Mathematica Slovaca

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A TOPOLOGICAL STRUCTURE
OF SOLUTION SETS TO EVOLUTION SYSTEMS

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(*Communicated by Michal Fečkan*)

ABSTRACT. In this paper we deal with the Peano phenomenon for general initial-boundary value problems of quasilinear evolution systems with arbitrary even order space derivatives. The nonlinearity is a continuous or continuously Fréchet differentiable function. Qualitative and quantitative structure of solution sets is studied by the theory of proper, Fredholm and Nemitskiĭ operators. These results can be applied to the different technical and natural science models.

Introduction

In this paper we shall study generic properties of quasilinear initial-boundary value problems for evolution systems of an even order with the continuous or continuous differentiable nonlinearities and the general boundary value conditions. In special Hölder spaces we use the Nikoľskiĭ decomposition theorem from [29; p. 233] for linear Fredholm operators, the global inversion theorem of [9], [6] and [7; pp. 42–43] and the A m b r o s e t t i solution quantitative results from [2; p. 216]. In the consideration on surjectivity, the generalized Leray-Schauder condition is employed which is similar to that one in [20]. In the case of nonlinear Fredholm operators we use the main Q u i n n and S m a l e theorem from [22] and [24].

The topological structure of solution sets for ordinary differential systems has been studied by many authors in [3], [4], [5], [8], [17], [27]. Such questions for second order partial differential equations was studied in the authors papers [12], [13].

The present results allow us to investigate different problems describing dynamics of mechanical processes (bending, vibration), physical-heating processes,

reaction-diffusion processes in chemical and biological technologies or in the ecology.

1. The formulation of problem, assumptions and spaces

The set $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ means a bounded domain with the boundary $\partial\Omega$. The real number T will be positive and $Q := (0, T] \times \Omega$, $\Gamma := (0, T] \times \partial\Omega$. If the multiindex $k = (k_1, \ldots, k_n)$ with $|k| = \sum_{i=1}^{n} k_i$, then we use the notation D_x^k for the differential operator $\frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$ and D_t for $\frac{\partial}{\partial t}$. If the modul $|k| = 0$, then D_x^k means an identity mapping. The symbol $\mathrm{cl}\, M$ means the closure of the set M in $\mathbb{R}^n$.

In this paper we consider the general system of $p \geq 1$ nonlinear differential equations (parabolic or non-parabolic type) of an arbitrary even order $2b$ (b is a positive integer) with p unknown functions in the column vector form $(u_1, \ldots, u_p)^T = u \colon \mathrm{cl}\, Q \to \mathbb{R}^p$. Its matrix form is given as follows:

$$A(t, x, \mathrm{D}_t, \mathrm{D}_x)u + f(t, x, \bar{\mathrm{D}}_x^\gamma u) = g(t, x) \qquad \text{for} \quad (t, x) \in Q, \qquad (1.1)$$

where

$$A(t, x, \mathrm{D}_t, \mathrm{D}_x)u := \mathrm{D}_t\, u - \sum_{|k|=2b} a_k(t, x)\, \mathrm{D}_x^k\, u - \sum_{0 \leq |k| \leq 2b-1} a_k(t, x)\, \mathrm{D}_x^k\, u,$$

and $\bar{\mathrm{D}}_x^\gamma u$ is a vector function whose components are derivatives $\mathrm{D}_x^\gamma u_l$ with the different multiindices $0 \leq |\gamma| \leq 2b - 1$ for $l = 1, \ldots, p$.

The system of boundary conditions is given by the vector equation with the bp components

$$B(t, x, \mathrm{D}_x)u\big|_{\mathrm{cl}\,\Gamma} := \big(B_1(t, x, \mathrm{D}_x)u, \ldots, B_{bp}(t, x, \mathrm{D}_x)u\big)^T\big|_{\mathrm{cl}\,\Gamma} = 0 \qquad (1.2)$$

in which

$$B_j(t, x, \mathrm{D}_x)u := \sum_{0 \leq |k| \leq r_j} b_{jk}(t, x)\, \mathrm{D}_x^k\, u$$

for an integer $0 \leq r_j \leq 2b - 1$ and $j = 1, \ldots, bp$.

Further the initial value homogeneous condition

$$u(0, x) = 0 \qquad \text{for} \quad x \in \bar{\Omega} \qquad (1.3)$$

is considered.

Here the given functions are the following mappings:

$$a_k := \big(a_k^{hl}\big)_{h,l=1}^p \colon \mathrm{cl}\, Q \to \mathbb{R}^{p^2} \qquad \text{for} \quad 0 \leq |k| \leq 2b$$

are $(p \times p)$-matrix functions;

$$b_{jk} = \left(b^1_{jk}, \ldots, b^p_{jk}\right) \colon \ \mathrm{cl}\,\Gamma \to \mathbb{R}^p \qquad \text{for} \quad 0 \le |k| \le r_j\,, \quad j = 1, \ldots, bp$$

are row vector functions;

$$f = (f_1, \ldots, f_p)^T \colon \ \mathrm{cl}\,Q \times \mathbb{R}^\kappa \to \mathbb{R}^p \quad \text{and} \quad g = (g_1, \ldots, g_p)^T \colon \ \mathrm{cl}\,Q \to \mathbb{R}^p$$

are column vector functions, where κ is a positive integer satisfying the inequality

$$\kappa \le \left[\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots \right.$$
$$\left. \cdots + \binom{n+|\gamma|-2}{|\gamma|-1} + \binom{n+|\gamma|-1}{|\gamma|} \right] p\,.$$

Under several supplementary assumptions, problem $(1.1)-(1.3)$ defines homeomorphism between some Hölder spaces. Now, we formulate these suppositions.

(P) A δ-uniform parabolic condition holds for system (1.1) in the sense of J. G. Petrovskiĭ, $\delta > 0$.

The system (1.1) and boundary condition (1.2) are connected by

(C) a δ^+-uniform complementary condition with $\delta^+ > 0$

and

(Q) a compatibility condition.

The coefficients of the operator $A(t, x, \mathrm{D}_t, \mathrm{D}_x)$ from (1.1) and of $B(t, x, \mathrm{D}_x)$ from (1.2) and the boundary $\partial\Omega$ satisfy

$(S^{l+\alpha})$ a smoothness condition for a nonnegative integer l and a number $\alpha \in (0, 1)$.

We shall be dealing with the Banach spaces of continuously differentiable functions $C_x^l(\mathrm{cl}\,Q, \mathbb{R}^p)$ and $C_{t,x}^{l/2b,l}(\mathrm{cl}\,Q, \mathbb{R}^p)$ and the Hölder spaces $C_x^{l+\alpha}(\mathrm{cl}\,Q, \mathbb{R}^p)$, $C_{t,x}^{(l+\alpha)/2b,l+\alpha}(\mathrm{cl}\,Q, \mathbb{R}^p)$ for a nonnegative integer l and $\alpha \in (0, 1)$.

For the exact definition of conditions (P), (C), (Q), $\left(S^{l+\alpha}\right)$ see [19; pp. 12 21] and for the definition of the spaces see [19; pp. 8–12] or [11].

The homeomorphism result for (1.1) (1.3) can be formulated as follows:

PROPOSITION 1.1. (See [19; p. 21] and [15; pp. 182–183].) *Let the conditions* (P), (C) *and* (S^α) *be satisfied for* $\alpha \in (0, 1)$. *Necessary and sufficient conditions for the existence and uniqueness of the solution*

$$u \in C_{t,x}^{(2b+\alpha)/2b, 2b+\alpha}\left(\mathrm{cl}\,Q, \mathbb{R}^p\right)$$

of linear problem $(1.1)-(1.3)$ *for* $f = 0$ *is*

$$g \in C_{t,x}^{\alpha/2b, \alpha}\left(\mathrm{cl}\,Q, \mathbb{R}^p\right)$$

and the compatibility condition (Q).

Moreover, there exists a constant $c > 0$ *independent of* g *such that*

$$c^{-1}\|g\|_{\alpha/2b, \alpha, Q, p} \le \|u\|_{(2b+\alpha)/2b, 2b+\alpha, Q, p} \le c\|g\|_{\alpha/2b, \alpha, Q, p}\,.$$

2. General results

In this part we remind some notions and assertions from the nonlinear functional analysis applied in the fundamental lemmas and theorems.

Throughout this paper we shall assume that X and Y are Banach spaces either both over the real or complex field.

In the Z e i d l e r books [31; pp. 365–366] and [32; pp. 667 668] we find definitions of the linear and nonlinear Fredholm operator.

The following proposition gives the necessary and sufficient condition for a linear operator to be Fredholm.

PROPOSITION 2.1. (S. M. Nikoľskiĭ, see [29; p. 233].) *A linear bounded operator* $A\colon X \to Y$ *is Fredholm of the zero index if and only if* $A = C + T$, *where* $C\colon X \to Y$ *is a linear homeomorphism and* $T\colon X \to Y$ *is a linear completely continuous operator.*

In the theory and applications of nonlinear operators, the notions as a *proper*, σ *-proper*, *closed*, *coercive operator* (for definitions, see books [31] and [32]) are very frequent. Their significant application give the following statements.

PROPOSITION 2.2. (The Ambrosetti theorem, [2; p. 216].) *Let* $F \in C(X, Y)$ *be a proper mapping. Then the cardinal number* $\operatorname{card} F^{-1}(q)$ *of the set* $F^{-1}(q)$ *is constant and finite* (*it may be zero*) *for every* q *taken from the same component* (*nonempty and connected subset*) *of the set* $Y \setminus F(\Sigma)$. *Here* Σ *means a closed set of all points* $u \in X$ *at which* F *is not locally invertible.*

A relation between the local invertibility and homeomorphism of X onto Y gives the global inverse mapping theorem.

PROPOSITION 2.3. ([9], [31; p. 174].) *Let* $F \in C(X, Y)$ *be a locally invertible mapping in* X. *Then* F *is a homeomorphism of* X *onto* Y *if and only if* F *is proper.*

The following propositions give necessary and sufficient conditions for the proper mapping.

PROPOSITION 2.4. (See [31; p. 176], [23; p. 49], [27; p. 20].) *Let* $F \in C(X, Y)$.

  (j) *If* F *is proper, then* F *is a nonconstant closed mapping.*
  (jj) *If* $\dim X = +\infty$ *and* F *is a nonconstant closed mapping, then* F *is proper.*

PROPOSITION 2.5. (See [23; pp. 58–59], [31; p. 498], [27; p. 20].) *Suppose that* $F \colon X \to Y$ *and* $F = F_1 + F_2$, *where*

   (i) $F_1 \colon X \to Y$ *is a continuous proper mapping on* X,
   (ii) $F_2 \colon X \to Y$ *is complete continuous.*

Then

   (j) *the restriction of the mapping* F *to an arbitrary bounded closed set in* X *is a proper mapping.*
   (jj) *If moreover,* F *is coercive, then* F *is a proper mapping.*

Now we can formulate some sufficient conditions for the surjectivity of an operator.

PROPOSITION 2.6. (See [27; pp. 24, 27].) *Let* X *be a real Banach space. Suppose*

   (i) $P = I - f \colon X \to X$ *is a condensing field, where* $I \colon X \to X$ *is the identity,*
   (ii) P *is coercive,*
   (iii) *there exists a strictly solvable field* $G = I - g \colon X \to X$ *and* $R > 0$ *such that for all solutions* $u \in X$ *of the equation*

$$P(u) = k \cdot G(u)$$

   and for all $k < 0$ *the estimation* $\|u\|_X < R$ *holds.*

Then the following statements are true:

   (j) P *is a proper mapping,*
   (jj) P *is strictly surjective,*
   (jjj) $\operatorname{card} F^{-1}(q)$ *is constant, finite and nonzero for every* q *from the same connected component of the set* $Y \setminus F(\Sigma)$, *where* Σ *is as in Proposition 2.2.*

The definition of a condensing field is understood in the sense given in [10; p. 69]. For the definition of a strict solvable field and strict surjective field see [29].

Remark 2.1. It is clear that an operator F is strictly surjective, then it is surjective and if F is strictly solvable, then it is also solvable. Moreover, if F is strictly surjective, then it is strictly solvable, too.

PROPOSITION 2.7. (The Schauder invariance of domain theorem, [31; p. 705].) *Let* $F \colon M(\subseteq X) \to X$ *is continuous and locally compact perturbation of identity on the open nonempty set* M *in the Banach space* X. *Then:*

   (j) *If* F *is locally injective on* M, *so* F *is an open mapping.*
   (jj) *If* F *is injective on* M, *so* F *is a homeomorphism from* M *onto the open set* $F(M)$.

For the compact perturbation of C^1-Fredholm operator we shall use the following proposition.

PROPOSITION 2.8. ([32; p. 672].) *Let $A\colon D(A) \subset X \to Y$ be a C^1-Fredholm operator on the open set $D(A)$ and $B\colon D(A) \to Y$ be a compact mapping from the class C^1. Then $A + B\colon D(A) \to Y$ is a Fredholm (possible nonlinear) operator with the same index as A at each point of $D(A)$.*

In the following propositions we use the notion of a *regular, singular, critical point* of an operator and a *regular, singular values* of operators. The reader finds these definitions in [32; p. 668] or [31; p. 184].

Also, we need a residual set. A subset of a topological space Z is called *residual* if and only if it is a countable intersection of dense and open subsets of Z.

By the Baire theorem in any complete metric space or locally compact Hausdorff topological space, a residual set is dense in this space.

The most important theorem for nonlinear Fredholm mappings is due to S. S m a l e [24; p. 862] and Q u i n n [22]. It is also in [7; pp. 11 12].

PROPOSITION 2.9. (A Smale-Quinn theorem.) *If $F\colon X \to Y$ is a Fredholm mapping (possible nonlinear) of the class $C^k(X, Y)$ in the Fréchet sense and either*

   X *has a countable basis* ($S\,m\,a\,l\,e$),

or

   F *is σ-proper* ($Q\,u\,i\,n\,n$),

then the set R_F of all regular values of F is residual in Y. Moreover, if F is proper, then R_F is open and dense set in Y.

A necessary and sufficient condition for a local diffeomorphism (see [31; p. 171]) is given in the following proposition.

PROPOSITION 2.10. (A Local inverse mapping theorem, [31; p. 172].) *Let $F\colon U(u_0) \subset X \to Y$ be a C^1-mapping in the Fréchet sense. Then F is a local C^1-diffeomorphism at u_0 if and only if u_0 is a regular point of F.*

PROPOSITION 2.11. ([23; p. 89].) *Let $\dim Y \geq 3$ and $F\colon X \to Y$ be a Fredholm mapping of the zero index. If $u_0 \in X$ is an isolated singular point of F, then F is locally invertible at u_0.*

To illustrate the following results we shall need estimations of a Green $p \times p$-matrix for linear problem $(1.1)-(1.3)$.

LEMMA 2.1. *Let the assumptions* (P), (C), (S^{α}) *be satisfied for* $\alpha \in (0,1)$. *Then for the Green matrix* G *of linear problem* (1.1)–(1.3) *with* $f = 0$ *we have*

$$|\mathrm{D}_t^{k_0} \mathrm{D}_x^k \, G(t,x;\tau,\xi)| \le c(t-\tau)^{-\mu} \|x - \xi\|_{\mathbb{R}^n}^{2b\mu - (n+2bk_0+|k|)} \mathbb{E} \qquad (2.1)$$

for $0 \le 2bk_0 + |k| \le 2b$ *and* $\mu \le (n + 2bk_0 + |k|)/2b$, *thereby* $0 \le \tau < t \le T$ *and* $x, \xi \in \mathrm{cl}\,\Omega$, $x \ne \xi$. *The positive constant* c *does not depend on* t, x, τ, ξ *and* $\mathbb{E}$ *means the* $p \times p$-*matrix consisting only from units,* $r = 2b/(2b-1)$.

P r o o f. Since $n + 2bk_0 + |k| - 2b\mu \ge 0$ and $\|x - \xi\|_{\mathbb{R}^n} < \mathrm{diam}\,\Omega$, so for $0 < \delta \le t - \tau \le T$ we obtain (2.1) by the estimation (see [15; pp. 182 183])

$$|\mathrm{D}_t^{k_0} \mathrm{D}_x^k \, G(t,x;\tau,\xi)|$$

$$\le c_1(t-\tau)^{-\frac{n+2bk_0+|k|}{2b}} \exp\left\{-c_2 \frac{\|x-\xi\|_{\mathbb{R}^n}^r}{(t-\tau)^{1/(2b-1)}}\right\}$$

$$\le c_1(t-\tau)^{-\mu} \|x - \xi\|_{\mathbb{R}^n}^{2b\mu - (n+2bk_0+|k|)} \left[\|x - \xi\|_{\mathbb{R}^n}^{2b}/(t-\tau)\right]^{(n+2bk_0+|k|-2b\mu)/2b} \times$$

$$\times \exp\left\{-c_2\left[\|x-\xi\|_{\mathbb{R}^n}^{2b}/(t-\tau)\right]^{1/(2b-1)}\right\}\mathbb{E}.$$

If $0 < t - \tau < \delta$, then from

$$\lim_{y \to +\infty} y^u \exp\{-cy^v\} = 0$$

for every $u, v \in \mathbb{R}$ and $c > 0$, we get estimation (2.1). $\qquad \square$

Remark 2.2.

a) For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the inequalities

$$c_n \sum_{i=1}^n |x_i| \le \|x\|_{\mathbb{R}^n} \le \sum_{i=1}^n |x_i| \qquad (2.2)$$

hold if $c_n \in \left(0, 1/(\sqrt{2}\,)^{n-1}\right)$, $n \in \mathbb{N}$ not depended of x.

b) Also, we see that the mild solution $u \in C_x^{|\gamma|}(\mathrm{cl}\,Q, \mathbb{R})$ of problem (1.1) (1.3) satisfies the column vector integro-differential equation.

$$u(t,x) = \int\limits_0^t \mathrm{d}\tau \int\limits_\Omega G(t,x;\tau,\xi)\left[g(\tau,\xi) - f\left(\tau,\xi,\bar{\mathrm{D}}^\gamma u(\tau,\xi)\right)\right]\,\mathrm{d}\xi$$

$$=: (Su)(t,x) \qquad \text{for} \quad (t,x) \in \mathrm{cl}\,Q, \qquad (2.3)$$

for $0 \le |\gamma| \le 2b - 1$ and conversely the solution $v \in C_x^{|\gamma|}(\mathrm{cl}\,Q, \mathbb{R}^p)$ satisfying (2.3) is a mild solution of (1.1)–(1.3).

3. Operator formulation and fundamental lemmas

Consider the following operators:

(i)

$$A \colon X \to Y \,, \tag{3.1}$$

where

$$(Au)(t,x) = A(t,x,\mathrm{D}_t,\mathrm{D}_x)u(t,x) = \mathrm{D}_t\, u(t,x) - \sum_{0 \le |k| \le 2b} a_k(t,x)\, \mathrm{D}_x^k\, u(t,x)$$

for $(t,x) \in \operatorname{cl} Q$, $u \in X$,

$$X = \Big\{ u \in X_\rho \colon\ \big(\forall j \in \{1,\ldots,bp\}\big)\big(B_j(t,x,\mathrm{D}_x)u|_\Gamma = 0\big)$$
$$\&\ \big(\forall x \in \operatorname{cl} Q\big)\big(u(0,x) = 0\big)\Big\} \subset C(\operatorname{cl} Q, \mathbb{R}^p)\,.$$

Here

$$X_\rho \subset C_{t,x}^{(2b+\alpha)/2b,\,2b+\alpha}\big(\operatorname{cl} Q, \mathbb{R}^p\big)$$

is the Banach space of continuous functions $u \colon \operatorname{cl} Q \to \mathbb{R}^p$ with the continuous derivatives $\mathrm{D}_x^k u$ for $|k| = 1,\ldots,2b$ and $\mathrm{D}_x^{k_0}\mathrm{D}_x^k u$ for $1 \le 2bk_0 + |k| \le 2b$ on $\operatorname{cl} Q$ and with the finite norm

$$\|u\|_{X_\rho} = \max_{l=1,\ldots,p} \Bigg[\sum_{0 \le 2bk_0 + |k| \le 2b} \sup_{(t,x) \in \operatorname{cl} Q} |\mathrm{D}_t^{k_0}\mathrm{D}_x^k u_l(t,x)| + \langle \mathrm{D}_t\, u_l \rangle_{x,\alpha,Q}^y$$

$$+ \sum_{|k|=2b} \langle \mathrm{D}_x^k\, u_l \rangle_{x,\alpha+\rho,Q}^y + \langle \mathrm{D}_t\, u_l \rangle_{t,\alpha/2b,Q}^s$$

$$+ \sum_{|k|=1}^{2b-1} \langle \mathrm{D}_x^k\, u_l \rangle_{t,(2b+\alpha-|k|)/2b,Q}^s + \sum_{|k|=2b} \langle \mathrm{D}_x^k\, u_l \rangle_{t,(\alpha+\rho)/2b,Q}^s \Bigg]\,,$$

where $\rho > 0$ and $\alpha + \rho < 1$. Further

$$Y := TX \subset C_{t,x}^{\alpha/2b,\,\alpha}\big(\operatorname{cl} Q, \mathbb{R}^p\big)$$

for $\alpha \in (0,1)$ with the norm

$$\|u\|_Y = \max_{l=1,\ldots,p}\Big[\sup_{(t,x)\in\operatorname{cl} Q} |u_l(t,x)| + \langle u_l \rangle_{x,\alpha,Q}^y + \langle u_l \rangle_{t,\alpha/2b,Q}^s \Big]\,.$$

For $v \colon \operatorname{cl} Q \to \mathbb{R}$ we understand

$$\langle v \rangle_{t,\mu,Q}^s := \sup_{\substack{(t,x),(s,x)\in\operatorname{cl} Q \\ t \ne s}} \frac{|v(t,x) - v(s,x)|}{|t-s|^\mu}\,,$$

$$\langle v \rangle_{x,\mu,Q}^y := \sup_{\substack{(t,x),(t,y)\in\operatorname{cl} Q \\ x \ne y}} \frac{|v(t,x) - v(t,y)|}{\|x-y\|_{\mathbb{R}^n}^\mu}\,.$$

(ii)   The *Nemitskiĭ operator*

$$N\colon X \to Y\,, \tag{3.2}$$

where

$$(Nu)(t,x) = (f \circ u)(t,x) = f\bigl(t,x,\bar{\mathrm{D}}_x^\gamma u(t,x)\bigr)$$

for $(t,x) \in \mathrm{cl}\, Q$, $u \in X$.

(iii)   The operator

$$F\colon X \to Y\,, \tag{3.3}$$

where

$$(Fu)(t,x) = (Au)(t,x) + (Nu)(t,x) \qquad \text{for} \quad (t,x) \in \mathrm{cl}\, Q\,, \quad u \in X\,.$$

Together with the solution sets of given problem $(1.1)-(1.3)$ we shall search the bifurcation points sets.

DEFINITION 3.1.

(i)   A couple $(u,g) \in X \times Y$ will be called the *bifurcation point* of (1.1) (1.3) if and only if u is a solution of this problem and there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset Y$ such that $\lim_{k\to\infty} g_k = g$ in Y and initial-boundary value problem $(1.1)-(1.3)$ with $g = g_k$ has at least two different solutions u_k, v_k for each $k \in \mathbb{N}$ and $\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = u$ in X.

(ii)   The set of all solutions $u \in X$ of $(1.1)-(1.3)$ (or the set of all functions $g \in Y$) such that (u,g) is a bifurcation point of $(1.1)-(1.3)$ will be called the *domain of bifurcation* (the *bifurcation range*) of $(1.1)-(1.3)$.

EXAMPLE 3.1.   The point $(u_r, 0) \in X \times Y$ for $r \in \langle 0, T \rangle$ is a bifurcation point of the Neumann problem (parabolic and non-parabolic)

$$\frac{\partial u}{\partial t} = \pm\frac{\partial^2 u}{\partial x^2} + f(t,x,u)\,, \qquad (t,x) \in (0,T\rangle \times \Omega = Q \subset \mathbb{R}^2\,, \tag{3.1*}$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0\,, \qquad\qquad t \in \langle 0, T \rangle\,, \tag{3.2*}$$

$$u(0,x) = 0\,, \qquad\qquad x \in \overline{\Omega}\,, \tag{3.3*}$$

for $f(t,x,u) = |u|^{1/2} - au$, $a > 0$.

Here for $r \in (0,T)$

$$u_r(t,x) = \begin{cases} 0 & \text{if } (t,x) \in \langle 0, r \rangle \times \overline{\Omega}\,, \\ \frac{1}{\alpha^2}\bigl(1 - \exp\{-\frac{a}{2}(t-r)\}\bigr)^2 & \text{if } (t,x) \in \bigl(r, T \rangle \times \overline{\Omega}\,. \end{cases}$$

The function $u_0(t,x) = \frac{1}{\alpha^2}\bigl(1 - \exp\{-at/2\}\bigr)^2$, $u_T(t,x) = 0$ are solutions of the given problem, too.

Really, there is the zero sequence $\{g_k\}_{k \in \mathbb{N}}$ of the right-hand side of (1.1) for which exist two different sequences of solutions

$$\{u_k\}_{k \in \mathbb{N}} = \left\{ u_{\frac{r(k+1)}{k+2}} \right\}_{k \in \mathbb{N}} \qquad \text{and} \qquad \{v_k\}_{k \in \mathbb{N}} = \left\{ v_{\frac{rk}{k+1}} v \right\}_{k \in \mathbb{N}}$$

with the same limit $u_r \in X$.

LEMMA 3.1.

   (j)   *The function $u \in X$ is a solution of initial-boundary value problem* (1.1) (1.3) *for $g \in Y$ if and only if $Fu = g$.*

   (jj)   *The couple $(u, g) \in X \times Y$ is a bifurcation point of* (1.1) (1.3) *if and only if $Fu = g$ and u is a point at which F is not locally invertible, i.e. $u \in \Sigma$.*

   P r o o f . The first assertion is clear.

   (jj)   If (u, g) is a bifurcation point of (1.1)–(1.3), then by Definition 3.1 we get $F(u) = g$, $F(u_k) = g_k = F(v_k)$, $u_k \neq v_k$. Thus F is not locally injective at u. Hence, F is not locally invertible at u, i.e. $u \in \Sigma$. To the contrary, if F is not locally invertible at u and $F(u) = g$, then F is not locally injective at u. Hence, it follows that the couple $(u, g) \in X \times Y$ is a bifurcation point of (1.1) (1.3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

   The following lemma gives sufficient conditions under which the operator A is of the Fredholm type.

LEMMA 3.2. *Let the operator A from* (3.1) *satisfy the smoothness hypothesis $\left(S^{\alpha+\rho}\right)$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$ (it has not to satisfy the conditions* (P), (C), (Q)). *Further let the condition*

   (A.1)   *There exists a linear homeomorphism $H \colon X \to Y$ such that*

$$Hu = \mathrm{D}_t\, u - H(t, x, \mathrm{D}_x)u\,, \qquad u \in X\,,$$

   where

$$H(t, x, \mathrm{D}_x)u = \sum_{|k|=2b} h_k(t,x) D_x^k u + \sum_{0 \le |k| \le 2b-1} h_k(t,x)\, \mathrm{D}_x^k\, u$$

   satisfies $\left(S^{\alpha+\rho}\right)$ for $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$

holds.

   Then

   (j) $\dim X = +\infty$.

   (jj)   *The operator $A \colon X \to Y$ is a linear bounded Fredholm operator of the zero-index.*

   P r o o f .

   (j)   The equation

$$\dim C_0^\infty(Q, \mathbb{R}) = +\infty$$

and the inclusion

$$C_0^\infty(Q, \mathbb{R}) \subset X$$

imply $\dim X = +\infty$.

(jj)   Since the coefficients a_k for $0 \leq |k| \leq 2b$ are continuous on the compact set $\operatorname{cl} Q$, there is a positive constant $K > 0$ such that

$$\|Au\|_Y \leq K \left(\| \mathrm{D}_t\, u\|_Y + \sum_{0 \leq |k| \leq 2b} \| \mathrm{D}_x^k u\|_Y \right) = K\|u\|_X$$

for all $u \in X$, whence the operator A is bounded on X.

If the operator A is a homeomorphism, then statement (jj) is clear.

If A is not the homeomorphism, then by the decomposition Nikoľskiĭ theorem from Proposition 2.1, it is sufficient to show that

$$Au = Hu + \big(H(t, x, \mathrm{D}_x) - A(t, x, \mathrm{D}_x)\big)u := Hu + Tu,$$

thereby the mapping $T\colon X \to Y$ is the linear completely continuous operator. It will be proved by generalized Ascoli-Arzelà theorem from [21; p. 31].

From the hypothesis $\big(S^{\alpha+\rho}\big)$, the equi-boundedness of

$$Tu = \sum_{|k|=2b} \big(h_k(t, x) - a_k(t, x)\big) \mathrm{D}_x^k u + \sum_{0 \leq |k| \leq 2b-1} \big(h_k(t, x) - a_k(t, x)\big) \mathrm{D}_x^k u$$

holds at the bounded set $S \subset X$, i.e. there is a constant $K_1(n, \alpha, T, \Omega) > 0$ such that $\|Tu\|_Y \leq K_1\|u\|_X$ for all $u \in S$.

Now for the equi-continuity of the set $TS \subset Y$ we have to prove the inequality (for every element u_l, $l = 1, \ldots, p$, of $u = (u_1, \ldots, u_p)$)

$$|(Tu)_l(t, x) - (Tu)_l(s, y)| + \frac{|(Tu)_l(t, x) - (Tu)_l(t, y)|}{\|x - y\|_{\mathbb{R}^n}^\alpha}$$
$$+ \frac{|(Tu)_l(t, x) - (Tu)_l(s, x)|}{|t - s|^{\alpha/2b}} < \varepsilon$$

for all $u \in S$ and $(t, x), (s, y), (t, y), (s, x) \in \operatorname{cl} Q$, $x \neq y$, $t \neq s$ for which the norms $\|x - y\|_{\mathbb{R}^n}$ and $|t - s|$ are sufficiently small, $\varepsilon > 0$.

With respect to $\left(S^{\alpha+\rho}\right)$ we obtain for the first member of the previous inequality

$$
\begin{aligned}
|(Tu)_l&(t,x) - (Tu)_l(s,y)| \\
&\leq \sum_{0 \leq |k| \leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)||\,\mathrm{D}_x^k\,u_l(t,x)| \\
&\quad + \sum_{|k|=2b} |h_k(s,y) - a_k(s,y)||\,\mathrm{D}_x^k\,u_l(t,x) - \mathrm{D}_x^k\,u_l(s,y)| \\
&\quad + \sum_{0 \leq |k| \leq 2b-1} |h_k(s,y) - a_k(s,y)||\,\mathrm{D}_x^k\,u_l(t,x) - \mathrm{D}_x^k\,u_l(s,y)| \\
&\leq K_2 \sum_{0 \leq |k| \leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)| \\
&\quad + K_3 \sum_{|k|=2b} |\,\mathrm{D}_x^k\,u_l(t,x) - \mathrm{D}_x^k\,u_l(s,y)| \\
&\quad + K_3 \sum_{0 \leq |k| \leq 2b-1} |\,\mathrm{D}_x^k\,u_l(t,x) - \mathrm{D}_x^k\,u_l(s,y)|\,,
\end{aligned}
$$

where K_2, K_3 are positive constants dependent only of n, α, T, Ω. For $|t-s| < \delta$, $\|x-y\|_{\mathbb{R}^n} < \delta$ with a sufficiently small $\delta > 0$ every member of the last inequality is smaller than a fixed arbitrary $\varepsilon > 0$. (Since $u \in S \subset X$, the number δ does not depend on u.)

For the second member we get by the condition $\left(S^{\alpha+\rho}\right)$ and using the mean value theorem

$$
\begin{aligned}
|(Tu)_l&(t,x) - (Tu)_l(t,y)|\|x-y\|_{\mathbb{R}^n}^{-\alpha} \\
&\leq K_2 \sum_{0 \leq |k| \leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(t,y)|\|x-y\|_{\mathbb{R}^n}^{-\alpha} \\
&\quad + K_3 \sum_{|k|=2b} \|\,\mathrm{D}_x^k\,u_l(t,x) - \mathrm{D}_x^k\,u_l(t,y)\| \cdot \|x-y\|_{\mathbb{R}^n}^{-\alpha} \\
&\quad + K_3 \sum_{0 \leq |k| \leq 2b-1} |\,\mathrm{D}_x^k\,u_l(t,x) - \mathrm{D}_x^k\,u_l(t,y)| \cdot \|x-y\|_{\mathbb{R}^n}^{-\alpha} \\
&\leq K\left(2\|x-y\|_{\mathbb{R}^n}^{\rho} + \|x-y\|^{1-\alpha}\right).
\end{aligned}
$$

In a similar way we have for the third member

$$|(Tu)_l(t,x) - (Tu)_l(s,x)||t-s|^{-\alpha/2b}$$

$$\leq K_2 \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,x)||t-s|^{-\alpha/2b}$$

$$+ K_3 \sum_{|k|=2b} |\,\mathrm{D}_x^k u_l(t,x) - \mathrm{D}_x^k u_l(s,x)||t-s|^{-\alpha/2b}$$

$$+ K_3 \sum_{0\leq|k|\leq 2b-1} |\,\mathrm{D}_x^k u_l(t,x) - \mathrm{D}_x^k u_l(s,x)||t-s|^{-\alpha/2b}$$

$$\leq K\left(2|t-s|^{\rho/2b} + |t-s|^{1-\alpha/2b} + \sum_{|k|=1}^{2b-1} |t-s|^{1-|k|/2b}\right).$$

By these three estimations the assertion (jj) is proved. $\qquad\square$

Remark 3.1. Necessary and sufficient conditions for the existence of a linear homeomorphism $H\colon X \to Y$ from the assumption (A.1) are given in Proposition 1.1. Concretely, for example, $Hu = \frac{\partial u}{\partial t} - \Delta u$, $u \in X$.

COROLLARY 3.1. *Let $\mathcal{L}$ mean the set of all linear differential operators $A = \mathrm{D}_t - A(t,x,\mathrm{D}_x)\colon X \to Y$ satisfying the hypothesis $\left(S^{\alpha+\rho}\right)$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$. Then for each $A \in \mathcal{L}$ the initial-boundary value homogeneous problem $Au = 0$, (1.2), (1.3) has a nontrivial solution, or any $A \in \mathcal{L}$ is a linear bounded Fredholm operator of the zero index.*

P r o o f . Really, if there exists an operator $A \in \mathcal{L}$ such that the problem $Au = 0$, (1.2), (1.3) has only trivial solution, then A is a homeomorphism from X onto Y (see Proposition 1.1). Then by Lemma 3.2 all operators of $\mathcal{L}$ are Fredholm of the zero index. $\qquad\square$

LEMMA 3.3. *Suppose*

(N.1) *The vector function $f \in C\big(\mathrm{cl}\,Q \times \mathbb{R}^\kappa, \mathbb{R}^p\big)$ satisfies the following local grown vector condition*

$$|f(t,x,u^\gamma) - f(s,y,v^\gamma)| \leq L\left[|t-s|^{\beta_1} + \|x-y\|_{\mathbb{R}^n}^{\beta_2} + \sum_{l=1}^{p}\sum_{0\leq|\gamma|\leq 2b-1} |u_l^\gamma - v_l^\gamma|^{\beta_{\gamma,l}}\right]\mathbb{J}$$

for (t,x,u^γ), (s,y,v^γ) from a compact subset of $\mathbb{R}^\kappa$ and $\beta_1 > \alpha/2b$, $\beta_2 > \alpha$, $\beta_{\gamma,l} > \alpha/(\alpha+\rho)$, $0 \leq |\gamma| \leq 2b-1$, $l = 1,\ldots,p$, where $L > 0$.

Then the Nemitskiĭ operator $N\colon X \to Y$ from (3.2) is completely continuous on X.

P r o o f . For any bounded set $S \subset X$ the operator N is equi-bounded in Y. Indeed, for all $u \in S$ using (N.1) the norm satisfies

$$
\begin{aligned}
\|Nu\|_Y \leq \max_{l=1,\dots,p} & \left[\sup_{(t,x)\in\mathrm{cl}\,Q} \left| f_l\!\left(t,x,\bar{\mathrm D}_x^\gamma u(t,x)\right) \right| \right. \\
& + L \sup_{\substack{(t,x),(t,y)\in\mathrm{cl}\,Q \\ x\neq y}} \frac{\|x-y\|_{\mathbb{R}^n}^{\beta_2} + \sum_{l=1}^{p}\sum_{0\leq|\gamma|\leq 2b-1} |\mathrm D_x^\gamma u_l(t,x) - \mathrm D_x^\gamma u_l(t,y)|^{\beta_{\gamma,l}}}{\|x-y\|_{\mathbb{R}^n}^{\alpha}} \\
& \left. + \sup_{\substack{(t,x),(s,x)\in\mathrm{cl}\,Q \\ t\neq s}} \frac{|t-s|^{\beta_1} + \sum_{l=1}^{p}\sum_{0\leq|\gamma|\leq 2b-1} |\mathrm D_x^\gamma u_l(t,x) - \mathrm D_x^\gamma u_l(s,x)|^{\beta_{\gamma,l}}}{|t-s|^{\alpha/2b}} \right].
\end{aligned}
$$

Hence, it is bounded by a positive constant $K\big(\Omega,T,L,\alpha,\beta_1,\beta_2,\beta_{\gamma,l}\big)$.

Also, for $|t-s|^2 + \|x-y\|_{\mathbb{R}^n}^2 < \delta^2$ with a sufficiently small $\delta > 0$ we get the equi-continuity of N. It is sufficient to prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$
\begin{aligned}
|(Nu)_l(t,x) - (Nu)_l(s,y)| & + \frac{|(Nu)_l(t,x) - (Nu)_l(t,y)|}{\|x-y\|_{\mathbb{R}^n}^{\alpha}} \\
& + \frac{|(Nu)_l(t,x) - (Nu)_l(s,x)|}{|t-s|^{\alpha/2b}} < \varepsilon
\end{aligned}
$$

is true for all $u \in S$ if both t, s and x, y are sufficiently near and $l = 1,\dots,p$. $\square$

LEMMA 3.4. *Let* $(S^{\alpha+\rho})$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho > 1$, (A.1), (N.1) *and an almost coercivity condition*

(F.1) *for each bounded set* $S \subset Y$ *there is a constant* $K^a > 0$ *such that for all solutions* $u \in X$ *of* (1.1)–(1.3) *with* $g \in S$ *the inequality*

$$
\|u\|_{a,Q} = \max_{l=1,\dots,p} \sum_{0\leq|k|\leq a} \sup_{(t,x)\in\mathrm{cl}\,Q} |\mathrm D_x^k u_l(t,x)| \leq K^a \tag{3.4}
$$

holds for $a = \max\{|\gamma|, r\}$. *Here* r *is an integer* $0 \leq r \leq 2b-1$ *for which the coefficients of operators* A *and* H *from* (3.1) *and* (A.1), *respectively, satisfy the relations* $a_k = h_k$ *for* $|k| = r+1,\dots,2b$ *and* $a_k \neq h_k$ *for at least one multiindex* k *with* $|k| = r$ *on* $\mathrm{cl}\,Q$

be satisfied. Then

(j) *F from* (3.3) *is coercive at* X.

(jj) *F is proper and continuous at* X.

Proof.

(j)   We need to prove that if the set $S \subset Y$ is bounded in Y, then the set of arguments $F^{-1}(S) \subset X$ is bounded in X.

By (3.4) and assumption (F.1) it follows that the set $F^{-1}(S)$ is bounded in the norm $\| \cdot \|_{a,Q}$. Hence, and by (N.1) one obtains the estimation $\|Nu\|_Y \leq K_4$ for all $u \in F^{-1}(S)$. From Lemma 3.2 (jj) also $\|Au\|_Y \leq \|Fu\|_Y + \|Nu\|_Y \leq K_5$ for any $u \in F^{-1}(S)$, where K_4, K_5 are positive constants.

On the other hand, condition (A.1) ensures the existence and uniqueness of the solution $u \in X$ of the linear equation $Hu = y$ for any $y \in Y$ and (see the Green representation of solution from (2.3) and [15; pp. 182–183] and estimation (2.1)) the estimation

$$\|u\|_X \leq K_6 \|y\|_Y \,, \qquad K_6 > 0 \,, \ u \in F^{-1}(S) \,, \tag{3.5}$$

is true.

Then for $u \in F^{-1}(S)$ we have

$$Hu = Au + \sum_{0 \leq |k| \leq 2b} \big(a_k(t,x) - h_k(t,x) \big) \, \mathrm{D}_x^k \, u \,.$$

With respect to (S^α) and (F.1)

$$\|y\|_Y = \|Hu\|_Y \leq \|Au\|_Y + \sum_{0 \leq |k| \leq r} \|a_k - h_k\|_Y \| \mathrm{D}_x^k \, u\|_Y$$

$$\leq K_5 + K_7 \|u\|_{r,Q} \leq K_5 + K_7 \|u\|_{a,Q} \leq K_5 + K_7 K^a \,, \qquad K_7 > 0 \,.$$

From this, and by (3.5),

$$\|u\|_X \leq K_6 (K_5 + K_7 K^a) \,, \qquad u \in F^{-1}(S) \,.$$

(jj)   Since $\dim X = +\infty$ and A is a nonconstant and closed mapping on X, then by Proposition 2.4 (jj) it is proper on X. From Lemma 3.3 the operator N is completely continuous on X. From (j) of this lemma, F is coercive on X. Proposition 2.5 (jj) concludes the proof of (jj) and the proof of Lemma 3.4.   □

In the following lemmas we shall consider the continuous nonlinearity f. Conditions for the continuous F-differentiability of the Nemitskiĭ operator N gives the following lemma.

LEMMA 3.5. *Let the Nemitskiĭ operator* $N \colon X \to Y$ *satisfy the condition* (N.1) *and*

(N.2)
$$\frac{\partial f}{\partial v_{\beta, l}} \in C\big(\operatorname{cl} Q \times \mathbb{R}^{\kappa}, \mathbb{R}^{p}\big)$$

for $l = 1, \ldots, p$ *and the multiindices* β *with the modul* $0 \leq |\beta| \leq 2b - 1$, *where* κ *represents the number of all components in the vector function* $\bar{\mathrm{D}}_{x}^{\beta} u$ *from* (1.1).

Then

(j) *the operator* N *is continuously Fréchet differentiable on* X, *i.e.* $N \in C^{1}(X, Y)$.

(jj) *If moreover* $\big(S^{\alpha + \rho}\big)$ *for* $\alpha \in (0, 1)$, $\rho > 0$, $\alpha + \rho < 1$ *holds, then* $F \in C^{1}(X, Y)$.

P r o o f .

(j)   We need to prove that the Fréchet derivative $N' \colon X \to L(X, Y)$ defined by the vector equation

$$N'(u)h(t, x) = \sum_{\substack{0 \leq |\beta| \leq 2b - 1 \\ \operatorname{card}\{\beta, l\} = \kappa \\ l = 1, \ldots, p}} \frac{\partial f}{\partial v_{\beta}}\big[t, x, \bar{\mathrm{D}}_{x}^{\gamma} u(t, x)\big] \mathrm{D}_{x}^{\beta} h_{l}(t, x) \qquad (3.6)$$

is continuos on X for every $u, h \in X$. Here $\beta = (\beta_{1}, \ldots, \beta_{n})$ represents every multiindex $\gamma = (\gamma_{1}, \ldots, \gamma_{n})$ appearing in the nonlinearity f. It is sufficient to show, for every fixed $v \in X$, the implication:

$$\big(\forall \varepsilon > 0\big)\big(\exists \delta(\varepsilon, v) > 0\big)\big(\forall u \in X\big)\Big(\|u - v\|_{X} < \delta \implies \|N'u - N'v\|_{L(X, Y)} < \varepsilon\Big)$$

i.e.

$$\sup_{h \in X, \|h\|_{X} \leq 1} \|N'(u)h - N'(v)h\|_{Y} < \varepsilon. \qquad (3.7)$$

Let us take an arbitrary $\varepsilon > 0$ and $u \in X$ such that $\|u - v\|_{X} < \delta$, i.e. $|\mathrm{D}_{t} u_{l}(t, x) - \mathrm{D}_{t} v_{l}(t, x)| < \delta$ and $|\mathrm{D}_{x}^{k} u_{l}(t, x) - D_{x}^{k} v_{l}(t, x)| < \delta$ for all multiindices $0 \leq |k| \leq 2b$ on $\operatorname{cl} Q$. Hence with respect to the uniform continuity of $\frac{\partial f}{\partial v_{\beta, l}}$ for $0 \leq |\beta| \leq 2b - 1$, $l = 1, \ldots, p$ on every compact of $\operatorname{cl} Q \times \mathbb{R}^{\kappa}$ we get the vector inequality

$$|N'(u)h(t, x) - N'(v)h(t, x)|$$

$$\leq \sum_{\substack{0 \leq |\beta| \leq 2b - 1 \\ \operatorname{card}\{\beta\} = \kappa \\ l = 1, \ldots, p}} \left| \frac{\partial f}{\partial v_{\beta, l}}\big[t, x, \bar{\mathrm{D}}_{x}^{\gamma} u(t, x)\big] - \frac{\partial f}{\partial v_{\beta, l}}\big[t, x, \bar{\mathrm{D}}_{x}^{\gamma} v(t, x)\big] \right| |\mathrm{D}_{x}^{\beta} h_{l}(t, x)| < \varepsilon \mathbb{J}$$

for $\|h\|_X \leq 1$ and all $(t, x) \in \operatorname{cl} Q$. It finishes the proof of (3.7).

(jj)   We easily see that Fréchet derivative $F' \colon X \to L(X, Y)$ is defined by the vector equation

$$F'(u)h(t, x) = \mathrm{D}_t\, h(t, x) - \sum_{0 \leq |k| \leq 2b} a_k(t, x)\, \mathrm{D}_x^k\, h(t, x) + N'(u)h(t, x)$$

for $u, h \in X$. Hence and by (j) we get $F \in C^1(X, Y)$. $\square$

LEMMA 3.6. *Let the hypotheses* $(S^{\alpha+\rho})$, $\alpha \in (0, 1)$, $\rho > 0$, $\alpha + \rho < 1$, (A.1), (N.1) *and* (N.2) *be satisfied. Then* $F = A + N \colon X \to Y$ *is a nonlinear Fredholm operator of the zero index on* X.

P r o o f .   According to Lemma 3.2 the operator $A \colon X \to Y$ is a linear continuous and C^1-Fredholm mapping of the zero index. By the statement of Lemma 3.3 the operator $N \colon X \to Y$ is compact. By Lemma 3.5 it belongs to the class C^1. Then the Proposition 2.8 implies that F is a nonlinear Fredholm operator with the zero index. $\square$

4. The solution set for continuous nonlinearities

The first results for that proper mapping F are given by the following theorem.

THEOREM 4.1. *Let hypotheses* $\left(S^{\alpha+\rho}\right)$ *for* $\alpha \in (0, 1)$, $\rho > 0$, $\alpha + \rho < 1$, (A.1), (N.1) *hold. Then*

    (a) *for any compact set of the right-hand sides* $g \in Y$ *of* (1.1) *the corresponding set of all solutions of* (1.1) – (1.3) *is a countable union of compact sets,*

    (b) *for* $u_0 \in X$ *there exists a neighbourhood* $U(u_0)$ *of* u_0 *and* $U\!\left(F(u_0)\right)$ *of* $F(u_0) \in Y$ *such that for each* $g \in U\!\left(F(u_0)\right)$ *there is a unique solution of* (1.1) – (1.3) *if and only if the operator* F *is locally injective at* u_0.

    (c) *Let moreover* (F.1) *hold. Then for any compact set of the right-hand sides* $g \in Y$ *from* (1.1), *the set of all solutions of* (1.1)   (1.3) *is compact (possible empty).*

P r o o f .

(a)   Since $F = A + N$ (see (3.3)), by the decomposition of $A = C + T$ (Proposition 2.1) we have $F = C + (T + N)$, where C is a continuous and proper mapping X onto Y (see Proposition 2.4), A is a Fredholm operator of the zero index, T and N are completely continuous mappings. Since X is a countable union of closed balls in X, by Proposition 2.5 (j) the operator F is σ-proper (continuous). Lemma 3.1 (j) implies assertion (a).

(b)   Suppose that F is injective in a neighbourhood $U(u_0)$ of $u_0 \in X$. From the decomposition (for H see Lemma 3.2)

$$F = H + (T + N)$$

we obtain $H^{-1}F = I + H^{-1}(T + N)$, which is a completely continuous and injective perturbation of the identity $I \colon X \to Y$ in $U(u_0)$. According to Proposition 2.7 (j) the set $H^{-1}F\big(U(u_0)\big)$ is open in X and the restriction $H^{-1}F\big|_{U(u_0)}$ is a homeomorphism of $U(u_0)$ onto $H^{-1}F\big(U(u_0)\big)$. Therefore F is locally invertible at u_0. Again by Lemma 3.1 (j) we obtain (b).

(c)   By Lemma 3.4 (jj) the operator $F \colon X \to Y$ is proper, which implies the given assertion and includes the proof of Theorem 4.1. $\square$

Further qualitative and quantitative properties of the set of solutions of (1.1)–(1.3) are given by the following theorem.

THEOREM 4.2. *Let hypotheses* $\big(S^{\alpha+\rho}\big)$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, (A.1), (N.1), (F.1) *hold. For solutions of* (1.1)–(1.3) *the following statements are true:*

(d)   *The set of solution for each $g \in Y$ is compact (possible empty).*

(e)   *The set $R(F) = \big\{ g \in Y : \text{ there exists at least one solution } u \in X \text{ of } (1.1)\text{–}(1.3) \big\}$ is closed and connected in Y.*

(f)   *The domain of bifurcation D_b is closed in X and the bifurcation range R_b is closed in Y. The set $F(X \setminus D_b)$ is open in Y.*

(g)   *If $Y \setminus R_b \neq \emptyset$, then each component of $Y \setminus R_b$ is a nonempty open set (i.e. domain).*

(h)   *If $Y \setminus R_b \neq \emptyset$, the number n_g of solutions is finite and constant (it may be zero) on each component of $Y \setminus R_b$, i.e. n_g is the same nonnegative integer for each g belonging to the same component of $Y \setminus R_b$.*

(i)   *If $R_b = \emptyset$, then the given problem has a unique solution $u \in X$ for each $g \in Y$ and this solution continuously depends on g as a mapping from Y onto X.*

(j)   *If $R_b \neq \emptyset$, then the boundary $\partial F(X \setminus D_b)$ is a subset of $F(D_b) = R_b$ $\big(\partial F(X \setminus D_b) \subset F(D_b) \big)$.*

P r o o f . Assertion (d) follows directly from Theorem 4.1 (c).

(e)   Take the sequence $\{g_n\}_{n \in \mathbb{N}} \subset R(F) \subset Y$ converging to $g \in Y$ as $n \to \infty$. By (d) there is a compact set of all solutions $\{u_\gamma\}_{\gamma \in I} \subset X$ (here I means an index set) of the equations $F(u) = g_n$ for $n = 1, 2, \dots$. Thus there exists a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}} \subset \{u_\gamma\}_{\gamma \in I}$ converging to $u \in X$ and $F(u_{n_k}) = g_{n_k} \to g$ in Y as $n \to \infty$. Since the mapping F is proper (Lemma 3.4 (jj)), by Proposition 2.4 (j) it is closed, whence $F(u) = g$, i.e. $g \in R(F)$. The set $R(F)$ is closed. $R(F) = F(X)$ is connected as a continuous image of the connected set X.

(f)   According to Lemma 3.1 (jj) $D_b = \Sigma$ and $R_b = F(D_b) = F(\Sigma)$. Since $X \setminus \Sigma$ is an open set, D_b is closed in X and its continuous image R_b is a closed set in Y.

Since $X \setminus D_b = X \setminus \Sigma$ is the set of all points at which the mapping F is locally invertible, to each $u_0 \in X \setminus D_b$ there exists a neighbourhood $U_1\big(F(u_0)\big) \subset F(X \setminus D_b)$. It means, the set $F(X \setminus D_b)$ is open.

(g)   The set $Y \setminus R_b = Y \setminus F(D_b) \neq \emptyset$ is open in Y. Then each its component is nonempty and open, too.

(h)   This directly follows from Proposition 2.2.

(i)   By $R_b = \emptyset$, we have $D_b = \emptyset$ and the mapping F is locally invertible in X. Proposition 2.5 (jj) asserts that F is a proper mapping. Then the global inverse mapping theorem (Proposition 2.3) implies that F is homeomorphism X onto Y.

(j)   From Lemma 3.1 (jj), $D_b = \Sigma$ and by (f), D_b and $F(D_b)$ are closed. Then $\partial F(X \setminus D_b) = \partial F(D_b) \subset F(D_b)$.

This finishes the proof of the theorem. $\qquad\qquad\qquad\qquad\qquad$ □

For the surjectivity of (1.1) – (1.3), see the following two theorems.

THEOREM 4.3. *Under the assumptions* $\big(S^{\alpha+\rho}\big)$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, (A.1), (N.1), (F.1) *each of the following conditions is sufficient to the solvability of problem* (1.1) – (1.3) *for each* $g \in Y$:

(k) *For each* $g \in R_b$ *there is a solution* $u \in X \setminus D_b$ *of* (1.1) (1.3).

(l) *The set* $Y \setminus R_b$ *is connected and there exists* $g \in R(F) \setminus R_b$ *(for* $R(F)$ *see Theorem* 4.2 (e)).

P r o o f. First of all we see that conditions (k) and (l) are mutually equivalent to the conditions

(k') $F(D_b) \subset F(X \setminus D_b)$

and

(l') $Y \setminus R_b$ is a connected set and $F(X \setminus D_b) \setminus R_b \neq \emptyset$,

respectively.

From the proof of Theorem 4.2 (f) we have $D_b = \Sigma$.

(k)   From (k') we have $F(X) = F(D_b) \cup F(X \setminus D_b) = F(X \setminus D_b)$. So $R(F) = F(X)$ is closed and connected in Y (Theorem 4.2 (e)) as well as open set in Y (see Theorem 4.2 (f)). Thus $R(F) = Y$, which implies the surjectivity of F.

(l)   By (h) of Theorem 4.2, card $F^{-1}\big(\{g\}\big)$ is a constant $k \geq 0$ for every g from the same component of $Y \setminus R_b$.

If $k = 0$ for all $g \in Y \setminus R_b$ such $F(X) = R_b$, whence $F(X \setminus D_b) \subset R_b$. It is a contradiction with (l'). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

THEOREM 4.4. *Let* $\left(S^{\alpha+\rho}\right)$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, (A.1), (N.1), (F.1) *hold together with the hypothesis:*

(S.1) *There exists a constant* $K^a > 0$ *such that all solutions* $u \in X$ *of the initial-boundary value problem for the equation*

$$Hu + \mu(Au - Hu + Nu) = 0\,, \qquad \mu \in (0,1)\,,$$

with data (1.2), (1.3) *fulfil inequality* (3.4) *from Lemma* 3.4. *H is the linear homeomorphism from hypothesis* (A.1).

Then

(m) *problem* (1.1)–(1.3) *has at least one solution for each* $g \in Y$,

(n) *the number* n_g *of solutions* (1.1)–(1.3) *is finite, constant and different from zero on each component of the set* $Y \setminus R_b$ *(for all g belonging to the same component of $Y \setminus R_b$).*

P r o o f.

(m)   It is sufficient to prove the surjectivity of $F \colon X \to Y$. By Lemma 3.2 (see the proof of (ii)) we can write

$$F = A + N = H + (T + N)\,.$$

The mapping

$$H^{-1}F = I + H^{-1}(T + N) \colon X \to X$$

is a completely continuous and condensing field (see [31; p. 496]).

Let $S \subset X$ be a bounded set. Then $H(S)$ is a bounded set in Y. From the coercivity of F (see Lemma 3.4(j)) the set $F^{-1}\big[H(S)\big] = \left(H^{-1}F\right)^{-1}(S)$ is bounded at X. Hence $H^{-1}F$ is coercive.

Now we show that condition (iii) from Proposition 2.6 is satisfied for the condensing and coercive field $P = H^{-1}F$. Take the strictly solvable field $G(u) = u$. Then the equation $P(u) = kG(u)$ implies

$$\left(H^{-1}F\right)(u) = ku\,.$$

Hence we get, for $u \in X$ and $k < 0$,

$$Hu + (1-k)^{-1}[Au - Hu + Nu] = 0$$

where $(1-k)^{-1} \in (0,1)$. With respect to condition (S.1)

$$\|u\|_{a,Q} \leq K^a$$

for $a = \max\{|\gamma|, r\}$, where $|\gamma| = 0, 1, \dots, 2b-1$ and $0 \leq r \leq 2b - 1$ are fixed. Using the same method as in Lemma 3.4(j) we obtain for all solutions of

$$(H^{-1}F)u = ku$$

the estimation $\|u\|_X \leq K_8$, $K_8 > 0$. By Proposition 2.6 we have the strict surjectivity of $H^{-1}F$ and so F. This proves (m).

(n)   From the surjectivity of F on X it follows $n_g \neq 0$. The other assertions of (n) follow from Theorem 4.2(h). $\qquad\square$

EXAMPLE 4.1. The initial-boundary value problem for the system of p equations can be a simple example illustrating results of this part.

$$\frac{\partial u_l}{\partial t} - K_l \frac{\partial^2 u_l}{\partial x^2} + f_l(u) = 0\,, \qquad (t,x) \in \langle 0,T\rangle \times \Omega \subset \mathbb{R} \times \mathbb{R},$$

$l = 1,\ldots,p$, with the conditions

$$\frac{\partial u_l}{\partial x}(t,0) = \frac{\partial u_l}{\partial x}(t,1) = 0\,, \qquad t \in \langle 0,T\rangle\,,$$
$$u_l(0,x) = 0\,, \qquad\qquad x \in \mathrm{cl}\,\Omega\,.$$

We take $K_l > 0$ and

$$f_l(u) = \begin{cases} u_l^{1/2} & \text{if } u_l \in \langle 0,a\rangle\,, \\ a^{1/2} & \text{if } u_l \in (a,\infty)\,, \\ 0 & \text{if } u_l \le 0 \end{cases}$$

for $l = 1,\ldots,p$. Assumption (A.1) is satisfied by Proposition 1.1. Condition (N.1) can be verified by elementary calculus. Supposition (F.1) follows from equation (2.3) and Green matrix estimations (2.1). Condition $\left(S^{\alpha+\rho}\right)$ holds for $0 < \alpha < 1/2$, $1/2 < \rho < 1$ and $\alpha + \rho < 1$ (for example $\alpha = 1/5$, $\rho = 3/5$).

5. The solution set for C^1-nonlinearities

With respect to the C^1-differentiability of the operator N from (3.2) we prove here several stronger results than in Section 4 for the solutions of (1.1)–(1.3).

THEOREM 5.1. *Suppose that* $\left(S^{\alpha+\rho}\right)$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *(A.1), (N.1), (N.2) and (F.1) be satisfied and R_b mean the bifurcation range of (1.1)–(1.3) from Definition 3.1. Then the set $Y \setminus R_b$ is open and dense in Y and thus the bifurcation range R_b of initial-boundary value problem (1.1)–(1.3) is nowhere dense in Y.*

P r o o f. The openess of $Y \setminus R_b$ follows from the statement (f) of Theorem 4.2.

From previous lemmas the operator $A\colon X \to Y$ is a linear continuous Fredholm mapping of the zero index and the Nemitskiĭ operator $N\colon X \to Y$ is compact and $N \in C^1(X,Y)$.

For every $u \in X$ the linear operator $N'\colon X \to Y$ from (3.6) is completely continuous on X. By the Nikoľskiĭ decomposition theorem (see Proposition 2.1) the operator $F'(u) = A + N'(u)\colon X \to Y$ is a linear Fredholm mapping of the zero index for each $u \in X$. By Lemma 3.5(jj) there is $F \in C^1(X,Y)$ and by Lemma 3.6 the F is a nonlinear Fredholm operator of the zero index.

According to the Banach open mapping theorem (see [30; p. 77]) the mutual equivalence is true: $F'(u)$ is a linear homeomorphism if and only if it is a bijective mapping. Since $F'(u)$ is a linear Fredholm mapping of the zero index for every $u \in X$, $F'(u)$ is bijective if and only if it is injective (in this case the injectivity implies surjectivity, see [31; p. 366, Proposition 8.14 (1)]). We see that $u \in X$ is a singular point of the Fredholm operator F if and only if u is a critical point of F.

From Proposition 2.10 we obtain that the set Σ (of all points $u \in X$ for which F is not locally invertible) is a subset of all critical points of F. Then, evidently Σ is a subset of all singular points S of F, i.e. $\Sigma \subset S$. Hence we get for the set of regular values R_F of the operator F the relations

$$R_F = Y \setminus F(S) \subset Y \setminus F(\Sigma) \subset Y \setminus R_b \subset Y,$$

where $R_b \subset F(\Sigma)$ is a bifurcation range of F.

Since $F: X \to Y$ is nonconstant closed mapping with $\dim X = \infty$, by Proposition 2.4 we obtain that F is a proper mapping. By the Proposition 2.9 (the Quinn version) the set R_F is residual, open and dense in Y. Hence $Y \setminus R_b$ is dense in Y, too. By Lemma 3.1 (jj) we can conclude the proof. $\square$

In the following results we shall deal with the linear problem in $h \in X$

$$Ah(t,x) + \sum_{\substack{0 \le |\beta| \le 2b-1 \\ \mathrm{card}\{|\beta|\} = \kappa}} \frac{\partial f}{\partial v_\beta}\big[t, x, \mathrm{D}_x^\gamma u(t,x)\big] \, \mathrm{D}_x^\beta h(t,x) = g(t,x) \qquad (5.1)$$

for $(t,x) \in Q$ and some fixed $u \in X$ with condition (1.2), (1.3). The left-hand side of equation (5.1) represents the Fréchet derivative $F'(u)h$ of the operator $F = A + N: X \to Y$.

THEOREM 5.2. *Let the hypotheses $\big(S^{\alpha+\rho}\big)$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, (A.1), (N.1), (N.2) and (F.1) be satisfied. Then:*

(o) *The number of solutions of (1.1)–(1.3) is constant and finite (it may be zero) on each connected component of the open set $Y \setminus F(S)$, i.e. for any g belonging to the same connected component of $Y \setminus F(S)$. Here S means the set of all critical points of the operator $F = A + N: X \to Y$.*

(p) *Let $u_0 \in X$ be a regular solution of (1.1)–(1.3) with the right-hand side $g_0 \in Y$. Then there exists a neighbourhood $U(g_0) \subset Y$ of g_0 such that for any $g \in U(g_0)$ initial-boundary value problem (1.1)–(1.3) has one and only one solution $u \in X$. This solution continuously depends on g.*

The associated linear problem (5.1), (1.2), (1.3) for $u = u_0$ has a unique solution $h \in X$ for any g from a neighbourhood $U(g_0)$ of $g_0 = F(u_0)$. This solution continuously depends on g.

(q)   *Denote by G the set of all right-hand sides $g \in Y$ of equation (1.1) for which the corresponding solutions $u \in X$ of $(1.1) - (1.3)$ are its critical points. Then G is closed nowhere dense in Y.*

(r)   *If the singular point set of $(1.1) - (1.3)$ is empty, then this problem has unique solution $u \in X$ for each $g \in Y$. It continuously depends on the right-hand side g.*

P r o o f .

(o)   In the proof of Theorem 5.1 we have showed that the set of all singular points of F is equal to the set of all critical points of F. Then the A m b r o s e t t i theorem (see Proposition 2.2) implies the statement (o).

(p)   Since $u_0 \in X \setminus S$, where S is a set of all singular (in our case all critical) points, by Proposition 2.10 the mapping F is a local C^1-diffeomorphism at u_0. This proves first part of (p) for $(1.1) - (1.3)$.

From F as the C^1-diffeomorphism it follows that $F' \in C(X,Y)$, $\left(F^{-1}\right)' \in C(X,Y)$, where $F'(u)h$ is the left-hand side of (5.1) and $\left(F^{-1}\right)'(Fu) = \left(F'(u)\right)^{-1}$ for every $u \in X$. Hence linear problem (5.1), (1.2), (1.3) for $u = u_0$ has a unique solution $h \in X$ for any $g \in U(g_0)$ with $g_0 = F(u_0)$. This solution continuously depends on the right-hand side g. The proof of (p) is completed.

(q)   In our case the equality $G = F(S)$ holds, where S is the set of all critical (all singular) points of F. By the S m a l e - Q u i n n theorem (Proposition 2.9) we obtain the expected results.

(r)   By Proposition 2.10, the operator $F\colon X \to Y$ is a local C^1-diffeomorphism at any point $u \in X$. Hence there follows the last assertion.   □

By point (p) of Theorem 5.2 we obtain the following corollary.

COROLLARY 5.1. *Let the hypotheses of Theorem 5.2 hold and moreover:*

(H.1) *Linear homogeneous problem (5.1), (1.2), (1.3) (for $g = 0$) has only zero solution $h = 0 \in X$ for any $u \in X$.*

Then initial-boundary value problem $(1.1) - (1.3)$ has a unique solution $u \in X$ for any $g \in Y$. Moreover, linear problem (5.1), (1.2), (1.3) has a unique solution $h \in X$ for any $u \in X$ and the right-hand side $g \in Y$ of (5.1). This solution continuously depends on g.

COROLLARY 5.2. *Let the assumptions of Theorem 5.2 be satisfied. Then we have:*

(s) *If the set S of all singular (in our case all critical) points of F is nonempty, then $\partial F(X \setminus S) \subset F(S)$.*

(t) *If $F(S) \subset F(X \setminus S)$, then problem $(1.1) - (1.3)$ has the solution $u \in X$ for any $g \in Y$, i.e. $R(F) = Y$ (F is a surjectivity of X onto Y).*

(u) *If $Y \setminus F(S)$ is connected and $X \setminus S \neq \emptyset$, then $R(F) = Y$ (the solvability of (1.1) (1.3) for any $g \in Y$).*

P r o o f. By Theorem 5.2(q) the set $F(S)$ is closed in Y, and by Proposition 2.9, $F(X \setminus S)$ is open in Y. Hence we have the equations

$$F(X) = F(S) \cup F(X \setminus S) = F(S) \cup \overline{F(X \setminus S)} = \overline{F(X)}, \qquad (5.2)$$

which implies that $F(X)$ is a closed set.

(s) Since $F \in C^1(X, Y)$, we get $\Sigma \subset S$ as in Theorem 5.1. Hence, and by Theorem 4.2(j)

$$\partial F(X \setminus S) \subset \partial F(X \setminus \Sigma) \subset F(\Sigma) \subset F(S).$$

(t) From the first equation of (5.2) we have $F(X) = F(X \setminus S)$ and so $R(F)$ is an open as well as a closed subset of the connected space Y. Thus $R(F) = Y$.

(u) Since $Y \setminus F(S)$ is connected, by Proposition 2.2 we obtain $\operatorname{card} F^{-1}(\{g\})$ $= \operatorname{const} =: k \geq 0$ for each $g \in Y \setminus F(S)$.

If k would be equal to 0, then $F(X) = F(S)$ and $F(X \setminus S) \subset F(S)$ and this is a contradiction with $X \setminus S \neq \emptyset$. Thus $k > 0$. □

THEOREM 5.3. *Suppose that hypotheses* $(S^{\alpha+\rho})$, $\alpha \in (0, 1)$, $\rho > 0$, $\alpha + \rho < 1$, *(A.1), (N.1), (N.2) and (F.1) hold together with the condition:*

(H.2) *Each point* $u \in X$ *is either a regular point or an isolated critical point of problem* (1.1) – (1.3).

Then to every $g \in Y$ *there exists one solution* $u \in X$ *of* (1.1) – (1.3). *It continuously depends on* g.

P r o o f. The associated operator $F \colon X \to Y$ is a proper C^1-Fredholm mapping of the zero index. By Proposition 2.10, F is a local C^1-diffeomorphism at a regular point of F. In the isolated singular point, by Proposition 2.11, F is locally invertible. Since F is proper, the global inverse mapping theorem (see Proposition 2.3) implies the statement of the theorem. □

EXAMPLE 5.1. Example 4.1 illustrates the results of Section 5 for $f_l(u) = \sin\left(\sum\limits_{i=1}^{l} u_i^2\right)$.

The studied models describe different natural science phenomena (a reaction-diffusion and environment models, a diffusive waves in fluid dynamics — the Burges equation, the wave propagation in a large number of biological and chemical systems — the Fisher equation, a nerve pulse propagation in nerve fibers and wall motion in liquid crystals).

The results of the present paper can be generalized also to the quasilinear parabolic and general evolution systems of the type (1.1) – (1.3). We can apply the Fredholm theory to hyperbolic equations modelling different nonlinear vibration problems, to a nonlinear dispersion (the nonlinear Klein-Gordan equation), a propagation of magnetic flux and the stability of fluid notions (the nonlinear Sine-Gordan equation) and so on.

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