Mathematica Slovaca

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Mathematica Slovaca, Vol. 55 (2005), No. 2, 203--216

Persistent URL: http://dml.cz/dmlcz/136914

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF GENERALIZED "FOOD-LIMITED" TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

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(*Communicated by Milan Medveď*)

ABSTRACT. Using a new method, we establish sufficient conditions which guarantee every solution of generalized "food-limited" type functional differential equation

$$x'(t) + \frac{\big(1+x(t)\big)\big(1-\lambda x(t)\big)}{1+\lambda} F\big(t, [x(\cdot)]^\alpha\big) = 0\,, \qquad t \geq 0\,,$$

to converge to zero as t tends to infinity, where $\alpha > 1$ is a ratio of two positive odd integers. The results in [LIU, Y. J.: *Global attractivity for a differential-difference population model*, Appl. Math. E-Notes **1** (2001), 56–64], [FENG, W.—ZHAO, A. M.—YAN, J. Y.: *Global attractivity of generalized delay Logistic equation*, Appl. Math. J. Chinese Univ. Ser. A **16** (2001), 136–142] are generalized and improved.

1. Introduction

Recently, there was an increasing interest into the study of the solutions of functional differential equations because of its importance both for the theory of functional differential equations and its ecological applications. For example, [1] investigate the asymptotic behavior of the so-called "food-limited" type functional differential equation

$$x'(t) + \frac{\big(1+x(t)\big)\big(1-\lambda x(t)\big)}{1+\lambda} F\big(t, x(\cdot)\big) = 0\,, \qquad t \geq 0\,, \tag{1}$$

where $\lambda > 0$, $F(t,\phi)$ is a continuous functional on $[0,+\infty) \times C_t$, C_t is the space of all continuous functions $\phi\colon \big[g(t), t\big] \to [-1,+\infty)$ endowed with norm

2000 Mathematics Subject Classification: Primary 34K15.
Keywords: functional differential equation, "food-limited" type, asymptotic behavior, solution.

Supported by National Natural Sciences Foundation of P. R. China.

$\|\phi\|_t = \sup\limits_{s\in[g(t),t]} |\phi(s)|$. $g\colon [0,+\infty) \to \mathbb{R}$ is a nondecreasing function and satisfies $g(t) < t$, $g(t) \to \infty$ $(t \to \infty)$. The following assumptions are set:

(A) F depends only on the value of ϕ on $[g(t),t]$, $F(t,0) \equiv 0$ and F fulfils

$$-p(t)M_t(\phi) \le F(t,\phi) \le p(t)M_t(\phi)\,, \qquad t \ge 0\,, \quad \phi \in C_t\,, \tag{2}$$

where $M_t(\phi) = \max\Big\{0, \sup\limits_{s\in[g(t),t]} \phi(s)\Big\}$, $p(t) \in C\big([0,+\infty),(0,+\infty)\big)$.

(B) For every $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that if $\inf\limits_{s\in[g(t),t]} \phi(s) \ge \varepsilon$, then

$$F(t,\phi) \ge \eta r(t)\,, \quad \text{or} \quad F(t,-\phi) \le -\eta r(t)\,, \qquad \text{for} \quad t \ge 0\,. \tag{3}$$

Let $\tau = g(0)$. The initial value condition of equation (1) is

$$x(t) = \phi(t)\,, \qquad t \in [-\tau, 0]\,, \tag{4}$$

where $\phi \in C\big([-\tau,0],\big[-1,\frac{1}{\lambda}\big)\big)$, $\phi(0) > -1$. It was proved in [1] that every solution of equation (1) tends to zero as t tends to infinity if

$$\int_0^{+\infty} r(t)\ \mathrm{d}t = +\infty\,, \tag{5}$$

and

$$\int_{g(t)}^{t} p(s)\ \mathrm{d}s \le \frac{3}{2}(1+\lambda)\,, \tag{6}$$

for sufficiently large t.

Equation (1) contains many ecological models as special cases. One may see [1], [2], [5], [6], [10]. However, it is easy to see that the following generalized "food-limited" type functional differential equation

$$x'(t) + \frac{\big(1+x(t)\big)\big(1-\lambda x(t)\big)}{1+\lambda} p(t)\big[x(\cdot)\big]^{\alpha} = 0\,, \qquad t \ge 0\,, \tag{7}$$

does not satisfy the condition (A) since $F(t,\phi) = p(t)\big[\phi(\cdot)\big]^{\alpha}$ and $\alpha > 1$, where α is a ratio of two odd integers, $\lambda \in [0,+\infty)$, g is defined in (1). Hence, the results in [1] cannot be applied to this equation. Equation (7) is deduced from the so-called "food-limited" type model, which was proposed in [2] as a generalized food-limited model in the form

$$N'(t) = p(t)N(t)\left(\frac{1-N(t-\tau)}{1+\lambda N(t-\tau)}\right)^{\alpha}, \qquad t \ge 0\,, \tag{8}$$

where α, λ, $p(t)$ are defined in (7). Transformation $-x(t) = \frac{1-N(t)}{1+\lambda N(t)}$ converts equation (8) into the form of (7). Equation (8), if $\alpha = 1$, becomes

$$N'(t) = p(t)N(t)\frac{1 - N(t-\tau)}{1 + \lambda N(t-\tau)}, \qquad t \geq 0,$$

which is called "food-limited" population model. This equation was studied by many authors, we refer the reader to [6]–[13] and the references therein.

Recently, [3] studied asymptotic behavior of equation (8). It was proved that if $\lambda \in (0,1]$, and

$$\int_0^{+\infty} p(t)\,\mathrm{d}t = +\infty, \qquad \lim_{t\to+\infty} \int_{g(t)}^{t} p(s)\,\mathrm{d}s \leq 3\lambda^{\alpha}, \tag{9}$$

then every positive solution of equation (8) tends to 1 as t tends to infinity. If $\lambda \in (1,+\infty)$, and

$$\int_0^{+\infty} p(t)\,\mathrm{d}t = +\infty, \qquad \lim_{t\to+\infty} \int_{g(t)}^{t} p(s)\,\mathrm{d}s \leq 3, \tag{10}$$

then every positive of equation (8) tends to 1 as t tends to infinity.

The asymptotic behavior of the equation

$$N'(t) = p(t)N(t)\big(1 - N(t-\tau)\big)^{\alpha}, \qquad t \geq 0, \tag{11}$$

was studied in [4]. Transformation $x(t) = N(t-\tau) - 1$ converts equation (11) into the form

$$x'(t) + p(t)\big(1 + x(t)\big)x^{\alpha}(t-\tau) = 0, \qquad t \geq \tau. \tag{12}$$

It was proved that if $\int_0^{+\infty} p(t)\,\mathrm{d}t = +\infty$, and for sufficiently large t, we have

$$\int_{t-\tau}^{t} p(s)\,\mathrm{d}s \leq h(\alpha) = \begin{cases} 1, & \alpha \in \big[1, G(1)\big], \\ M(\alpha), & \alpha \in \big(G(1), +\infty\big). \end{cases} \tag{13}$$

Then every positive solution of equation (11) tends to 1 as t tends to infinity, where

$$G(x) = 1 + \frac{\ln(\mathrm{e}^{x}+1) - \ln(2x)}{\ln(\mathrm{e}^{x}-1)}, \qquad M(x) = \begin{cases} 1, & 1 \leq G(1), \\ G^{-1}(x), & x \geq G(1). \end{cases}$$

It is showed that $0 < M(\eta) < 1/2 + \ln 2$ in [4].

Motivated by papers mentioned above, it is of significance to study the asymptotic behavior of solutions of equation (7). The purpose of this paper is to improve conditions (9), (10) and (13). And these results also generalize and improve the results in [3], [4]. The proofs here are simple and the methods different from those in [1], [3], [4].

For generality, we consider the following equation

$$x'(t) + \frac{\big(1+x(t)\big)\big(1-\lambda x(t)\big)}{1+\lambda} F\big(t, [x(\cdot)]^{\alpha}\big) = 0\,, \qquad t \geq 0\,, \tag{14}$$

where α is defined in (7), $F(t,\phi)$ is defined in (1), λ is defined in (1). The main results are the following:

THEOREM 1. *Suppose that $\lambda \in (0,1]$, (A) and (B) hold. In addition, for*

$$M = \big[2\alpha u(1-u)^{\alpha-1}(1+u)^{-\alpha-1}\big]^{-1}\,, \qquad u = \alpha - \sqrt{\alpha^2-1}\,,$$

if

$$\int_0^{+\infty} r(t)\ \mathrm{d}t = +\infty\,, \tag{15}$$

$$\limsup_{t\to+\infty} \int_{g(t)}^{t} p(s)\ \mathrm{d}s \leq \frac{3}{2} M \lambda^{\alpha}\,, \tag{16}$$

then every solution of equation (14) tends to zero as t tends to infinity.

THEOREM 2. *Suppose that $\lambda \in (1,+\infty)$, (A), (B) and (15) hold, and*

$$\limsup_{t\to+\infty} \int_{g(t)}^{t} p(s)\ \mathrm{d}s \leq \frac{3}{2} M\,. \tag{17}$$

Then every solution of equation (14) tends to zero as t tends to infinity. M is defined in Theorem 1.

THEOREM 3. *Suppose that there is $\delta > 0$ such that*

$$\limsup_{t\to+\infty} \int_{g(t)}^{t} p(s)\ \mathrm{d}s \leq \delta \left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1}\,, \qquad \int_0^{+\infty} p(s)\ \mathrm{d}s = +\infty\,, \tag{18}$$

and

$$\left(\delta - \frac{1}{2}\right)\left[\frac{\delta}{\delta-\frac{1}{2}}\left(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1\right)^{\alpha} - \frac{1}{2}\right] \leq 1\,. \tag{19}$$

Then every solution of equation (11) tends to 1 as t tends to infinity.

Remark 1. By the definition of M, it is easy to see that if $\alpha > 1$, then $M > 2$. Hence Theorems 1, 2 improve [3; Theorems A, B].

Remark 2. By (19), we see that if $\delta = (\frac{\alpha-1}{\alpha})^{\alpha-1}\ln 2 + \frac{1}{2}$, then one gets

$$\left(\delta - \frac{1}{2}\right)\left[\frac{\delta}{\delta - \frac{1}{2}}\left(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1\right)^{\alpha} - \frac{1}{2}\right]$$
$$= \delta - \frac{1}{2}\delta + \frac{1}{4} = \frac{1}{2}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\ln 2 + \frac{1}{2}.$$

So

$$\delta\left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1} = \ln 2 + \frac{1}{2}\left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1} > \frac{1}{2} + \ln 2.$$

Thus Theorem 3 improves the results in [4]. The method here is different and simpler from that in [4].

2. Preliminary Lemmas

Lemma 1. *Suppose that* $\lambda \in [0,1]$ *and* (A) *holds. Then for the initial value condition* (4) *equation* (1) *has unique solution* $x(t;0,\phi)$ *which is defined on* $[0,+\infty)$ *and satisfies* $-1 < x(t;0,\phi) < \frac{1}{\lambda}$. *(If* $\lambda = 0$, *then* $\frac{1}{\lambda} = +\infty$.)

Lemma 2. *Suppose that* $\lambda \in [0,1]$, (2), (A), (B) *and* (5) *hold. Then every non-oscillatory solution of* (14) *approaches zero as* t *tends to infinity.*

Lemma 3. *Suppose that* $\lambda \in [0,1]$, (A) *holds, and there is a positive number* M *such that for sufficiently large* t

$$\int_{g(t)}^{t} p(s)\,\mathrm{d}s \le M \tag{20}$$

is valid. Then every oscillatory solution of (14) *is bounded below away from* -1 *and is bounded above away from* $\frac{1}{\lambda}$, *i.e. there are constants* $A > 0$ *and* $B > 0$ *such that* $-1 < A \le x(t) \le B < +\infty$ *for all* t.

The proofs of Lemmas 1, 2 and 3 are similar to those of [1; Lemmas 2.1, 2.2, 2.3] and then are omitted.

Lemma 4. *The following inequalities*

$$\Big(\frac{1-\mathrm{e}^x}{1+\mathrm{e}^x}\Big)^\alpha \le -\frac{1}{M}x\,, \qquad x \le 0\,, \tag{21}$$

$$\Big(\frac{1-\mathrm{e}^x}{1+\mathrm{e}^x}\Big)^\alpha \ge -\frac{1}{M}x\,, \qquad x \ge 0\,, \tag{22}$$

$$(1-\mathrm{e}^x)^\alpha \le -\Big(\frac{\alpha-1}{\alpha}\Big)^{\alpha-1} x\,, \qquad x \le 0\,, \tag{23}$$

$$(1-\mathrm{e}^x)^\alpha \ge -\frac{(\mathrm{e}^A-1)^\alpha}{A}x\,,, \qquad 0 \le x \le A \tag{24}$$

hold, where M is defined in Theorem 1, A is a positive parameter.

Proof. We only prove (21) and (24), the proofs of (22) and (23) are exactly similar and therefore omitted. Let

$$f(x) = \frac{1}{M}x + \Big(\frac{1-\mathrm{e}^x}{1+\mathrm{e}^x}\Big)^\alpha .$$

Then

$$f'(x) = \frac{1}{M} - 2\alpha\,\mathrm{e}^x(1-\mathrm{e}^x)^{\alpha-1}(1+\mathrm{e}^x)^{-\alpha-1}.$$

Again, let

$$g(u) = u(1-u)^{\alpha-1}(1+u)^{-\alpha-1}.$$

One has

$$g'(u) = (1-u)^{\alpha-2}(1+u)^{-\alpha-2}(u^2-2\alpha u+1)\,.$$

So $g(u)$ has a unique maximum value point $u_0 = \alpha - \sqrt{\alpha^2-1}$ in $(0,1)$. If $x \le 0$, then we get $f'(x) \ge \frac{1}{M} - 2\alpha u_0(1-u_0)^{\alpha-1}(1+u_0)^{-\alpha-1} = 0$. Thus (21) holds.

Now, we prove (24). Let

$$f(x) = (1-\mathrm{e}^x)^\alpha.$$

Then $f'(x) = -\alpha\,\mathrm{e}^x(1-\mathrm{e}^x)^{\alpha-1}$, $f''(x) = \alpha\,\mathrm{e}^x(1-\mathrm{e}^x)^{\alpha-2}(-1+\alpha\,\mathrm{e}^x)$. If $0 \le x \le A$, then $f(x) \ge -\frac{(\mathrm{e}^A-1)^\alpha}{A}x$ since $f''(x) \le 0$, $f'(x) \le 0$. This completes the proof. □

3. Proofs of Theorems

Proof of Theorem 1. Suppose that $x(t) = x(t;0,\phi)$ is a solution of equation (14). By Lemma 1, $x(t)$ exists on $[0,+\infty)$ and satisfies $-1 < x(t) < \frac{1}{\lambda}$ for all $t \ge 0$. By Lemma 2, every non-oscillatory solution tends to zero as t tends to infinity, it suffices to prove that every oscillatory solution of equation (14) tends to zero. Let

$$\limsup_{t\to+\infty} x(t) = u\,, \qquad \liminf_{t\to+\infty} x(t) = v\,. \tag{25}$$

Then by Lemma 3, we have $-1 < v \le 0 \le u < \frac{1}{\lambda}$. For any positive number $\varepsilon < \min\{v+1,\ \frac{1}{\lambda} - u\}$, choose T sufficiently large such that

$$\int\limits_{g(t)}^{t} r(s)\ \mathrm{d}s \le \Big(\frac{3}{2} + \varepsilon\Big) M\lambda^{\alpha}\,, \qquad t \ge T\,, \tag{26}$$

and

$$v_1 = v - \varepsilon < x\big(g(t)\big) < u + \varepsilon = u_1\,, \qquad t \ge T\,. \tag{27}$$

It follows from (14), (A), (27) that

$$\frac{x'(t)}{\big(1+x(t)\big)\big(1-\lambda x(t)\big)} = -F\Big(t,\ \big[x\big(g(t)\big)\big]^{\alpha}\Big) \le -\frac{r(t)}{1+\lambda} v_1^{\alpha}\,, \qquad t \ge T\,. \tag{28}$$

Similarly, we get

$$\frac{x'(t)}{\big(1+x(t)\big)\big(1-\lambda x(t)\big)} \ge -\frac{r(t)}{1+\lambda} u_1^{\alpha}\,, \qquad t \ge T\,. \tag{29}$$

Now, we choose increasing sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ such that

$$g(p_n) > T\,, \quad x'(p_n) \ge 0\,, \quad x(p_n) > 0 \quad x(p_n) \to u\,, \quad n \to +\infty$$

and

$$g(q_n) > T\,, \quad x'(q_n) \le 0\,, \quad x(q_n) < 0\,, \quad x(q_n) \to v\,, \quad n \to +\infty\,.$$

From (A) and (14), there exist numbers ξ_n and η_n such that $x(\xi_n) = x(\eta_n) = 0$ and $x(t) > 0$ for $t \in (\xi_n, p_n]$ and $x(t) < 0$ for $t \in (\eta_n, q_n]$.

For $s \le \xi_n$, integrating (28) from s to ξ_n, we get

$$-x(s) \le \frac{1 - \mathrm{e}^{v_1^{\alpha} \int_s^{\xi_n} r(s)\ \mathrm{d}s}}{1 + \lambda\,\mathrm{e}^{v_1^{\alpha} \int_s^{\xi_n} r(s)\ \mathrm{d}s}}\,. \tag{30}$$

By (14), applying (30), we get for $t \in [\xi_n, p_n]$

$$\begin{aligned}\frac{x'(t)}{\big(1+x(t)\big)\big(1-\lambda x(t)\big)} &= -\frac{1}{1+\lambda}F\Big(t, \big[x(g(t))\big]^\alpha\Big)\\ &\le \frac{r(t)}{1+\lambda}\max\Big\{0, \sup_{s\in[g(t),t]}\big[-x(s)\big]^\alpha\Big\}\\ &= \frac{r(t)}{1+\lambda}\max\Big\{0, \sup_{s\in[g(t),\xi_n]}\big[-x(s)\big]^\alpha\Big\}\\ &\le \frac{r(t)}{1+\lambda}\left(\frac{1-\mathrm{e}^{v_1^\alpha\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s}}{1+\lambda\mathrm{e}^{v_1^\alpha\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s}}\right)^\alpha\\ &\le \frac{r(t)}{(1+\lambda)\lambda^\alpha}\left(\frac{1-\mathrm{e}^{v_1^\alpha\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s}}{1+\mathrm{e}^{v_1^\alpha\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s}}\right)^\alpha\\ &\le -\frac{r(t)}{(1+\lambda)\lambda^\alpha}\frac{1}{M}v_1^\alpha\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s\,.\end{aligned}$$

For $t \in [\xi_n, p_n]$, we have

$$\frac{x'(t)}{\big(1+x(t)\big)\big(1-\lambda x(t)\big)} \le \min\left\{-v_1^\alpha\frac{r(t)}{(1+\lambda)}\,,\ -\frac{r(t)}{(1+\lambda)\lambda^\alpha}\frac{1}{M}v_1^\alpha\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s\right\}. \tag{31}$$

Now, we consider two cases.

Case 1: $\int_{\xi_n}^{p_n} r(s)\,\mathrm{d}s \le M\lambda^\alpha$.

Integrating (31) from ξ_n to p_n, applying (26), we get

$$\begin{aligned}\ln\frac{1+x(p_n)}{1-\lambda x(p_n)} &\le -\frac{v_1^\alpha}{\lambda^\alpha M}\int_{\xi_n}^{p_n} r(t)\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s\,\mathrm{d}t\\ &= -\frac{v_1^\alpha}{\lambda^\alpha M}\int_{\xi_n}^{p_n} r(t)\left(\int_{g(t)}^{t} r(s)\,\mathrm{d}s - \int_{\xi_n}^{t} r(s)\,\mathrm{d}s\right)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\le -\frac{v_1^\alpha}{\lambda^\alpha M}\int_{\xi_n}^{p_n} r(t)\Big(\Big(\frac{3}{2}+\varepsilon\Big)M\lambda^\alpha - \int_{\xi_n}^{t} r(s)\,\mathrm{d}s\Big)\,\mathrm{d}t\\
&= -\Big(\frac{3}{2}+\varepsilon\Big)v_1^\alpha\int_{\xi_n}^{p_n} r(t)\,\mathrm{d}t - \frac{v_1^\alpha}{\lambda^\alpha M}\int_{\xi_n}^{p_n} r(t)\int_{\xi_n}^{t} r(s)\,\mathrm{d}s\,\mathrm{d}t\\
&= -v_1^\alpha\left[\Big(\frac{3}{2}+\varepsilon\Big)\int_{\xi_n}^{p_n} r(t)\,\mathrm{d}t - \frac{1}{\lambda^\alpha M}\frac{1}{2}\Big(\int_{\xi_n}^{p_n} r(t)\,\mathrm{d}t\Big)^2\right]\\
&\le -v_1^\alpha\Big(\Big(\frac{3}{2}+\varepsilon\Big)M\lambda^\alpha - \frac{1}{\lambda^\alpha M}\frac{1}{2}(M\lambda^\alpha)^2\Big)\\
&= -v_1^\alpha M\lambda^\alpha(1+\varepsilon)\,.
\end{aligned}
$$

Case 2: $\int_{\xi_n}^{p_n} r(s)\,\mathrm{d}s > M\lambda^\alpha$.

Choose $s_n \in (\xi_n, p_n)$ such that $\int_{s_n}^{p_n} r(s)\,\mathrm{d}s = \lambda^\alpha M$. Integrating (31), we get from (26)

$$
\begin{aligned}
&\ln\frac{1+x(p_n)}{1-\lambda x(p_n)}\\
&\le -v_1^\alpha\int_{\xi_n}^{s_n} r(t)\,\mathrm{d}t - \frac{v_1^\alpha}{M\lambda^\alpha}\int_{s_n}^{p_n} r(t)\int_{g(t)}^{\xi_n} r(s)\,\mathrm{d}s\,\mathrm{d}t\\
&\le -v_1^\alpha\int_{\xi_n}^{s_n} r(t)\,\mathrm{d}t - \frac{v_1^\alpha}{M\lambda^\alpha}\Big(\Big(\frac{3}{2}+\varepsilon\Big)M\lambda^\alpha\int_{s_n}^{p_n} r(t)\,\mathrm{d}t - \int_{s_n}^{p_n} r(t)\int_{\xi_n}^{t} r(s)\,\mathrm{d}s\,\mathrm{d}t\Big)\\
&\le -v_1^\alpha\Bigg\{\int_{\xi_n}^{s_n} r(t)\,\mathrm{d}t - \Big(\frac{3}{2}+\varepsilon\Big)M\lambda^\alpha\\
&\qquad\qquad - \frac{1}{M\lambda^\alpha}\frac{1}{2}\left[\Big(\int_{\xi_n}^{p_n} r(s)\,\mathrm{d}s\Big)^2 - \Big(\int_{\xi_n}^{s_n} r(s)\,\mathrm{d}s\Big)^2\right]\Bigg\}\\
&\le -v_1^\alpha\left[\int_{\xi_n}^{s_n} r(t)\,\mathrm{d}t - \Big(\frac{3}{2}+\varepsilon\Big)M\lambda^\alpha - \frac{1}{2}\Big(\int_{\xi_n}^{p_n} r(t)\,\mathrm{d}t + \int_{\xi_n}^{s_n} r(t)\,\mathrm{d}t\Big)\right]\\
&= -(1+\varepsilon)M\lambda^\alpha v_1^\alpha\,.
\end{aligned}
$$

Then it follows that

$$\ln\frac{1+x(p_n)}{1-\lambda x(p_n)} \le -(1+\varepsilon)M\lambda^\alpha v_1^\alpha .$$

Let $n\to+\infty$ and $\varepsilon\to 0$; we get

$$\ln\frac{1+u}{1-\lambda u} \le -M\lambda^\alpha v^\alpha . \tag{32}$$

Then

$$u^\alpha \le -\left(\frac{1-\mathrm{e}^{-M\lambda v^\alpha}}{1+\lambda\,\mathrm{e}^{-M\lambda^\alpha v^\alpha}}\right)^\alpha .$$

If $v=0$, then $u=0$. Thus the proof completes. If $v\ne 0$, then $v<0$. Thus by (21), we get

$$u^\alpha \le -\frac{1}{\lambda^\alpha}\left(\frac{1-\mathrm{e}^{-M\lambda^\alpha v^\alpha}}{1+\mathrm{e}^{-M\lambda^\alpha v^\alpha}}\right)^\alpha < \frac{1}{\lambda^\alpha}\frac{1}{M}(-M\lambda^\alpha v^\alpha) = -v^\alpha .$$

On the other hand, if $s\le\eta_n$, integrating (29) from s to η_n, we get

$$-x(s) \ge \frac{1-\mathrm{e}^{u_1^\alpha\int_s^{\eta_n} r(s)\,\mathrm{d}s}}{1+\lambda\,\mathrm{e}^{u_1^\alpha\int_s^{\eta_n} r(s)\,\mathrm{d}s}} .$$

Applying (A) and (22), we get for $t\in[\eta_n,q_n]$

$$\begin{aligned}
\frac{x'(t)}{\big(1+x(t)\big)\big(1-\lambda x(t)\big)} &\ge -\frac{1}{1+\lambda}F\Big(t,\big[x\big(g(t)\big)\big]^\alpha\Big)\\
&\ge -\frac{r(t)}{1+\lambda}\max\Big\{0,\ \sup_{s\in[g(t),t]}\big[x(s)\big]^\alpha\Big\}\\
&= -\frac{r(t)}{1+\lambda}\max\Big\{0,\ \sup_{s\in[g(t),\eta_n]}\big[x(s)\big]^\alpha\Big\}\\
&\ge \frac{r(t)}{1+\lambda}\left(\frac{1-\mathrm{e}^{u_1^\alpha\int_{g(t)}^{\eta_n} r(s)\,\mathrm{d}s}}{1+\lambda\,\mathrm{e}^{u_1^\alpha\int_{g(t)}^{\eta_n} r(s)\,\mathrm{d}s}}\right)^\alpha\\
&\ge \frac{r(t)}{(1+\lambda)\lambda^\alpha}\left(\frac{1-\mathrm{e}^{u_1^\alpha\int_{g(t)}^{\eta_n} r(s)\,\mathrm{d}s}}{1+\mathrm{e}^{u_1^\alpha\int_{g(t)}^{\eta_n} r(s)\,\mathrm{d}s}}\right)^\alpha\\
&\ge -\frac{r(t)}{(1+\lambda)\lambda^\alpha}\frac{1}{M}u_1^\alpha\int_{g(t)}^{\eta_n} r(s)\,\mathrm{d}s .
\end{aligned}$$

Then by a similar fashion of Case 1, we get

$$\frac{x'(t)}{\bigl(1+x(t)\bigr)\bigl(1-\lambda x(t)\bigr)} \geq \max\Bigl\{-\tfrac{u_1^\alpha}{1+\lambda}r(t)\,,\ -\tfrac{u_1^\alpha}{M(1+\lambda)\lambda^\alpha}r(t)\int\limits_{g(t)}^{\eta_n} r(s)\,\mathrm{d}s\Bigr\}. \tag{33}$$

Now, we consider two cases similar to those of Cases 1 and 2. We get

$$\ln\frac{1+x(q_n)}{1-\lambda x(q_n)} \geq -(1+\varepsilon)M\lambda^\alpha u_1^\alpha\,.$$

Let $n \to +\infty$ and $\varepsilon \to 0$. Then

$$v \geq -\frac{1-\mathrm{e}^{-M\lambda^\alpha u^\alpha}}{1+\lambda\,\mathrm{e}^{-M\lambda^\alpha u^\alpha}}\,.$$

By (22), one has $v^\alpha \geq -u^\alpha$. Thus, combining $u^\alpha < -v^\alpha$, we get $u^\alpha < u^\alpha$, a contradiction. The proof is complete. □

P r o o f o f T h e o r e m 2 . Let $\lambda x(t) = -y(t)$, then (14) becomes

$$y'(t) + \frac{1}{1+\frac{1}{\lambda}}\Bigl(1-\frac{1}{\lambda}y(t)\Bigr)\bigl(1+y(t)\bigr)F\Bigl(t,\ \frac{1}{\lambda^\alpha}\bigl[y(\cdot)\bigr]^\alpha\Bigr) = 0\,.$$

Since $0 < \frac{1}{\lambda} < 1$, we get Theorem 2 from Theorem 1. □

P r o o f o f T h e o r e m 3 . Let $1 - N(t) = -x(t)$, then (11) becomes

$$x'(t) + p(t)\bigl(1+x(t)\bigr)\bigl[x(t-\tau)\bigr]^\alpha = 0\,, \qquad t \geq 0\,. \tag{34}$$

Similar to the proof of Theorem 1, if $x(t)$ is non-oscillatory, then from Lemma 2 it follows that $x(t)$ tends to zero as t tends to infinity. If $x(t)$ is oscillatory, let

$$\limsup_{t\to+\infty} x(t) = u\,, \qquad \liminf_{t\to+\infty} x(t) = v\,. \tag{35}$$

By Lemma 3, we have $-1 < v \leq 0 \leq u < +\infty$. Similarly as in Theorem 1, we have

$$\frac{x'(t)}{1+x(t)} \leq -p(t)v_1^\alpha\,, \qquad t \in [\xi_n, p_n]\,, \tag{36}$$

$$\frac{x'(t)}{1+x(t)} \geq -p(t)u_1^\alpha\,, \qquad t \in [\eta_n, q_n]\,, \tag{37}$$

where ξ_n, η_n, p_n, q_n and ε, u_1, v_1 are defined similarly as in Theorem 1. Then, for $t \in [\xi_n, p_n]$, integrating (36) from $t-\tau$ to ξ_n, we get

$$\ln\bigl(1+x(t-\tau)\bigr) \geq v_1^\alpha \int\limits_{t-\tau}^{\xi_n} p(s)\,\mathrm{d}s\,.$$

We get

$$\frac{x'(t)}{1+x(t)} \le -p(t)\Big(\frac{\alpha-1}{\alpha}\Big)^{\alpha-1} v_1^\alpha \int_{t-\tau}^{\xi_n} p(s)\,\mathrm{d}s\,.$$

There are two cases to be considered.

Case 1: $\int_{\xi_n}^{p_n} p(s)\,\mathrm{d}s \le \big(\frac{\alpha}{\alpha-1}\big)^{\alpha-1}$.

Case 2: $\int_{\xi_n}^{p_n} p(s)\,\mathrm{d}s > \big(\frac{\alpha}{\alpha-1}\big)^{\alpha-1}$.

Similarly to the arguments of Cases 1 and 2 in the proof of Theorem 1, all these cases imply that

$$\ln(1+u) \le -\Big(\delta-\frac{1}{2}\Big)\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1} v^\alpha\,. \tag{38}$$

If $v=0$, then $u=0$. This completes the proof. If $v \ne 0$, then $v<0$. Since $v \in (-1,0)$, together with (38), we get

$$u \le \mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1\,.$$

Choose T sufficiently large such that

$$x(t-\tau) < \mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1 + \varepsilon\,, \qquad t \ge T\,. \tag{39}$$

On the other hand, from (37), we get for $s \le \eta_n$,

$$-x(s) \ge 1 - \mathrm{e}^{u_1^\alpha \int_s^{\eta_n} r(t)\,\mathrm{d}t}\,. \tag{40}$$

By (23) and (24), if $0 \le \mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1 + \varepsilon$, it is easy to prove

$$-y^\alpha \ge -\frac{\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1 + \varepsilon\Big)^\alpha}{A} \ln(1+y)\,,$$

where $A = \big(\delta-\frac{1}{2}\big)\big(\frac{\alpha}{\alpha-1}\big)\alpha - 1$. Hence

$$\begin{aligned}\frac{x'(t)}{1+x(t)} &\ge -p(t)\frac{\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1 + \varepsilon\Big)^\alpha}{A} \ln\big(1+x(t-\tau)\big) \\ &\ge -u_1^\alpha \frac{\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1 + \varepsilon\Big)^\alpha}{A} p(t) \int_{t-\tau}^{\eta_n} p(s)\,\mathrm{d}s\,.\end{aligned}$$

We get for $t \in [\eta_n, q_n]$,

$$\frac{x'(t)}{1+x(t)} \ge \max\Big\{-u_1^\alpha p(t),\ -u_1^\alpha \frac{\big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} - 1+\varepsilon\big)^\alpha}{A} p(t) \int_{t-\tau}^{\eta_n} p(s)\,\mathrm{d}s\Big\}. \tag{41}$$

We consider two cases.

Case 1: $\int_{\eta_n}^{q_n} p(s)\,\mathrm{d}s \le A\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} -1+\varepsilon\Big)^{-\alpha}$.

Case 2: $\int_{\eta_n}^{q_n} p(s)\,\mathrm{d}s \ge A\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}} -1+\varepsilon\Big)^{-\alpha}$.

Similarly to Cases 1 and 2 in the proof of Theorem 1, it follows that

$$\ln(1+x(q_n)) \ge -\left[\delta\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1}\frac{\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}}{A}-\frac{1}{2}\right]\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1+\varepsilon\Big)^{-\alpha}Au_1^{\alpha}.$$

Let $n\to+\infty$ and $\varepsilon\to 0$. It follows that

$$\ln(1+v) \ge -\left[\delta\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1}\frac{\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}}{A}-\frac{1}{2}\right]\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{-\alpha}Au^{\alpha}.$$

Let $\ln(1+u)=x$, $\ln(1+v)=y$. We get

$$x \le \Big(\delta-\frac{1}{2}\Big)\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1}(1-\mathrm{e}^{y})^{\alpha}.$$

$$y \ge \left[\delta\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1}\frac{\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}}{A}-\frac{1}{2}\right]\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{-\alpha}A(1-\mathrm{e}^{x})^{\alpha}.$$

By (23) and (24), we get

$$\begin{aligned}
y &\ge \left[\frac{\delta}{\delta-\frac{1}{2}}\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}-\frac{1}{2}\right]\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{-\alpha}A\left(-\frac{(\mathrm{e}^{A}-1)^{\alpha}}{A}x\right)\\
&= -\left[\frac{\delta}{\delta-\frac{1}{2}}\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}-\frac{1}{2}\right](\mathrm{e}^{A}-1)^{\alpha}x\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{-\alpha}\\
&\ge -\left[\frac{\delta}{\delta-\frac{1}{2}}\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}-\frac{1}{2}\right]\Big(\delta-\frac{1}{2}\Big)\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1}(1-\mathrm{e}^{y})^{\alpha}\\
&> \Big(\delta-\frac{1}{2}\Big)\left[\frac{\delta}{\delta-\frac{1}{2}}\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}-\frac{1}{2}\right]\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1}\Big(\frac{\alpha-1}{\alpha}\Big)^{\alpha-1}y.
\end{aligned}$$

It follows from $y<0$ that

$$1 < \Big(\delta-\frac{1}{2}\Big)\left[\delta\Big(\frac{\alpha}{\alpha-1}\Big)^{\alpha-1}\frac{\Big(\mathrm{e}^{(\delta-\frac{1}{2})(\frac{\alpha}{\alpha-1})^{\alpha-1}}-1\Big)^{\alpha}-1}{A}-\frac{1}{2}\right].$$

This contradicts (19). The proof is complete. $\square$

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Received August 7, 2002
Revised December 15, 2003

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