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AN ABSTRACT UNIFORM BOUNDEDNESS RESULT

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ABSTRACT. We prove a uniform boundedness result for spaces which have a family of projection operators satisfying certain properties. The result is used to show that the space of Pettis integrable functions is barrelled.

In [DFP1] Drewnowski, Florencio and Paúl established an abstract uniform boundedness result and used the result to establish the fact that the space of Pettis integrable functions with respect to a finite measure is a barrelled space. They later extended their result to an arbitrary measure in [DFP2]. Their proof is based on the fact that the space of Pettis integrable functions has a family of "good projections". The projections in this case, as well as other typical applications to function spaces, are multiplications by characteristic functions. In this section we establish a result similar to the ones in [DFP1] and [DFP2] and also use the result to establish the barrelledness of the space of Pettis integrable functions.

Throughout this section let E be a Hausdorff locally convex TVS and let $\mathcal{A}$ be an algebra of subsets of S. We assume that there exists a map $P\colon \mathcal{A} \to L(E)$, the space of continuous linear operators on E. We denote the value of P at $A \in \mathcal{A}$ by P_A; if $y' \in E'$, $x \in E$, let $y'Px$ be the set function $A \mapsto \langle y', P_A x\rangle$ from $\mathcal{A}$ into $\mathbb{R}$ and let $v(y'Px)$ denote the variation of $y'Px$. We assume that P satisfies the following additivity properties:

(i) $P_\phi = 0$, $P_S = I$,

(ii) P is finitely additive.

We consider the following additional properties for P:

(D) For every $y' \in E'$, $x \in E$, the finitely additive set function $y'Px$ satisfies the decomposition property:
for every $\varepsilon > 0$ there exists a partition $\{B_1, \dots, B_k\}$ of S with $B_i \in \mathcal{A}$ such that $v(y'P_{B_i}x) < \varepsilon$ for $i = 1, \dots, k$ (Rao and Rao refer to the decomposition property (D) as "strongly continuous" ([RR])).

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Remark 1.

(a) If $y'Px$ is bounded and non-atomic for every $y' \in E'$, $x \in E$ and $\mathcal{A}$ is a σ-algebra, then (D) is satisfied ([RR; 5.1.6]).

(b) If $\mathcal{A}$ is a σ-algebra and μ is a σ-finite, non-atomic measure on $\mathcal{A}$ such that $y'Px$ is μ-continuous (i.e., $\lim_{\mu(A)\to 0} y'P_A x = 0$), then (a) (and, therefore (D)) holds; in particular, if the E-valued set function $P_\bullet x$ is μ-continuous for every $x \in E$, then (a) holds.

We further consider a gliding hump property for P:

(GHP) If $\{A_j\}$ is a pairwise disjoint sequence from $\mathcal{A}$, $\{x_j\}$ is a null sequence from E and H is a countable $\sigma(E',E)$ bounded subset from E', then there is an increasing sequence $\{n_j\}$ such that the series $\sum_{j=1}^{\infty} P_{A_{n_j}} x_{n_j}$ is $\sigma(E,H)$ convergent to some $x \in E$.

Further, we say that P satisfies the *strong* (GHP) property if the series above converges in the original topology of E.

Remark 2. For example, let E be a complete metrizable space whose topology is generated by a quasi-norm $|\ |$, and suppose that $\{P_{A_j}\}$ is equicontinuous for every pairwise disjoint sequence $\{A_j\} \subset \mathcal{A}$. If $\{x_j\}$ is a null sequence in E, then $P_{A_j} x_j \to 0$ in E so there is a subsequence such that $\sum P_{A_{n_j}} x_{n_j}$ converges in E. Hence, strong (GHP) is satisfied.

For an example where (GHP) is satisfied but strong (GHP) is not, let ℓ^∞ be equipped with $\sigma(\ell^\infty, ba)$. Let $\mathcal{A}$ be the power set of $\mathbb{N}$ and for $A \in \mathcal{A}$ define $P_A\colon \ell^\infty \to \ell^\infty$ by $P_A x = C_A x$ for $x \in \ell^\infty$ where C_A is the characteristic function of A and $C_A x$ is the coordinatewise product of C_A and x. Then (i) and (ii) are satisfied. Then $\mathrm{e}^j \to 0$ in $\sigma(\ell^\infty, ba)$ but no subseries $\sum \mathrm{e}^{n_j}$ is $\sigma(\ell^\infty, ba)$ convergent so strong (GHP) is not satisfied. However, let $x^k \to 0$ in $\sigma(\ell^\infty, ba)$, let $\{A_k\} \subset \mathcal{A}$ be pairwise disjoint and let $\{\nu_j\} \subset ba$. By Drewnowski's Lemma ([D], [DU; I.6]) there is a subsequence $\{A_{n_j}\}$ such that each ν_i is countably additive on the σ-algebra Σ generated by $\{A_{n_j}\}$. Since $x^k \to 0$ in $\sigma(\ell^\infty, ba)$, there exists $M > 0$ such that $\|x^k\|_\infty \le M$. Let x be the coordinatewise sum of $\sum_{k=1}^{\infty} C_{A_{n_k}} x^{n_k}$. We claim that $x = \sum_{k=1}^{\infty} C_{A_{n_k}} x^{n_k}$ in the topology $\sigma\big(\ell^\infty, \{\nu_j\}\big)$. This follows since for every j,
$$\left|\left\langle \nu_j, x - \sum_{k=1}^{i} C_{A_{n_k}} x^{n_k} \right\rangle\right| = \left| \int_{\mathbb{N}} \sum_{k=i+1}^{\infty} C_{A_{n_k}} x^{n_k} \,\mathrm{d}\nu_j \right| \le M|\nu_j|\Big(\bigcup_{k=i+1}^{\infty} A_{n_k} \Big) \to 0 \text{ as}$$
$i \to \infty$ by the countable additivity of ν_j on Σ. Hence, (GHP) is satisfied.

We give further examples of spaces, including the space of Pettis integrable functions, satisfying (GHP) later.

THEOREM 3. *Assume that P satisfies* (D) *and* (GHP). *If $B \subset E'$ is $\sigma(E', E)$ bounded, then B is $\beta(E', E)$ (strongly) bounded, i.e., E is a Banach-Mackey space* ([Wi; 10.4]).

Proof. Suppose the conclusion fails. Then there exists a null sequence $\{x_j\}$ in E such that

$$\sup\{|\langle y', x_j\rangle| : y' \in B, \ j \in \mathbb{N}\} = \infty .$$

Pick $y'_1 \in B$, n_1 such that $|\langle y'_1, x_{n_1}\rangle| = |y'_1 P_S x_{n_1}| > 2$. From (D), there is a partition $\{B_1, \dots, B_k\}$ of S such that $v(y'_1 P_{B_i} x_{n_1}) < 1$ for $i = 1, \dots, k$. From (ii) and the $\sigma(E', E)$ boundedness of B, we may assume that

$$\sup\{|y' P_{B_1} x_j| : y' \in B, \ j > n_1\} = \infty .$$

Set $A_1 = S \setminus B_1$ and note from (ii) that $|y'_1 P_{A_1} x_{n_1}| > 1$.

Now if we treat B_1 as S was treated above there exist a partition (A_2, B_2) of B_1, $y'_2 \in B$ and $n_2 > n_1$ such that $\sup\{|y' P_{B_2} x_j| : y' \in B, \ j > n_2\} = \infty$ and $|y'_2 P_{A_2} x_{n_2}| > 2$. Continuing this construction produces a pairwise disjoint sequence $\{A_j\}$ from $\mathcal{A}$, $\{y'_j\} \subset B$ and a subsequence $\{x_{n_j}\}$ such that $|y'_j P_{A_j} x_{n_j}| > j$ for every j.

Now consider the matrix $M = [m_{ij}] = \left[\frac{1}{i} y'_i P_{A_j} x_{n_j}\right]$. The columns of M converge to 0 by the $\sigma(E', E)$ boundedness of B. Given any increasing sequence $\{r_j\}$ by (GHP) there is a subsequence $\{s_j\}$ such that the series $\sum_{j=1}^{\infty} P_{A_{s_j}} x_{n_{s_j}}$ is $\sigma(E, \{y'_i\})$ convergent to some $x \in E$. Hence, $\sum_{j=1}^{\infty} m_{is_j} = \langle \frac{1}{i} y'_i, x\rangle \to 0$ and M is a $\mathcal{K}$-matrix ([AS; §2]). By the Antosik-Mikusinski Theorem ([AS; 2.2]) the diagonal of M converges to 0 contradicting the construction above. □

We now give two examples which point out the importance of conditions (D) and (GHP).

EXAMPLE 4. Let E be an arbitrary Hausdorff locally convex *TVS*. Let $\mathcal{P}$ be the power set of $\mathbb{N}$ and define $P \colon \mathcal{P} \to L(E)$ by $P_A = I$ if $1 \in A$ and $P_A = 0$ if $1 \notin A$ (P is an operator version of the Dirac measure at 1). Then $y'Px$ is a "Dirac measure" with mass $\langle y', x\rangle$ at 1. So property (D) clearly fails to hold. Note, however, that conditions (i), (ii) and (GHP) do hold. Thus, if we take E to be any space which is not a Banach-Mackey space, Theorem 3 will fail.

EXAMPLE 5. Let $\mathcal{B}$ be the Borel sets in $[0,1]$, and let E be the space of all $\mathcal{B}$-simple functions equipped with the L^2-norm with respect to Lebesgue measure. For $A \in \mathcal{B}$ let P_A be the projection defined by $P_A f = C_A f$. Since fPg is non-atomic for every $f, g \in L^2[0,1]$, condition (D) is satisfied (as well as (i) and (ii)). However, condition (GHP) fails (take any pairwise disjoint sequence of Borel sets $\{A_j\}$ with positive measure and set $f_j = \frac{1}{j} C_{A_j}$). Clearly E is not a Banach-Mackey space.

Drewnowski, Florencio and Paúl proved results analogous to Theorem 3 in [DFP1] and [DFP2]. They assume that $\mathcal{A}$ is a σ-algebra and the map $P\colon \mathcal{A} \to L(E)$ is projection-valued. The conditions imposed on the map P are quite different from those in (D) and (GHP). In particular, in [DFP2] they assume that $P(\mathcal{A})$ is equicontinuous. On the other hand, they do not require any condition analogous to condition (D). Condition (D) effectively limits the applications of Theorem 3 to non-atomic measures. Condition (GHP) can be viewed as a continuous version of a gliding hump property for sequence spaces, called the zero gliding hump property (see [LS]).

We now give an application of Theorem 3 to the space of Pettis integrable functions. Let X be a Banach space, let Σ be a σ-algebra of subsets of S with μ a measure on Σ. A function $f\colon S \to X$ is said to be weakly measurable if $x'f$ is measurable for every $x' \in X'$ and is said to be weakly μ-integrable if $x'f$ is μ-integrable for every $x' \in X'$. If f is weakly μ-integrable, then for every $A \in \Sigma$, $x' \mapsto \int\limits_A x'f \, d\mu$ defines a continuous linear functional $x''_A \in X''$ (apply the Closed Graph Theorem to show the linear map $F\colon X' \to L^1(\mu)$, $F(x') = x'f$, is continuous and then observe that $x''_A = F'(C_A)$). The element x''_A is sometimes called the Gelfand or Dunford integral of f over A and is denoted by $\int\limits_A f \, d\mu$. Let $\mathcal{G}^1(\mu, X)$ be the space of all Gelfand integrable functions; equip $\mathcal{G}^1(\mu, X)$ with the semi-norm $\|f\|_1 = \sup\Big\{ \int\limits_S |x'f| \, d\mu : \|x'\| \le 1 \Big\}$ (this quantity is finite by the continuity of the map F defined above). A function f is said to be Pettis integrable if f is Gelfand integrable and $\int\limits_A f \, d\mu \in X$ for every $A \in \Sigma$. Let $\mathcal{P}^1(\mu, X)$ be the space of all Pettis integrable functions equipped with the semi-norm $\| \ \|_1$. (See [DU] or [HP] for discussions of these integrals.) It is known that, in general, $\mathcal{P}^1(\mu, X)$ is not complete ([Pe; 9.4]). However, using Theorem 3 we show that $\mathcal{P}^1(\mu, X)$ is barrelled if μ is σ-finite and non-atomic.

For $A \in \Sigma$ define a projection P_A on $\mathcal{P}^1(\mu, X)$ by $P_A f = C_A f$. The map P obviously satisfies conditions (i) and (ii) above. We first show that P satisfies the strong (GHP) condition.

THEOREM 6. *$\mathcal{P}^1(\mu, X)$ satisfies the strong* (GHP) *property.*

Proof. Let $\{A_j\}$ be a pairwise disjoint sequence from Σ and let $\{f_j\}$ converge to 0 in $\mathcal{P}^1(\mu, X)$. Pick a subsequence $\{n_j\}$ satisfying $\|f_{n_j}\|_1 < 1/2^j$. Let f be the pointwise sum of the series $\sum\limits_{j=1}^{\infty} C_{A_{n_j}} f_{n_j}$; f is obviously weakly μ-measurable.

If $x' \in X'$, then $x'f = \sum_{j=1}^{\infty} C_{A_{n_j}} x'f_{n_j}$ and $|x'f| = \sum_{j=1}^{\infty} C_{A_{n_j}} |x'f_{n_j}|$ pointwise, and $\int_S |x'f| \, d\mu = \sum_{j=1}^{\infty} \int_{A_{n_j}} |x'f_{n_j}| \, d\mu \le \|x'\| \sum_{j=1}^{\infty} \|f_{n_j}\|_1 < \infty$ implies that f is weakly μ-integrable.

We next claim that $\int_A f \, d\mu \in X$ for $A \in \Sigma$. Since $\sum_{j=1}^{\infty} \left\| \int_{A_{n_j} \cap A} f_{n_j} \, d\mu \right\| \le \sum_{j=1}^{\infty} \|f_{n_j}\|_1 < \infty$, the series $\sum_{j=1}^{\infty} \int_{A_{n_j} \cap A} f_{n_j} \, d\mu$ converges to some $x_A \in X$ by the completeness of X. Therefore, $\langle x', x_A \rangle = \sum_{j=1}^{\infty} \int_{A_{n_j} \cap A} x'f_{n_j} \, d\mu$. Since $|x'f| \ge \left| \sum_{j=1}^{n} C_{A_{n_j}} x'f_{n_j} \right|$ for every n, the Dominated Convergence Theorem implies that $\int_A x'f \, d\mu = \sum_{j=1}^{\infty} \int_{A_{n_j} \cap A} x'f_{n_j} \, d\mu$. Hence, $\int_A f \, d\mu = x_A \in X$ as desired, and f is Pettis μ-integrable.

Last, we claim that the series $\sum_{j=1}^{\infty} C_{A_{n_j}} f_{n_j}$ converges to f is the norm of $\mathcal{P}^1(\mu, X)$. This follows from

$$\begin{aligned}
\left\| f - \sum_{j=1}^{n} C_{A_{n_j}} f_{n_j} \right\|_1 &= \sup \left\{ \int_S \left| x' \left(\sum_{j=n+1}^{\infty} C_{A_{n_j}} f_{n_j} \right) \right| d\mu : \|x'\| \le 1 \right\} \\
&\le \sup \left\{ \sum_{j=n+1}^{\infty} \int_{A_{n_j}} |x'f_{n_j}| \, d\mu : \|x'\| \le 1 \right\} \\
&\le \sum_{j=n+1}^{\infty} \|f_{n_j}\|_1 \to 0 .
\end{aligned}$$

□

The proof of Theorem 6 also establishes:

COROLLARY 7. *The subspace of $\mathcal{P}^1(\mu, X)$ consisting of the strongly μ-measurable (countably valued) functions satisfies the strong* (GHP) *property.*

We next consider property (D).

PROPOSITION 8. *If μ is σ-finite and non-atomic, then $\mathcal{P}^1(\mu, X)$ satisfies* (D).

P r o o f. Fix $f \in \mathcal{P}^1(\mu, X)$. The indefinite Pettis integral of f is μ-continuous, i.e.,

$$\lim_{\mu(A)\to 0} \left\| \int_A f \, \mathrm{d}\mu \right\| = \lim_{\mu(A)\to 0} \sup\left\{ \left| \int_A x' f \, \mathrm{d}\mu \right| : \|x'\| \le 1 \right\} = 0$$

([Pe], [HP]) so by [DS; III.1.5 and III.2.15],

$$\lim_{\mu(A)\to 0} \sup\left\{ \int_A |x'f| \, \mathrm{d}\mu : \|x'\| \le 1 \right\} = \lim_{\mu(A)\to 0} \|C_A f\|_1 = 0 \,.$$

The result now follows from Remark 1(b). □

From Proposition 8, Corollary 7 and Theorem 3, we obtain:

COROLLARY 9. *If μ is a σ-finite, non-atomic measure, then $\mathcal{P}^1(\mu, X)$ and the subspace of strongly measurable (countably valued) functions is barrelled.*

D r e w n o w s k i, F l o r e n c i o and P a ú l generalize Corollary 9 to arbitrary (σ-finite) measures by decomposing the measure into its non-atomic and purely atomic parts (see [DFP1] or [DFP2]).

We can also use Theorem 3 to establish an interesting barrelledness result for the space of Bochner integrable functions. We refer the reader to [DU], [HP] for the basic properties of the Bochner integrable which we employ. Let Y be a subspace of X and let $L^1(\mu, X)$ be the space of X-valued Bochner μ-integrable functions equipped with the norm $\|f\| = \int_S \|f(\cdot)\| \, \mathrm{d}\mu$. Let $L^1(\mu, Y)$ be the subspace of $L^1(\mu, X)$ consisting of the Y-valued functions. For $A \in \Sigma$ let P_A be the projection on $L^1(\mu, X)$ defined by $P_A f = C_A f$. The function P obviously satisfies conditions (i) and (ii). We consider conditions (D) and (GHP).

PROPOSITION 10. *$L^1(\mu, Y)$ satisfies the strong* (GHP) *property.*

P r o o f. Let $\{f_k\}$ be a null sequence in $L^1(\mu, Y)$ and let $\{A_k\}$ be a pairwise disjoint sequence from Σ. Pick a subsequence $\{n_k\}$ such that $\|f_{n_k}\| < 1/2^k$. Let f be the pointwise limit of the series $\sum_{k=1}^{\infty} C_{A_{n_k}} f_{n_k}$. Then f is clearly strongly measurable, Y-valued, and since $\int_S \|f(\cdot)\| \, \mathrm{d}\mu = \sum_{k=1}^{\infty} \int_{A_{n_k}} \|f_{n_k}(\cdot)\| \, \mathrm{d}\mu \le \sum_{k=1}^{\infty} 1/2^k$, the series converges to f in $L^1(\mu, Y)$. □

PROPOSITION 11. *If μ is σ-finite and non-atomic, then $L^1(\mu, Y)$ satisfies* (D).

P r o o f. For $f \in L^1(\mu, Y)$, the map $A \mapsto C_A f$ is μ-continuous so the result follows from Remark 1(b). □

We thus have

COROLLARY 12. *If μ is σ-finite and non-atomic, then $L^1(\mu, Y)$ is barrelled.*

This is an interesting result in the sense that even though Y may not be barrelled the space $L^1(\mu, Y)$ is barrelled. More general results are given in [DFP1] and [DFP2].

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