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FRACTAL DIMENSION OF SETS INDUCED BY BASES OF IMAGINARY QUADRATIC FIELDS

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ABSTRACT. In an imaginary quadratic number field certain bases form a canonical number system. These bases induce a tiling of the algebraic numbers of this field. Each piece of the tiling consists of all numbers with fixed integer part. We map this tiling onto the two dimensional real vector space and determine the fractal dimension of the boundary of its pieces.

1. Introduction

Let $\mathbb{K} = \mathbb{Q}(\theta)$ be a number field of degree d and $\mathbb{Z}_{\mathbb{K}}$ the ring of its algebraic integers. Let $N(\theta)$ be the norm of θ, $\alpha \in \mathbb{Z}_{\mathbb{K}}$ and $\mathcal{N} = \{0, 1, \dots |N(\alpha)| - 1\}$. Then we call the pair $\{\alpha, \mathcal{N}\}$ a canonical number system, if any $\gamma \in \mathbb{Z}_{\mathbb{K}}$ has a unique representation

$$\gamma = c_0 + c_1\alpha + \dots + c_h\alpha^h; \qquad c_j \in \mathcal{N} \ (j = 0, \dots, h), \quad c_h \neq 0 \text{ if } h \neq 0.$$

α is called the base of this canonical number system and $\mathcal{N}$ is the set of its digits (cf. Kovács–Pethő [8]).

A number field $\mathbb{Q}(\theta)$ is called a canonical number field, if there exists a base $\alpha \in \mathbb{Z}_{\mathbb{K}}$ such that $\{\alpha, \mathcal{N}\}$ forms a canonical number system (cf. [7]). In particular any quadratic field is a canonical number field (bases can be computed explicitly (cf. [4], [5])).

In the ring of the Gaussian integers $\mathbb{Z}[\mathrm{i}]$ the bases of canonical number systems are given by $b = -n \pm \mathrm{i}$ (cf. [6]). Each of these bases gives rise to a tiling of the plane. One piece of this tiling consists of all complex numbers with a fixed integral part in their b-adic expansion. The tile with integer part zero is called the fundamental region. It is defined by

$$\mathcal{F} = \left\{ z \mid z = \sum_{j=-\infty}^{-1} c_j b^j \right\}$$

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where c_j are arbitrary b-adic digits. The boundaries of these pieces have fractal dimension. Their fractal dimension was computed to be

$$H_n = \frac{2\log\lambda_n}{\log(n^2+1)}, \tag{1.1}$$

where λ_n is the positive solution of

$$\lambda^3 - (2n-1)\lambda^2 - (n-1)^2\lambda - (n^2+1) = 0$$

by W. J. Gilbert in [2]. In Figure 1 one can see the fundamental region of the base $b = -1 + \mathrm{i}$, whose boundary has the fractal dimension $1.5236\ldots$ by Gilbert's calculations.

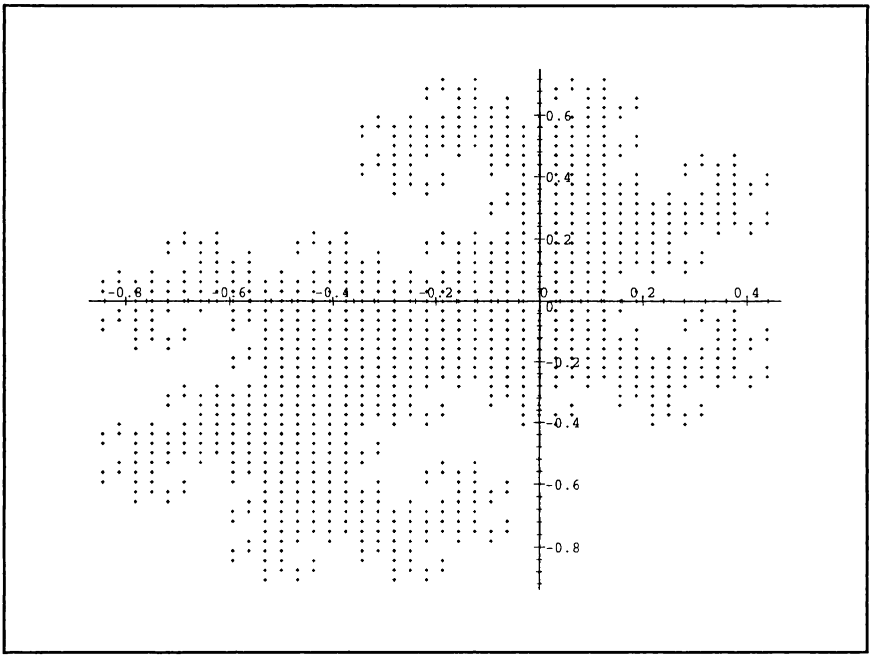

FIGURE 1.

We will extend this result to imaginary quadratic fields. The bases of canonical number systems in quadratic fields were characterized by I. Kátai and B. Kovács in [4] and [5]. In particular for imaginary quadratic fields $\mathbb{Q}(\mathrm{i}\sqrt{D})$ (D squarefree) we have (cf. [5])

$$\begin{aligned} &b = -A \pm \mathrm{i}\sqrt{D}, && -D \not\equiv 1, \quad 0 < 2A \le A^2 + D \ge 2 \qquad \text{or} \\ &b = \frac{1}{2}\left(-B \pm \mathrm{i}\sqrt{D}\right), && -D \equiv 1, \quad B \equiv 1(2), \quad 0 < B \le \frac{1}{4}(B^2 + D) \ge 2. \end{aligned} \tag{1.2}$$

The fundamental regions of these number systems and the tilings they induce are defined in the same way as in the Gaussian case. In the same way as discussed in [1; pp. 112–115] we map the numbers of the number field into the two dimensional real vector space $\mathbb{R}^2$. This mapping maps the number $\beta = a + b\,\mathrm{i}\sqrt{D}$ to $\left(a, b\sqrt{D}\right)$. In Figure 2 the fundamental region for $b = -2 + \sqrt{2}\,\mathrm{i}$ is shown. Similar results were obtained recently by K á t a i (cf. [3]) for another class of number systems.

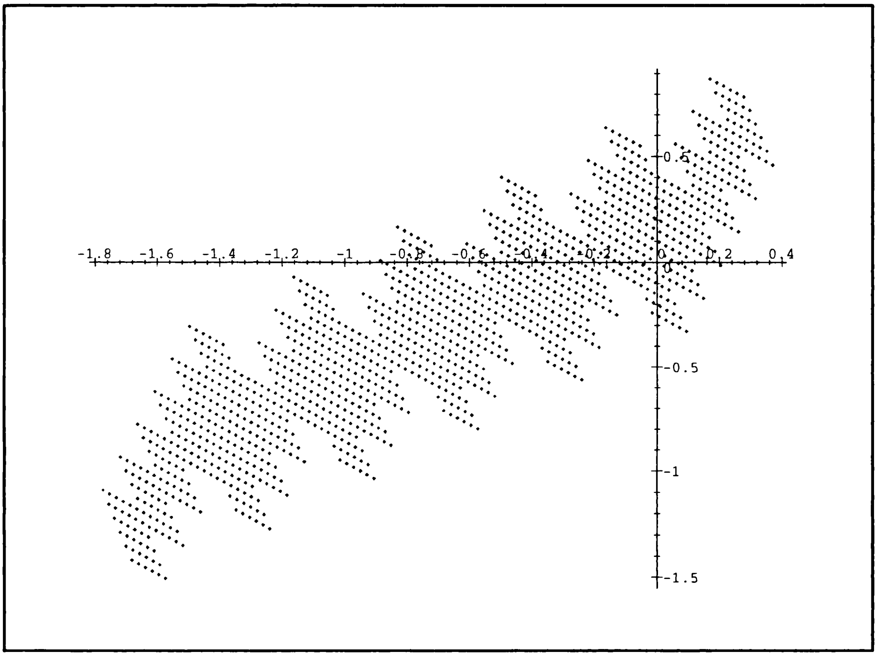

FIGURE 2.

2. Approximation of the boundary length

We will now construct the tiling induced by one of the bases b given in (1.2). An approximation of a tile will consist of the set of points with given integral part and a fixed number of negative powers. We divide up the plane into rectangles, whose centers are the algebraic integers of the field in discussion. The size of the rectangles is $1 \times \sqrt{D}$ for all the bases.

Let $N(b)$ be the norm of the base b. The kth approximation of the tiles consists of rectangles, whose side lengths are multiplied by $\left(N(b)\right)^{\frac{k}{2}}$ and whose centers are numbers of the form $(r_q r_{q-1} \cdots r_0 . r_{-1} \cdots r_{-k})_b$, where the integral part is fixed. If k tends to infinity, we get the desired tiling. The area of each tiling remains constant during the approximation process. However, as we shall see later, the length of the boundary increases towards infinity.

First we determine a formula for the length of the boundary of the kth approximation of a tile. Let $\mathcal{G}_k$ be the union of all rectangles of size $1 \times \sqrt{D}$ whose center is an algebraic integer with an expansion not exceeding k digits. The length of the boundary of the kth approximation of a tile is then exactly $(N(b))^{-\frac{k}{2}}$ times the length of the boundary of $\mathcal{G}_k$. So we can confine ourselves to the study of $\mathcal{G}_k$. The tile $\mathcal{G}_k$ consists of rectangles of length $N(b)$ and height $\sqrt{D}$. These rectangles cover the plane in a way as shown in Figure 3 for a special case.

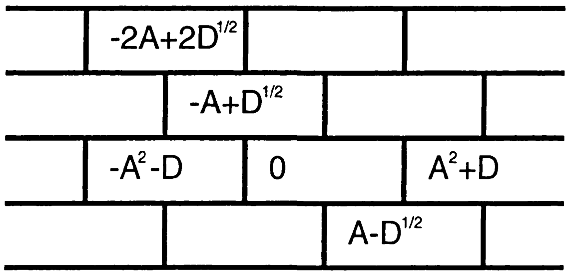

FIGURE 3.

The framework is constructed as shown in Figure 4. The basic edges, we call them X, Y and Z, will be used in the following construction. X and Y are the lengths of the two parts of the horizontal edge of the rectangle. Z is the length of the vertical one. Figure 4 shows the parts for a special case. Next we investigate, how these basic edges change from one approximation step to the next. It is clear that each of the rectangles in Figure 3 changes to a “staircase” consisting of $N(b)$ rectangular “steps”. Our next question is to determine how the boundary of this staircase emerges from the edges X, Y and Z. Since each staircase is self-similar to all the others it suffices to consider only one of them. We take the one containing 0 and call it $\mathcal{S}$. To be able to count the steps arising from X and Y, respectively, we need the b-adic expansion of -1. It is given by

$$-1 = b^2 + 2Ab + \big(N(b) - 1\big) \tag{2.1}$$

if $D \not\equiv -1(4)$. If $D \equiv -1(4)$ one obtains

$$-1 = b^2 + Bb + \big(N(b) - 1\big) . \tag{2.2}$$

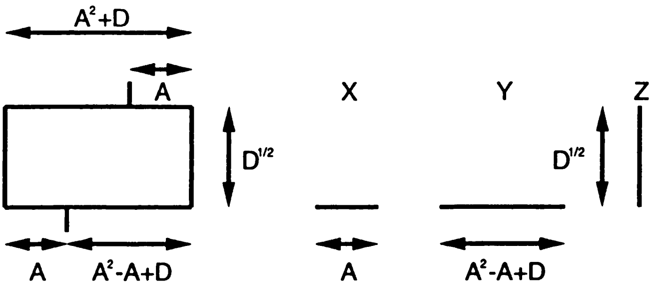

FIGURE 4.

From these representations we get information about the position of the staircases in the neighbourhood of $\mathcal{S}$ relative to $\mathcal{S}$. Note that the coefficient of b in (2.1) and (2.2) determines the number of steps that emerge from X. Since the total number of steps is $N(b)$ we know the number of the remaining steps emerging from Y. Table 2 shows, how the $k+1$st step of approximation of the boundary of the fundamental region emerges from the kth step.

kth approximation	$k+1$st approximation (1st case)
X	$(2A-1)\times X+2A\times Z$
Y	$(N(b)-2A+1)\times X+(N(b)-2A)\times Z$
Z	Y

kth approximation	$k+1$st approximation (2nd case)
X	$(B-1)\times X+B\times Y$
Y	$(N(b)-B+1)\times X+(N(b)-B)\times Z$
Z	Y

TABLE 1.

Remark 2.1. The two cases are related to the different expansions of -1 stated in (2.1) and (2.2).

With the help of Table 1 we are able to establish a matrix recurrence that gives us the number of X, Y and Z edges after $k+1$ steps:

$$\begin{pmatrix} a_{k+1} \\ b_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} 2A-1 & 0 & 2A \\ N(b)-2A+1 & 0 & N(b)-2A \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}. \tag{2.3}$$

We refer to the matrix occuring in (2.3) by T_b, to the vectors by $v_b^{(k)}$. In the second case one obtains a similar matrix. The general equation is

$$v_b^{(k+1)} = T_b v_b^{(k)} . \tag{2.4}$$

The initial value v_1 is given by $(2,2,2)^T$ because the first step of the approximation is a rectangle containing two pieces of every edge type. Then the length of the boundary of $\mathcal{G}_k$ is given by

$$g_{k,1} = \begin{pmatrix} A \\ N(b) - A \\ \sqrt{-D} \end{pmatrix} v_b^{(k)} = \begin{pmatrix} A \\ N(b) - A \\ \sqrt{-D} \end{pmatrix} T_b^{k-1} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \tag{2.5}$$

in the case $D \not\equiv 1(4)$. For the case $D \equiv 1(4)$ $g_{k,2}$ is defined in the same way. From this it follows from the definition of $\mathcal{G}_k$ that the length of the boundary of the kth approximation of a tile is given by

$$s_{k,i} = N(b)^{-\frac{k}{2}} g_{k,i} \qquad (i = 1,2). \tag{2.6}$$

Let $p_b(\lambda)$ be the characteristic polynomial of T_b. One easily verifies (for example by inserting the value $\sqrt{N(b)}$ in $p_b(\lambda)$) that there exists a positive eigenvalue λ_b with $\lambda_b > \sqrt{N(b)}$. Combining this with (2.5) and (2.6) one gets

$$\lim_{k\to\infty} s_{k,i} = \infty \qquad (i = 1,2).$$

3. The fractal dimension of the boundary

We are now in a position to give the fractal dimension of the boundary of the tiles induced by bases of imaginary quadratic number fields. It turns out to be the natural generalization of Gilbert's result (cf. [2]):

THEOREM 3.1. *Let b be the base of a canonical number system of an imaginary quadratic field. $p_b(\lambda)$ shall be the characteristic polynomial of the matrix T_b occuring in (2.4). Let λ_b be the dominant eigenvalue of $p_b(\lambda)$. Then the fractal dimension of the boundary of a tile consisting of all complex numbers having the same integral part in their b-adic expansion is given by*

$$H_b = \frac{2\log\lambda_b}{\log N(b)} . \tag{3.1}$$

H_b is greater than 1 for all bases b.

Remark 3.1. Of course it is possible to calculate $p_b(\lambda)$ explicitly from T_b. For bases of the shape $b = -A \pm \mathrm{i}\sqrt{D}$ we have

$$p_b(\lambda) = \lambda^3 - (2A-1)\lambda^2 - \big(N(b) - 2A\big)\lambda - N(b) . \tag{3.2}$$

For the case $b = \frac{1}{2}\left(-B \pm \mathrm{i}\, D^{\frac{1}{2}}\right)$ we obtain

$$p_b(\lambda) = \lambda^3 - (B-1)\lambda^2 - \big(N(b) - B\big)\lambda - N(b)\,. \tag{3.3}$$

Proof of the theorem. As mentioned above, $p_b(\lambda)$ has a positive dominant eigenvalue $\lambda_b > \sqrt{N(b)}$. From this it follows that $H_b > 1$. Now we have to prove that H_b is the fractal dimension of the boundary $\mathcal{B}$ of a piece of our tiling. The d-dimensional measure is a constant times $\lim\limits_{k\to\infty} \lambda_b^k N(b)^{-\frac{kd}{2}}$. It follows that if $\lambda_b N(b)^{-\frac{d}{2}} > 1$ this measure will be infinite, while for $\lambda_b N(b)^{-\frac{d}{2}} < 1$ it will be zero. So the fractal dimension of $\mathcal{B}$ is given by a quantity $\tilde{d}$ with

$$N(b)^{\frac{\tilde{d}}{2}} = \lambda_b\,.$$

Solving this equation for $\tilde{d}$ we get $\tilde{d} = H_b$ and the theorem is proved. □

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