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ALGEBRAICALLY UNRELATED SEQUENCES

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ABSTRACT. The paper deals with the so-called algebraically unrelated sequences. The criterion and some applications for infinite series of rational numbers are included.

1. Introduction

One approach to prove the algebraic independence of numbers is due to Mahler. For instance, Becker in [1] proved the algebraic independence of the values of certain series or Nishioka in [7] proved the algebraic independence of values of special functions. A nice survey of this kind of results can be found in the book of Nishioka in [6].

Other methods are described in Bundschuh [2], Hančl [4] or Nettler [5].

The following is an immediate consequence of Bundschuh's Criterion for algebraic independence in [3].

THEOREM 1.1. *Let K be a positive integer. Assume that $\{a_{j,n}\}_{n=1}^{\infty}$ $(j = 1,2,\dots,K)$ are sequences of positive integers and $\{b_{j,n}\}_{n=1}^{\infty}$ $(j = 1,2,\dots,K)$ are sequences of integers. Suppose that $g\colon \mathbb{N} \to \mathbb{R}$ is the function such that*

$$\lim_{n\to\infty} g(n) = \infty$$

and for infinitely many positive integers n and for every $k = 1,2,\dots,K$

$$g(n)\sum_{j=1}^{k-1}\left|\sum_{i=n+1}^{\infty}\frac{b_{j,i}}{a_{j,i}}\right| < \left|\sum_{i=n+1}^{\infty}\frac{b_{k,i}}{a_{k,i}}\right| \le \frac{1}{\left(\prod_{j=1}^{k}\operatorname{lcm}(a_{j,1},\dots,a_{j,n})\right)^{g(n)}}\,, \tag{1}$$

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where $\operatorname{lcm}(x_1,\dots,x_n)$ *is the least common multiply of the numbers* $x_1,\dots,x_n$. *Then the numbers* $\sum_{n=1}^{\infty}\frac{b_{j,n}}{a_{j,n}}$ $(j=1,2,\dots,K)$ *are algebraically independent over the rational numbers.*

2. Algebraically unrelated sequences

DEFINITION 2.1. Let $\{a_{i,n}\}_{n=1}^{\infty}$ $(i=1,\dots,K)$ be sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the numbers $\sum_{n=1}^{\infty}\frac{1}{a_{1,n}c_n},\dots,\sum_{n=1}^{\infty}\frac{1}{a_{K,n}c_n}$ are algebraically independent, then the sequences $\{a_{i,n}\}_{n=1}^{\infty}$ $(i=1,\dots,K)$ are *algebraically unrelated.*

THEOREM 2.1. *Let* K *be a positive integer. Assume that* ε, ε_1, ε_2 *and* ε_3 *are positive real numbers such that*

$$(1-\varepsilon_1)\varepsilon_2(1+\varepsilon)>1 \qquad \text{and} \qquad \varepsilon_2<1<\varepsilon_3\,. \tag{2}$$

Let $\{a_{i,n}\}_{n=1}^{\infty}$ *and* $\{b_{i,n}\}_{n=1}^{\infty}$ $(i=1,\dots,K)$ *be sequences of positive integers with* $\{a_{1,n}\}_{n=1}^{\infty}$ *is nondecreasing, such that*

$$\limsup_{n\to\infty}\frac{1}{n}\log\log a_{1,n}=\infty\,, \tag{3}$$

$$\lim_{n\to\infty}\frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}}=0 \qquad \text{for all} \quad j,i\in\{1,\dots,K\}\,,\quad i>j\,, \tag{4}$$

and for every sufficiently large positive integer n *and for every* $i=1,2,\dots,K$

$$n^{1+\varepsilon}<a_{1,n}\,, \tag{5}$$

$$b_{i,n}<a_{i,n}^{\varepsilon_1}\,, \tag{6}$$

$$a_{1,n}^{\varepsilon_2}<a_{i,n}<a_{1,n}^{\varepsilon_3}\,. \tag{7}$$

Then the sequences $\{\frac{a_{i,n}}{b_{i,n}}\}_{n=1}^{\infty}$ $(i=1,\dots,K)$ *are algebraically unrelated.*

EXAMPLE 2.1. Let K be a positive integer. As an immediate consequence of Theorem 2.1 we obtain that the sequences

$$\left\{\frac{2^{n!}+in^2}{n^i+1}\right\}_{n=1}^{\infty}\,, \qquad i=1,2,\dots,K\,,$$

are algebraically unrelated.

EXAMPLE 2.2. Let K be a positive integer. As an immediate consequence of Theorem 2.1 we obtain that the sequences

$$\left\{\frac{n^{n^n} + n^n i + 1}{2^{2^{in}} + n^{in}}\right\}_{n=1}^{\infty}, \qquad i = 1, 2, \dots, K,$$

are algebraically unrelated.

OPEN PROBLEM 2.1. Let K be a positive integer greater than one. Are the sequences

$$\left\{\frac{2^{n!} + in^2}{2^i + (-1)^n}\right\}_{n=1}^{\infty}, \qquad i = 1, 2, \dots, K,$$

algebraically unrelated?

3. Proofs

LEMMA 3.1. *Let ε be a positive real number and ε_1 be a nonnegative real number satisfying*

$$(1-\varepsilon_1)(1+\varepsilon) > 1. \tag{8}$$

Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers with $\{a_n\}_{n=1}^{\infty}$ is nondecreasing such that for every sufficiently large positive integer n

$$n^{1+\varepsilon} < a_n \tag{9}$$

and

$$b_n \le a_n^{\varepsilon_1}. \tag{10}$$

Then there exists a positive real number β such that for every sufficiently large positive integer n

$$\sum_{j=n}^{\infty} \frac{b_j}{a_j} < \frac{1}{a_n^{\beta}}. \tag{11}$$

Proof. From (10) we obtain

$$\sum_{j=n}^{\infty} \frac{b_j}{a_j} \le \sum_{j=n}^{\infty} \frac{1}{a_j^{1-\varepsilon_1}} = \sum_{n \le j \le a_n^{\frac{1-\varepsilon_1}{2}}} \frac{1}{a_j^{1-\varepsilon_1}} + \sum_{j > a_n^{\frac{1-\varepsilon_1}{2}}} \frac{1}{a_j^{1-\varepsilon_1}}. \tag{12}$$

Now we will estimate the right side of inequality (12). For the first summand we have the estimation

$$\sum_{n \le j \le a_n^{\frac{1-\varepsilon_1}{2}}} \frac{1}{a_j^{1-\varepsilon_1}} < \frac{a_n^{\frac{1-\varepsilon_1}{2}}}{a_n^{1-\varepsilon_1}} = \frac{1}{a_n^{\frac{1-\varepsilon_1}{2}}}. \tag{13}$$

Now we will estimate the second summand of inequality (12). From (8) and (9) we obtain that for every sufficiently large positive integer n

$$\sum_{j>a_n^{\frac{1-\varepsilon_1}{2}}} \frac{1}{a_j^{1-\varepsilon_1}} < \sum_{j>a_n^{\frac{1-\varepsilon_1}{2}}} \frac{1}{j^{(1+\varepsilon)(1-\varepsilon_1)}} < \int_{a_n^{\frac{1-\varepsilon_1}{2}}}^{\infty} \frac{2\,\mathrm{d}x}{x^{(1+\varepsilon)(1-\varepsilon_1)}}$$
$$\leq \frac{1}{\left(a_n^{\frac{1-\varepsilon_1}{2}}\right)^{\frac{(1+\varepsilon)(1-\varepsilon_1)-1}{2}}} = \frac{1}{a_n^{\frac{(1-\varepsilon_1)(1+\varepsilon)(\frac{\varepsilon}{1+\varepsilon}-\varepsilon_1)}{4}}}\,. \tag{14}$$

Let us put

$$\beta = \frac{1}{2}\min\left\{\tfrac{1-\varepsilon_1}{2},\ \tfrac{1}{4}(1-\varepsilon_1)(1+\varepsilon)\left(\tfrac{\varepsilon}{1+\varepsilon}-\varepsilon_1\right)\right\}. \tag{15}$$

Then inequalities (12), (13) and (14) and equation (15) imply (11). The proof of Lemma 3.1 is complete. □

Proof of Theorem 2.1. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then there is a bijective map $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that the sequence $\left\{c_{\sigma(n)}a_{1,\sigma(n)}\right\}_{n=1}^{\infty}$ is non-decreasing.

Now we will prove that the sequences $\left\{c_{\sigma(n)}a_{i,\sigma(n)}\right\}_{n=1}^{\infty}$ and $\left\{b_{i,\sigma(n)}\right\}_{n=1}^{\infty}$ ($i = 1,\dots,K$) will satisfy conditions (2)–(7). Condition (2) holds. From $c_{\sigma(n)}a_{1,\sigma(n)} \geq a_{1,n}$, $c_{\sigma(n)}a_{i,\sigma(n)} \geq a_{i,\sigma(n)}$ ($i = 1,\dots,K$) and $\varepsilon_2 < 1 < \varepsilon_3$ we obtain conditions (3), (5), (6) and (7). If $i > j$, then

$$\lim_{n\to\infty} \frac{c_{\sigma(n)}a_{i,\sigma(n)}b_{j,\sigma(n)}}{c_{\sigma(n)}a_{j,\sigma(n)}b_{i,\sigma(n)}} = \lim_{n\to\infty} \frac{a_{i,\sigma(n)}b_{j,\sigma(n)}}{a_{j,\sigma(n)}b_{i,\sigma(n)}} = \lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{a_{j,n}b_{i,n}} = 0\,.$$

Hence (4) holds.

Thus it suffices to prove if K, ε, ε_1, ε_2, ε_3 and the sequences $\{a_{i,n}\}_{n=1}^{\infty}$, $\{b_{i,n}\}_{n=1}^{\infty}$ ($i = 1,\dots,K$) satisfy all conditions stated in Theorem 2.1, then the numbers $\sum\limits_{n=1}^{\infty} \frac{b_{1,n}}{a_{1,n}}, \dots, \sum\limits_{n=1}^{\infty} \frac{b_{K,n}}{a_{K,n}}$ are algebraically independent over the rational numbers. To establish this, we verify condition (1) of Theorem 1.1. Let R be a large positive real number.

1. We prove that for every $k = 2,\dots,K$ and for every sufficiently large positive integer n

$$\sum_{i=n+1}^{\infty} \frac{b_{k,i}}{a_{k,i}} - R\sum_{j=1}^{k-1}\sum_{i=n+1}^{\infty} \frac{b_{j,i}}{a_{j,i}} > 0\,. \tag{16}$$

From (4) we obtain that for every sufficiently large positive integer n

$$\sum_{i=n+1}^{\infty} \frac{b_{k,i}}{a_{k,i}} - R\sum_{j=1}^{k-1} \sum_{i=n+1}^{\infty} \frac{b_{j,i}}{a_{j,i}} = \sum_{j=1}^{k-1} \sum_{i=n+1}^{\infty} \left(\frac{1}{k-1} \frac{b_{k,i}}{a_{k,i}} - R\frac{b_{j,i}}{a_{j,i}} \right)$$
$$= \sum_{j=1}^{k-1} \sum_{i=n+1}^{\infty} \frac{b_{k,i}}{a_{k,i}} \left(\frac{1}{k-1} - R\frac{b_{j,i}a_{k,i}}{a_{j,i}b_{k,i}} \right) > 0 .$$

Thus (16) holds.

2. Now we prove that there exist infinitely many positive integers n such that for every $k = 1, 2, \ldots, K$

$$\sum_{i=n+1}^{\infty} \frac{b_{k,i}}{a_{k,i}} \le \frac{1}{\left(\prod_{j=1}^{k} \prod_{i=1}^{n} a_{j,i} \right)^R} . \tag{17}$$

Inequalities (6) and (7) imply that for every $k = 2, \ldots, K$ and for every sufficiently large positive integer n

$$\sum_{j=n+1}^{\infty} \frac{b_{k,j}}{a_{k,j}} \le \sum_{j=n+1}^{\infty} \frac{1}{a_{k,j}^{1-\varepsilon_1}} \le \sum_{j=n+1}^{\infty} \frac{1}{a_{1,j}^{(1-\varepsilon_1)\varepsilon_2}} \le 2 \sum_{j=n+1}^{\infty} \frac{1}{\left[a_{1,j}^{(1-\varepsilon_1)\varepsilon_2}\right] + 1} ,$$

where $[x]$ is the greatest integer less than or equal x. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive integers such that either $\{a_n\}_{n=1}^{\infty} = \{a_{1,n}\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{b_{1,n}\}_{n=1}^{\infty}$ or $\{a_n\}_{n=1}^{\infty} = \left\{[a_{1,n}^{(1-\varepsilon_1)\varepsilon_2}]+1\right\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$.

Now we will prove that the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy all assumptions stated in Lemma 3.1. This is obvious for $\{a_n\}_{n=1}^{\infty} = \{a_{1,n}\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{b_{1,n}\}_{n=1}^{\infty}$. For $\{a_n\}_{n=1}^{\infty} = \left\{[a_{1,n}^{(1-\varepsilon_1)\varepsilon_2}] + 1\right\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$ we have $b_n = 1 = a_n^0$. Thus (10) holds. Conditions (2) and (5) imply $a_n = \left[a_{1,n}^{(1-\varepsilon_1)\varepsilon_2}\right] + 1 > a_{1,n}^{(1-\varepsilon_1)\varepsilon_2} > n^{(1-\varepsilon_1)\varepsilon_2(1+\varepsilon)}$, where $(1-\varepsilon_1)\varepsilon_2(1+\varepsilon) > 1$. Hence (9) holds. Condition (8) has the form $(1-\varepsilon_1)\varepsilon_2(1+\varepsilon) > 1$ and immediately follows from (2).

Hence Lemma 3.1 implies that there exists positive real number β such that for every $k = 1, 2, \ldots, K$ and for every sufficiently large positive integer n

$$\sum_{i=n+1}^{\infty} \frac{b_{k,i}}{a_{k,i}} \le \frac{1}{a_{1,n+1}^{\beta}} . \tag{18}$$

Let S be a sufficiently large positive integer such that

$$S > 1 + \frac{(1+(k-1)\varepsilon_3)R}{\beta} . \tag{19}$$

Then (3) implies that

$$\limsup_{n\to\infty} a_{1,n}^{\frac{1}{S^n}} = \infty\,. \tag{20}$$

From this we obtain that for infinitely many positive integers n

$$a_{1,n+1}^{\frac{1}{S^{n+1}}} > \max_{j=1,\dots,n} a_{1,j}^{\frac{1}{S^j}}\,, \tag{21}$$

otherwise there exist n_0 such that for every positive integer n with $n > n_0$

$$a_{1,n+1}^{\frac{1}{S^{n+1}}} \le \max_{j=1,\dots,n} a_{1,j}^{\frac{1}{S^j}} = \max_{j=1,\dots,n_0} a_{1,j}^{\frac{1}{S^j}}\,,$$

which contradicts (20). Now inequality (21) implies that for infinitely many positive integers n

$$\begin{aligned} a_{1,n+1} &> \Big(\max_{j=1,\dots,n} a_{1,j}^{\frac{1}{S^j}}\Big)^{S^{n+1}} > \Big(\max_{j=1,\dots,n} a_{1,j}^{\frac{1}{S^j}}\Big)^{(S-1)(S^n+S^{n-1}+\dots+S)} \\ &= \left(\prod_{j=1}^{n}\Big(\max_{j=1,\dots,n} a_{1,j}^{\frac{1}{S^j}}\Big)^{S^j}\right)^{S-1} \ge \left(\prod_{j=1}^{n} a_j\right)^{S-1}. \end{aligned} \tag{22}$$

From (18), (19) and (22) we obtain that for infinitely many sufficiently large positive integers n

$$\left(\prod_{j=1}^{k}\prod_{i=1}^{n} a_{j,i}\right)^R \sum_{i=n+1}^{\infty} \frac{b_{k,i}}{a_{k,i}} < \frac{a_{1,n+1}^{\frac{(1+(k-1)\varepsilon_3)R}{S-1}}}{a_{1,n+1}^{\beta}} = a_{1,n+1}^{\frac{(1+(k-1)\varepsilon_3)R}{S-1}-\beta} < 1\,.$$

This and (16) imply (1) and the proof of Theorem 2.1 is complete. □

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