have that $\alpha_n p^n \to 0$ with respect to the p-adic topology, but $\alpha_n p^n \not\to 0$ with respect to the topology $\mathcal{T}_\mu$. □

Notice that certain sequences like the $\big(\alpha_n p^n\big)_{n\in\mathbb{N}}$ in the proof above are used in [4; p. 165], [6; p. 71] and [11; p. 50] to show that neither $\overline{\mathbb{Q}}_p$ nor $\mathbb{Q}_p^{\mathrm{unram}}$ are complete fields with the p-adic topology. The idea in these books is to prove that

$$\Big(\sum_{n=1}^{m} \alpha_n p^n\Big)_{m\in\mathbb{N}}$$

is a Cauchy sequence without limit. A similar reasoning is not possible in $\big(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu\big)$.

We say that a set is *sequentially closed* if it contains the limits of all convergent sequences taking values in the set. A topological space is called *sequential* if every sequentially closed subset is closed.

LEMMA 8. *The topological field* $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ *is sequential.*

P r o o f . We reason by the way of contradiction. If $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is not sequential, then there exists a sequentially closed subset F such that $0 \in \overline{F} \setminus F$. There is not any sequence with their values in F converging to zero. There exists $t_1 > 1$ such that $B\lfloor t_1, 1\rfloor \cap F = \emptyset$, otherwise there would be a sequence in F converging to zero. If we assume that $F \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor m, 2\rfloor\big) \neq \emptyset$ for all $m > t_1$, then there exist

$$\alpha_m \in F \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor m, 2\rfloor\big) \qquad \text{for all} \quad m > t_1 .$$

We have that $\alpha_m = \beta_m + \gamma_m$ with $\beta_m \in B\lfloor t_1, 1\rfloor$ and $\gamma_m \in B\lfloor m, 2\rfloor$. Thus the sequence $\gamma_m \to 0$, and since $B\lfloor t_1, 1\rfloor$ is compact, after taking subsequences, we get that $\beta_m \to \alpha \in B\lfloor t_1, 1\rfloor$. Hence $\alpha_m \to \alpha \in B\lfloor t_1, 1\rfloor$. As F is sequentially closed, then $\alpha \in F$, which contradicts the fact that $B\lfloor t_1, 1\rfloor \cap F = \emptyset$. Therefore there exists $t_2 > t_1$ such that $F \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor t_2, 2\rfloor\big) = \emptyset$. In the same way, we get a strictly increasing sequence of natural numbers $t_1 < t_2 < \dots < t_s$ such that

$$F \cap \big(B\lfloor t_1, 1\rfloor + B\lfloor t_2, 2\rfloor + \ldots + B\lfloor t_s, s\rfloor\big) = \emptyset$$

for all $s \in \mathbb{N}$. We define $f \in \mathcal{F}$ such that $f(n) = t_n$ for all $n \in \mathbb{N}$ and we take the zero neighbourhood W_f. It is clear that $W_f \cap F = \emptyset$. Since $0 \in \overline{F}$, we have arrived to a contradiction. □

THEOREM 9. *The topological field* $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ *is complete.*

P r o o f . We reason by the way of contradiction. We assume that $(\alpha_i)_{i \in I}$ is a Cauchy net without limit, where I is a directed set. There exists $j \in I$ such that $\alpha_i - \alpha_j \in \overline{B}(0, p^{-1})$ for all $i \geq j$, and $(\alpha_i - \alpha_j)_{i \in I}$ is also a Cauchy net without limit. Therefore we assume that $\{\alpha_i\}_{i \in I} \subseteq \overline{B}(0, p^{-1})$. Since $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is separable, there exists a countable dense subset of $\overline{B}(0, p^{-1})$, which we denote by $\{\gamma_n\}_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$ we consider the Cauchy net $(\gamma_n - \alpha_i)_{i \in I}$, which has no limit. The sets A_n, defined in (10), are compact. Hence, for all $n \in \mathbb{N}$, there exists $i_n \in I$ such that the set

$$S_n = \overline{\{\gamma_n - \alpha_i : \ i \geq i_n\}} = \gamma_n - \overline{\{\alpha_i : \ i \geq i_n\}} \subseteq \overline{B}(0, p^{-1})$$

satisfies $S_n \cap A_n = \emptyset$. Let $S = \bigcup_{n \in \mathbb{N}} S_n$; since $0 \notin S_n$ for all n, then $0 \notin S$. Let us show that $0 \in \overline{S}$. Let $W_f \subset \overline{B}(0, p^{-1})$ be a neighbourhood of zero, there exists $i_f \in I$ such that $\alpha_i - \alpha_j \in W_f$ for all $i, j \geq i_f$. We fix $j \geq i_f$. There exists $\gamma_n \in \alpha_j + W_f$, that is $\gamma_n - \alpha_j \in W_f$. Let $i \geq \sup\{i_f, i_n\}$, then we have that $\gamma_n - \alpha_i \in S_n$ and $\alpha_i - \alpha_j \in W_f$. We obtain

$$\gamma_n - \alpha_i = (\gamma_n - \alpha_j) - (\alpha_i - \alpha_j) \in W_f - W_f = W_f .$$

Consequently $W_f \cap S_n \neq \emptyset$, and so $W_f \cap S \neq \emptyset$. We have proven that $0 \in \overline{S}$.

Since S is not closed and $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is sequential, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ contained in S which converges to an element $h \notin S$. Notice that $h \in \overline{B}(0,p^{-1})$. Considering that each S_n is closed and $S = \bigcup_{n\in\mathbb{N}} S_n$, we get a subsequence $(h_m)_{m\in\mathbb{N}}$ such that $h_m \in S_{n(m)}$ with $n(m+1) > n(m)$ for all m. By Lemma 6, there exists l such that $h_m - h \in A_l$ for all $m \geq m_l$. Since $S \subseteq \overline{B}(0,p^{-1}) = \bigcup_{l\in\mathbb{N}} A_l$, there exists t such that $h \in A_t$. Thus $h_m = (h_m - h) + h \in A_l + A_t = A_s$, where $s = \max\{l,t\}$. As $h_m \in S_{n(m)}$, then $h_m \notin A_{n(m)}$. We reach a contradiction for $n(m) \geq s$. □

THEOREM 10. *Each intermediate field $\mathbb{Q}_p \subseteq K \subseteq \overline{\mathbb{Q}}_p$ is complete with the subspace topology inherited from $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$. That is, K is closed in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$.*

P r o o f . We have seen in Lemma 3 that, if the extension $K/\mathbb{Q}_p$ is finite, then the p-adic topology coincides with the subspace topology obtained from $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$. The result follows taking into account that K is complete with the p-adic topology.

Now we consider the case in which the extension $K/\mathbb{Q}_p$ is infinite. Observe that, in the previous results in this article, we have only used the fact that $\overline{\mathbb{Q}}_p$ is an infinite algebraic extension of $\mathbb{Q}_p$, and we have not used properly the fact that $\overline{\mathbb{Q}}_p$ is algebraically closed. Therefore all the previous results are true for the field K with the subspace topology $\mathcal{T}_\mu|_K$, in particular, the fact that $(K, \mathcal{T}_\mu|_K)$ is complete. □

All the previous results can be rewritten for $\mathbb{Q}_p^{\mathrm{unram}}$, the maximal unramified extension of the p-adic field. In this specific case, there is exactly one intermediate extension $\mathbb{Q} \subseteq K \subset \mathbb{Q}_p^{\mathrm{unram}}$ of each degree $[K : \mathbb{Q}_p] = n$. Hence, the fields K_n, defined in (8), can be taken as the unique unramified extensions of $\mathbb{Q}_p$ of degree $n!$.

We recall that a topological space X is a *Baire space* if any countable union of closed subsets having no interior point cannot have an interior point; in particular, such a countable union cannot be equal to X.

THEOREM 11. *The topological field $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is not a Baire space.*

P r o o f . We have seen that there are a countable collection $\{K_n\}_{n\in\mathbb{N}}$ of finite field extensions of $\mathbb{Q}_p$ such that

$$\overline{\mathbb{Q}}_p = \bigcup_{n\in\mathbb{N}} K_n .$$

Since each neighbourhood of zero W_f contains elements of arbitrarily large degree over $\mathbb{Q}_p$, we have that $\overset{\circ}{K}_n = \emptyset$. As each K_n is closed, we conclude that $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is not a Baire space. □

There is a similar reasoning in [14; p. 129] and [16; p. 43] in order to prove that $(\overline{\mathbb{Q}}_p, |\ |_p)$ is not a Baire space, and therefore is not complete.

COROLLARY 12. *The topological field $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is not a first countable topological space.*

Proof. Each first countable Hausdorff topological group is metrizable, and each complete metric space is a Baire space. Applying Theorems 9 and 11, we conclude that our topological field is not first countable. □

We recall that a subset S of a commutative topological ring R is *bounded* if given any neighbourhood V of zero, there exists a neighbourhood U of zero such that $SU \subseteq V$. If R is a nondiscretely topologized field, this is equivalent to saying that given any neighbourhood V of zero, there exists a nonzero element $x \in R$ such that $Sx \subseteq V$ (see [15; p. 42, Theorem 3] or [17; p. 26, Lemma 12]).

A ring topology on R is *locally bounded* if there exists a bounded neighbourhood of zero. A topological field K is locally bounded if and only if there exists a neighbourhood of zero V such that $\{aV : a \in K \setminus \{0\}\}$ is a fundamental system of zero neighbourhoods.

LEMMA 13. *If $(K, \mathcal{T})$ is a topological field locally bounded and separable, then it satisfies the first axiom of countability.*

Proof. There exists a neighbourhood of zero V such that $\mathcal{B} = \{aV : a \in K \setminus \{0\}\}$ is a neighbourhood base at zero consisting of bounded neighbourhoods. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a dense subset in $(K, \mathcal{T})$. Let us see that $\{\gamma_n V : n \in \mathbb{N}\}$ is a base of zero neighbourhoods. Given $aV \in \mathcal{B}$, there exists $bV \in \mathcal{B}$ such that $bV + bV \subseteq aV$. Since V is bounded, there exists a neighbourhood of zero W such that $WV \subseteq bV$. There exists γ_n such that $\gamma_n - b \in W$. We conclude that

$$\gamma_n V \subseteq (\gamma_n - b)V + bV \subseteq WV + bV \subseteq bV + bV \subseteq aV .$$

□

COROLLARY 14. *The topological field $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is locally unbounded.*

Proof. Consider that $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ is separable (Lemma 4) and does not satisfy the first axiom of countability (Corollary 12). □

In [12; Table 2] Mutylin raised the question if there exists a complete, not locally bounded and not first countable field extension of the p-adic number field $\mathbb{Q}_p$ (see also [17; p. 256]). We have seen that the topological field $\left(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu\right)$ satisfies those properties.

An element α in a topological field is *topologically nilpotent* if the sequence $(\alpha^n)_{n\in\mathbb{N}}$ converges to zero.

In [9], [10] we introduced some locally unbounded topological fields having topologically nilpotent elements. Our field $\left(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu\right)$ has topologically nilpotent elements (for instance, p) and is locally unbounded. In fact, each element in the open disk $B(0,1)$ is topologically nilpotent. In [3; p. 147] it is proven the following result:

> *Let K be a locally bounded topological field with a topologically nilpotent element, then K possesses a topologically nilpotent neighbourhood of zero (and consequently, K satisfies the first axiom of countability).*

With this result and Corollary 12, we have another proof of Corollary 14.

LEMMA 15. *Every automorphism $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$ is continuous.*

Proof. For each $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$, and each $\alpha \in \overline{\mathbb{Q}}_p$ we have that $\deg(\alpha) = \deg\left(\sigma(\alpha)\right)$ and $v_p(\alpha) = v_p\left(\sigma(\alpha)\right)$. Hence for each neighbourhood of zero Z_f, defined in (12), we have that $\sigma(Z_f) = Z_f$. □

We recall that every $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}_p/\mathbb{Q}_p\right)$ can be extended to an automorphism of $\mathbb{C}_p$ which is an isometry, and therefore continuous with the p-adic topology ([11; p. 55]). The next result shows a characterization of the field topology $\mathcal{T}_\mu$ with respect to the p-adic topology.

THEOREM 16. *Let $\{L_j\}$ be the family of all finite field extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. The topology $\mathcal{T}_\mu$ is the finest ring topology on $\overline{\mathbb{Q}}_p$ among those ring topologies whose subspace topology in each L_j is the p-adic topology.*

Proof. Let $\mathcal{T}_1$ be a ring topology whose restriction to each L_j is the p-adic topology. Let V_0 be any neighbourhood of zero for $\mathcal{T}_1$; it suffices to show that there exists a zero neighbourhood W_f for $\mathcal{T}_\mu$ such that $W_f \subseteq V_0$. Each set $A_n = K_n \cap \overline{B}\left(0,p^{-1}\right)$, defined in (10), is also compact with respect to the topology $\mathcal{T}_1$, and therefore is bounded in $\left(\overline{\mathbb{Q}}_p, \mathcal{T}_1\right)$. There exists a family of neighbourhoods of zero $\{V_n\}_{n\geq 0}$ for $\mathcal{T}_1$ which satisfies the following conditions: $V_{n+1} + V_{n+1} \subseteq V_n$, $V_{n+1} \subseteq V_n$ and $A_n V_{n+1} \subseteq V_n$ for all $n \geq 0$. Since $p^n \to 0$ with respect to $\mathcal{T}_1$, for each $n \in \mathbb{N}$ there exists $t_n \in \mathbb{N}$ such that $p^m \in V_{n+1}$ for all $m \geq t_n$. We choose the numbers t_n such that $t_n < t_{n+1}$ for all $n \in \mathbb{N}$. We also have that

$$A_n p^{t_n} \subseteq V_n \qquad \text{for all} \quad n .$$

We inductively get

$$V_1 + V_1 \subseteq V_0 ,$$
$$V_1 + V_2 + V_2 \subseteq V_0 ,$$
$$\vdots$$
$$V_1 + V_2 + \cdots + V_{s-1} + V_s + V_s \subseteq V_0 \qquad \text{for all} \quad s \in \mathbb{N} .$$

Hence,

$$A_1 p^{t_1} + A_2 p^{t_2} + \cdots + A_s p^{t_s} \subseteq V_0 \qquad \text{for all} \quad s \in \mathbb{N} .$$

Since $A_n p^{t_n} = B\lfloor t_n{+}1, n\rfloor$, we rewrite the above expression as

$$\sum_{n=1}^{s} B\lfloor t_n{+}1, n\rfloor \subseteq V_0 \qquad \text{for all} \quad s \in \mathbb{N} .$$

We consider $f \in \mathcal{F}$ such that $f(n) = t_n + 1$ for all $n \in \mathbb{N}$ and the corresponding zero neighbourhood W_f. It is clear that $W_f \subseteq V_0$. □

3. Comment on analytic functions

In p-adic analysis, we usually deal with analytic functions defined either in K, a finite field extension of $\mathbb{Q}_p$, or in $\mathbb{C}_p$. Since $\overline{\mathbb{Q}}_p$ is a complete topological field with the topology $\mathcal{T}_\mu$, we look briefly at the possibility of defining analytic functions on it; although it seems that there is not any gain by doing p-adic analysis in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ instead of $\mathbb{C}_p$. We only study a rather specific case.

Let $\mathbb{C}_p\{X\}$ be the algebra of analytic functions defined on the closed disk $\overline{B}(0,1) \subset \mathbb{C}_p$, that is,

$$\mathbb{C}_p\{X\} = \Big\{ f(x) = \sum_{n=0}^{\infty} a_n x^n : \ \big(\forall n \in \mathbb{N} \cup \{0\}\big)(a_n \in \mathbb{C}_p), \ \lim_{n\to\infty} |a_n|_p = 0 \Big\} .$$

We recall that $\mathbb{C}_p\{X\}$ is a complete algebra with the norm

$$\begin{aligned} \|f(x)\| &= \max\big\{ |a_n|_p : \ n \in \mathbb{N} \cup \{0\} \big\} = \max\big\{ |f(x)|_p : \ |x|_p \le 1 \big\} \\ &= \max\big\{ |f(x)|_p : \ |x|_p = 1 \big\} . \end{aligned}$$

See, for instance, [4; Chap. 6], [14; Chap. 6] or [16; p. 121]. We have seen in Lemma 6 that a sequence $(a_n)_{n\in\mathbb{N}}$ converging to zero in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ has the degrees of its terms bounded. That is, there exists K, a finite field extension of $\mathbb{Q}_p$, such that $a_n \in K$ for all $n \in \mathbb{N}$. Consequently, in order to guarantee the convergence in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$, we are lead to define the following subring of $\mathbb{C}_p\{X\}$.

DEFINITION 17. Let E_p be the subring of $\mathbb{C}_p\{X\}$ consisting of those functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for which there exists K_f, a finite field extension of $\mathbb{Q}_p$ depending on f, such that $a_n \in K_f$ for all $n \in \mathbb{N}$.

For each finite field extension $K/\mathbb{Q}_p$, we consider the ring of analytic functions $K\{X\}$ which converge in $K \cap \overline{B}(0,1)$. The ring of functions E_p is the direct limit of the rings $K\{X\}$. In fact, the most commonly used analytic functions in p-adic analysis satisfy the requirements in Definition 17. Furthermore, these functions usually satisfy that $a_n \in \mathbb{Q}_p$ for all n.

Notice that any $f(x) = \sum_{n=0}^{\infty} a_n x^n \in E_p$ satisfies that $a_n \to 0$ with respect to the topology $\mathcal{T}_\mu$. Let K be a finite field extension of $\mathbb{Q}_p$ such that $a_n \in K$ for all n. Then, for each $\alpha \in \overline{\mathbb{Q}}_p \cap \overline{B}(0,1)$, we have that $f(\alpha) \in K[\alpha]$, and therefore $f(\alpha) \in \overline{\mathbb{Q}}_p$. We consider every $f \in E_p$ as a function

$$f\colon \overline{\mathbb{Q}}_p \cap \overline{B}(0,1) \to \overline{\mathbb{Q}}_p .$$

We need the following result of general topology whose proof is elementary.

LEMMA 18. *Let X be a topological space, and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X such that*

$$X = \bigcup_{n\in\mathbb{N}} X_n , \qquad X_n \subseteq X_{n+1} \quad \textit{for all } n \in \mathbb{N}.$$

Let $f\colon X \to X$ be a map such that $f(X_n) \subseteq X_n$ for all n. If each map $f|_{X_n} : X_n \to X_n$ is continuous with respect to the subspace topology in X_n, then f is continuous.

Proof. Use the fact that f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A. □

COROLLARY 19. *If both sets, $\overline{\mathbb{Q}}_p \cap \overline{B}(0,1)$ and $\overline{\mathbb{Q}}_p$, are endowed with the topology $\mathcal{T}_\mu$, then every $f \in E_p$ is continuous.*

Proof. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in E_p$. There exists $\alpha \in \overline{\mathbb{Q}}_p$ such that $a_n \in \mathbb{Q}_p[\alpha]$ for all n. We consider the sequence of fields $(K_n[\alpha])_{n\in\mathbb{N}}$, where each K_n is the field defined in (8). Each $K_n[\alpha]$ is complete and closed in $\overline{\mathbb{Q}}_p$, and $f(K_n[\alpha]) \subseteq K_n[\alpha]$. Hence f is continuous. □

The previous result is obviously valid if $\overline{\mathbb{Q}}_p \cap \overline{B}(0,1)$ and $\overline{\mathbb{Q}}_p$ are provided with the p-adic topology, in fact, in this situation, every $f \in E_p$ has continuous derivatives of all orders in $\overline{B}(0,1) \subset \overline{\mathbb{Q}}_p$.

Since $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ does not satisfy the first axiom of countability, in order to avoid nets, we restrict ourselves to deal with sequential differentiability. Let $X \subset \overline{\mathbb{Q}}_p$ be an open set for $\mathcal{T}_\mu$. We say that a function $f\colon X \to \overline{\mathbb{Q}}_p$ is *sequentially differentiable* at $a \in X$, with derivative $f'(a)$, if the limit

$$\lim_{n\to\infty} \frac{f(a+h_n) - f(a)}{h_n} = f'(a) \tag{13}$$

holds for all sequences $(h_n)_{n\in\mathbb{N}}$ converging to zero in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$. By the same token, we can define analogous concepts like continuous sequential differentiability and sequential C^∞ functions.

LEMMA 20. *If we consider the field $\overline{\mathbb{Q}}_p$ equipped with the topology $\mathcal{T}_\mu$, then every $f \in E_p$ is sequentially C^∞ in $\overline{B}(0,1)$.*

Let $f(x) = \sum\limits_{m=0}^{\infty} a_m x^m \in E_p$. Each formal derivative $f^{(n)} \in E_p$, and so it is a continuous function. For each sequence $(h_n)_{n\in\mathbb{N}}$, $h_n \to 0$, by Lemma 6, there is a finite field extension $K/\mathbb{Q}_p$ such that $a, a_m, h_n \in K$ for all $n, m \in \mathbb{N}$. If we consider K provided with the p-adic topology, then the function f is analytic in $K \cap \overline{B}(0,1)$. Therefore, we have the limit (13), with f' being the formal derivative of f, and $f'(a) \in K \subset \overline{\mathbb{Q}}_p$. This limit also holds in $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$. This result extends for all derivatives $f^{(n)}$.

Schikhof [16; §42, §43] shows the different behavior of analytic functions in locally compact p-adic fields (i.e., finite field extensions of $\mathbb{Q}_p$) and in $\mathbb{C}_p$. The behavior of analytic functions belonging to E_p with respect to the topological field $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ seems to be more similar to the analytic functions defined in $\mathbb{C}_p$. The possible reasons are that $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}_p$ endowed with the p-adic topology, and neither $(\mathbb{C}_p, |\ |_p)$ nor $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$ are locally compact. It seems possible to translate other results from p-adic analysis in $\mathbb{C}_p$ to our topological field $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$, for instance, the p-adic Weierstrass preparation theorem. As another example, we can define a function $\log\colon \overline{\mathbb{Q}}_p^\times \to \overline{\mathbb{Q}}_p$, sharing the same properties with the Iwasawa logarithm, translated from $(\mathbb{C}_p, |\ |_p)$ to $(\overline{\mathbb{Q}}_p, \mathcal{T}_\mu)$. The basic facts of the Iwasawa logarithm in $\mathbb{C}_p$ can be found in [11; Chap. 4] or [14; Chap. 5].

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