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ABSOLUTE COMPARISON THEOREMS FOR DOUBLE WEIGHTED MEAN AND DOUBLE CESÀRO MEANS

B. E. RHOADES

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ABSTRACT. In a recent paper [SARIGÖL, M. A.—BOR, H.: *On two summability methods*, Math. Slovaca **43** (1993), 317–325] the authors showed that the Cesàro means of order α are absolutely k-stronger than weighted means satisfying the condition $P_n = O(n^\alpha p_n)$, $0 < \alpha < 1$. It is the purpose of this paper to extend this result to double summability.

Let $\{s_{jk}\}$ denote a double sequence. The mn-term of the $(\overline{N}, p_{ij})$-transform of the sequence $\{s_{jk}\}$ is defined by

$$T_{mn} := \frac{1}{P_{mn}} \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij} s_{ij},$$

where

$$P_{mn} := \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij}.$$

The mn-term of the (C, α, β)-transform of a sequence $\{s_{jk}\}$ is defined by

$$\sigma_{mn}^{\alpha\beta} := \frac{1}{E_m^\alpha E_n^\beta} \sum_{i=0}^{m} \sum_{j=0}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} s_{ij},$$

where

$$E_n^\alpha := \binom{n+\alpha}{\alpha}.$$

A double sequence $\{p_{ij}\}$ is factorable if there exist single sequences $\{p_i\}$ and $\{q_j\}$ such that $p_{ij} = p_i q_j$. We restrict our attention to weighted mean methods

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generated by factorable sequences, since it was shown in [1] that the condition of being factorable is necessary in order to find the inverse of the transform. For any double sequence $\{u_{ij}\}$, $\Delta_{10}u_{ij} := u_{ij} - u_{i+1,j}$, $\Delta_{01}u_{ij} := u_{ij} - u_{i,j+1}$, and $\Delta_{11}u_{ij} := u_{ij} - u_{i,j+1} - u_{i+1,j} + u_{i+1,j+1}$.

THEOREM 1. *Let $0 < \alpha, \beta < 1$. If $\{p_{ij}\}$ is factorable, nondecreasing, and*

$$\frac{P_{mn}}{p_{mn}(m+1)^{\alpha}(n+1)^{\beta}} = O(1), \tag{1}$$

then $|\overline{N}, p_{ij}|_k$ summability implies $|C, \alpha, \beta|_k$ summability, $k \geq 1$.

Proof. Since $\{p_{ij}\}$ is factorable, we shall assume that we may write p_{ij} in the form $p_i q_j$, where $\{p_i\}$ and $\{q_j\}$ are positive nondecreasing sequences satisfying the conditions of the theorem. Then

$$T_{mn} = \frac{1}{P_m Q_n} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j s_{ij}. \tag{2}$$

Let $\{s_{ij}\}$ be absolutely k-summable by the weighted mean method defined by (2). This means that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (mn)^{k-1} |\Delta_{11} T_{mn}|^k < \infty .$$

We shall now use (1) to obtain explicit expressions for the a_{ij} in terms of the T_{ij}. Using (2) with $m = 0$ we obtain

$$T_{0n} = \frac{1}{Q_n} \sum_{j=0}^{n} q_j s_{0j}.$$

$$Q_n T_{0n} - Q_{n-1} T_{0,n-1} = q_n s_{0n}. \tag{3}$$

Using (2) with $m = 1$, $n > 1$, yields

$$T_{1n} = \frac{1}{P_1 Q_n} \sum_{i=0}^{1} \sum_{j=0}^{n} p_i q_j s_{ij}.$$

$$P_1(Q_n T_{1n} - Q_{n-1} T_{1,n-1}) = \sum_{i=0}^{1} p_i q_n s_{in}.$$

$$\frac{P_1(Q_n T_{1n} - Q_{n-1} T_{1,n-1})}{q_n} = p_0 s_{0n} + p_1 s_{1n} = (p_0 + p_1) s_{0n} + p_1 \sum_{k=0}^{n} a_{0k}.$$

Using (3),

$$p_1 \sum_{k=0}^{n} a_{1k} = \frac{P_1(Q_nT_{1n} - Q_{n-1}T_{1,n-1})}{q_n} - \frac{P_1(Q_nT_{0n} - Q_{n-1}T_{0,n-1})}{q_n}$$
$$= P_1\left[\frac{Q_n}{q_n}(T_{1n} - T_{1,n-1}) + T_{1,n-1} - \frac{Q_n}{q_n}(T_{0n} - T_{0,n-1}) - T_{0,n-1}\right]$$
$$= P_1\left[\frac{Q_n}{q_n}\Delta_{11}T_{0,n-1} - \Delta_{10}T_{0,n-1}\right].$$

Thus

$$\begin{aligned} a_{1n} &= \frac{P_1}{p_1}\left[\frac{Q_n}{q_n}\Delta_{11}T_{0,n-1} - \Delta_{10}T_{0,n-1} - \frac{Q_{n-1}}{q_{n-1}}\Delta_{11}T_{0,n-2} + \Delta_{10}T_{0,n-2}\right] \\ &= \frac{P_1}{p_1}\left[\frac{Q_n}{q_n}\Delta_{11}T_{0,n-1} - \frac{Q_{n-1}}{q_{n-1}}\Delta_{11}T_{0,n-2} + \Delta_{11}T_{0,n-2}\right]. \end{aligned} \tag{4}$$

Similarly, for $m > 1$,

$$a_{m1} = \frac{Q_1}{q_1}\left[\frac{P_m}{p_m}\Delta_{11}T_{m-1,0} - \frac{P_{m-1}}{p_{m-1}}\Delta_{11}T_{m-2,0} + \Delta_{11}T_{m-2,0}\right]. \tag{5}$$

Using (2) for $m, n > 1$,

$$P_mQ_nT_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} p_iq_js_{ij},$$

$$P_mQ_nT_{mn} - P_{m-1}Q_nT_{m-1,n} = \sum_{j=0}^{n} p_mq_js_{mj},$$

$$P_mQ_{n-1}T_{m,n-1} - P_{m-1}Q_{n-1}T_{m-1,n-1} = \sum_{j=0}^{n-1} p_mq_js_{mj},$$

and hence

$$P_mQ_nT_{mn} - P_{m-1}Q_nT_{m-1,n} - P_mQ_{n-1}T_{m,n-1} + P_{m-1}Q_{n-1}T_{m-1,n-1} = p_mq_ns_{mn}.$$

$$\begin{aligned} s_{mn} &= \frac{P_mQ_n}{p_mq_n}T_{mn} - \left(\frac{P_m - p_m}{p_m}\right)\frac{Q_n}{q_n}T_{m-1,n} - \left(\frac{Q_n - q_n}{q_n}\right)\frac{P_m}{p_m}T_{m,n-1} \\ &\quad + \left(\frac{P_m - p_m}{p_m}\right)\left(\frac{Q_n - q_n}{q_n}\right)T_{m-1,n-1} \\ &= \frac{P_mQ_n}{p_mq_n}\Delta_{11}T_{m-1,n-1} - \frac{Q_n}{q_n}\Delta_{01}T_{m-1,n-1} - \frac{P_m}{p_m}\Delta_{10}T_{m-1,n-1} + T_{m-1,n-1}. \end{aligned}$$

$$
\begin{aligned}
a_{mn} = \Delta_{11}\left(\frac{P_{m-1}Q_{n-1}}{p_{m-1}q_{n-1}}\Delta_{11}T_{m-2,n-2}\right) &+ \frac{Q_n}{q_n}\Delta_{11}T_{m-2,n-1} \\
&-\frac{Q_{n-1}}{q_{n-1}}\Delta_{11}T_{m-2,n-2} + \frac{P_m}{p_m}\Delta_{11}T_{m-1,n-2} \\
&-\frac{P_{m-1}}{p_{m-1}}\Delta_{11}T_{m-2,n-2} + \Delta_{11}T_{m-2,n-2}
\end{aligned} \tag{6}
$$

In a similar manner it can be shown that

$$
a_{11} = \frac{P_1 Q_1}{p_1 q_1}\Delta_{11}T_{00}\,. \tag{7}
$$

Let $t_{mn}^{\alpha\beta}$ denote the mn-term of the (C, α, β)-transform in terms of $\{mna_{mn}\}$ where a_{mn} is expressed in terms of T_{mn}. Then, to prove the comparison, it will be sufficient to show that

$$
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}|t_{mn}^{\alpha\beta}|^k < \infty\,.
$$

Using (4) – (7),

$$
\begin{aligned}
t_{mn}^{\alpha\beta} &:= \frac{1}{E_m^\alpha E_n^\beta}\sum_{i=0}^{m}\sum_{j=0}^{n}E_{m-1}^{\alpha-1}E_{n-j}^{\beta-1}ija_{ij} \\
&= \frac{1}{E_m^\alpha E_n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}ija_{ij} \\
&= \frac{1}{E_m^\alpha E_n^\beta}\left[\sum_{j=1}^{n}E_{m-1}^{\alpha-1}E_{n-j}^{\beta-1}ja_{1j} + \sum_{i=2}^{m}\sum_{j=1}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}ija_{ij}\right] \\
&= \frac{1}{E_m^\alpha E_n^\beta}\Bigg[E_{m-1}^{\alpha-1}E_{n-1}^{\beta-1}a_{11} + \sum_{j=2}^{n}E_{m-1}^{\alpha-1}E_{n-j}^{\beta-1}ja_{1j} + \sum_{i=2}^{m}E_{m-i}^{\alpha-1}E_{n-1}^{\beta-1}ia_{i1} \\
&\qquad + \sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}ija_{ij}\Bigg] \\
&= w_1 + w_2 + w_3 + w_4\,, \qquad \text{say}\,.
\end{aligned}
$$

$$
\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_1|^k = O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(mn)^{-k-1} = O(1)\,.
$$

$$
\begin{aligned}
w_2 = \frac{1}{E_m^\alpha E_n^\beta}\sum_{j=2}^{n}E_{m-1}^{\alpha-1}E_{n-j}^{\beta-1}j\Bigg(\frac{P_1}{p_1}\Bigg[\frac{Q_j}{q_j}\Delta_{11}T_{0,j-1} &- \frac{Q_{j-1}}{q_{j-1}}\Delta_{11}T_{0,j-2} \\
&+\Delta_{11}T_{0,j-2}\Bigg]\Bigg)
\end{aligned}
$$

$$= w_{21} + w_{22}, \qquad \text{say}.$$

$$w_{21} = \frac{P_1}{p_1}\frac{E_{m-1}^{\alpha-1}}{E_m^\alpha E_n^\beta}\left[\sum_{j=2}^{n} E_{n-j}^{\beta-1} j \frac{Q_j}{q_j}\Delta_{11}T_{0,j-1} - \sum_{j=2}^{n} E_{n-j}^{\beta-1} j \frac{Q_{j-1}}{q_{j-1}}\Delta_{11}T_{0,j-2}\right]$$

$$= \frac{P_1}{p_1}\frac{E_{m-1}^{\alpha-1}}{E_m^\alpha E_n^\beta}\left[\frac{nQ_n}{q_n}\Delta_{11}T_{0,n-1} - 2E_{n-2}^{\beta-1}\frac{Q_1}{q_1}\Delta_{11}T_{00} + \sum_{j=2}^{n-1}\left(jE_{n-j}^{\beta-1} - (j+1)E_{n-j-1}^{\beta-1}\right)\frac{Q_j}{q_j}\Delta_{11}T_{0,j-1}\right]$$

$$= w_{211} + w_{212} + w_{213}, \qquad \text{say}.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{211}|^k = \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{P_1}{p_1}\frac{E_{m-1}^{\alpha-1}}{E_m^\alpha E_n^\beta}\frac{nQ_n}{q_n}\Delta_{11}T_{0,n-1}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty} m^{-k-1}\sum_{n=2}^{\infty} n^{k-1}\left(\frac{Q_n}{n^\beta q_n}\right)^k |\Delta_{11}T_{0,n-1}|^k$$

$$= O(1)\sum_{n=2}^{\infty} n^{k-1}|\Delta_{11}T_{0,n-1}|^k = O(1).$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{212}|^k = O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{m-1}^{\alpha-1}E_{n-2}^{\beta-1}}{E_m^\alpha E_n^\beta}\Delta_{11}T_{00}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(mn)^{-k-1} = O(1).$$

From [2; p. 320], $jE_{n-j}^{\beta-1} - E_{n-j-1}^{\beta-1}(j+1) = jE_{n-j}^{\beta-2} - E_{n-j-1}^{\beta-1}$. Write $w_{213} = w_{2131} + w_{2132}$.

Using Hölder's inequality and the results on the last line of page 322 and on page 323 of [2],

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{2131}|^k = O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{m-1}^{\alpha-1}}{E_m^\alpha E_n^\beta}\sum_{j=2}^{n-1} jE_{n-j}^{\beta-2}\frac{Q_j}{q_j}\Delta_{11}T_{0,j-1}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty} m^{-k-1}\sum_{n=2}^{\infty} n^{-1}\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}|E_{n-j}^{\beta-2}|\left(\frac{jQ_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\right]\times$$

$$\times\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}|E_{n-j}^{\beta-2}|\right]^{k-1}$$

$$= O(1)\sum_{j=2}^{\infty}\left(\frac{jQ_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\frac{1}{(E_{j+1}^{\beta})^{k-1}}\sum_{n=j+1}^{\infty}\frac{|E_{n-j}^{\beta-2}|}{nE_n^{\beta}}$$

$$= O(1)\sum_{j=2}^{\infty}\left(\frac{jQ_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k j^{-\beta(k-1)-2}$$

$$= O(1)\sum_{j=2}^{\infty} j^{k+\beta-2}|\Delta_{11}T_{0,j-1}|^k = O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{2132}|^k$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{m-1}^{\alpha-1}}{E_m^{\alpha}E_n^{\beta}}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\frac{Q_j}{q_j}\Delta_{11}T_{0,j-1}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty} m^{-k-1}\sum_{n=2}^{\infty} n^{-1}\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\left(\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\right]\times$$

$$\times\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\right]^{k-1}$$

$$= O(1)\sum_{j=2}^{\infty}\left(\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\sum_{n=j+1}^{\infty}\frac{E_{n-j-1}^{\beta-1}}{nE_n^{\beta}}$$

$$= O(1)\sum_{j=2}^{\infty} j^{\beta k}|\Delta_{11}T_{0,j-1}|^k j^{-1} = O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{2132}|^k$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{m-1}^{\alpha-1}}{E_m^{\alpha}E_n^{\beta}}\sum_{j=2}^{n} jE_{n-j}^{\beta-1}\Delta_{11}T_{0,j-2}\right|$$

$$= O(1)\sum_{m=2}^{\infty} m^{-k-1}\sum_{n=2}^{\infty} n^{-1}\left[\frac{1}{E_n^\beta}\sum_{j=2}^{n} j^k E_{n-j}^{\beta-1}|\Delta_{11}T_{0,j-2}|^k\right]\times$$

$$\times\left[\frac{1}{E_n^\beta}\sum_{j=2}^{n} E_{n-j}^{\beta-1}\right]^{k-1}$$

$$= O(1)\sum_{j=2}^{\infty} j^k|\Delta_{11}T_{0,j-2}|^k\sum_{n=j}^{\infty}\frac{E_{n-j}^{\beta-1}}{nE_n^\beta}$$

$$= O(1)\sum_{j=2}^{\infty} j^{k-1}|\Delta_{11}T_{0,j-2}|^k = O(1)\,.$$

Since w_3 is w_2 with the roles of p_i and q_j interchanged, it follows that

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_3|^k = O(1)\,.$$

From (6), we can write $w_4 = w_{41} + w_{42} + w_{43} + w_{44} + w_{45} + w_{46}$.

$$w_{41} = \frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n} ijE_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\Delta_{11}\left(\frac{P_{i-1}Q_{j-1}}{p_{i-1}q_{j-1}}\Delta_{11}T_{i-2,j-2}\right)$$

$$= \frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m} iE_{m-i}^{\alpha-1}\left[\sum_{j=2}^{n} jE_{n-j}^{\beta-1}\frac{P_{i-1}Q_{j-1}}{p_{i-1}q_{j-1}}\Delta_{11}T_{i-2,j-2}\right.$$

$$-\sum_{j=2}^{n} jE_{n-j}^{\beta-1}\frac{P_{i-1}Q_j}{p_{i-1}q_j}\Delta_{11}T_{i-2,j-1} - \sum_{j=2}^{n} jE_{n-j}^{\beta-1}\frac{P_iQ_{j-1}}{p_iq_{j-1}}\Delta_{11}T_{i-1,j-2}$$

$$\left.+\sum_{j=2}^{n} jE_{n-j}^{\beta-1}\frac{P_iQ_j}{p_iq_j}\Delta_{11}T_{i-1,j-1}\right]$$

$$= \frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m} iE_{m-i}^{\alpha-1}\left[2E_{n-2}^{\beta-1}\frac{P_{i-1}Q_1}{p_{i-1}q_1}\Delta_{11}T_{i-2,0} - \frac{nP_{i-1}Q_n}{p_{i-1}q_n}\Delta_{11}T_{i-2,n-1}\right.$$

$$+\sum_{j=2}^{n-1}((j+1)E_{n-j-1}^{\beta-1} - jE_{n-j}^{\beta-1})\frac{P_{i-1}Q_j}{p_{i-1}q_j}\Delta_{11}T_{i-2,j-1}$$

$$-2E_{n-2}^{\beta-1}\frac{P_iQ_1}{p_iq_1}\Delta_{11}T_{i-1,0} + \frac{nP_iQ_n}{p_iq_n}\Delta_{11}T_{i-1,n-1}$$

$$\left.-\sum_{j=2}^{n-1}((j+1)E_{n-j-1}^{\beta-1} - jE_{n-j}^{\beta-1})\frac{P_iQ_j}{p_iq_j}\Delta_{11}T_{i-1,j-1}\right]$$

$$= \frac{1}{E_m^\alpha E_n^\beta}\left[2E_{n-2}^{\beta-1}\left(\sum_{i=2}^{m} iE_{m-i}^{\alpha-1}\frac{P_{i-1}Q_1}{p_{i-1}q_1}\Delta_{11}T_{i-2,0} - \sum_{i=2}^{m} iE_{m-i}^{\alpha-1}\frac{P_iQ_1}{p_iq_1}\Delta_{11}T_{i-1,0}\right)\right.$$

$$+\sum_{j=2}^{n-1}\left((j+1)E_{n-j-1}^{\beta-1}-jE_{n-j}^{\beta-1}\right)\left(\sum_{i=2}^{m}iE_{m-i}^{\alpha-1}\frac{P_{i-1}Q_j}{p_{i-1}q_j}\Delta_{11}T_{i-2,j-1}\right.$$

$$\left.-\sum_{i=2}^{m}iE_{m-i}^{\alpha-1}\frac{P_iQ_j}{p_iq_j}\Delta_{11}T_{i-1,j-1}\right)$$

$$\left.-n\left(\sum_{i=2}^{m}iE_{m-i}^{\alpha-1}\frac{P_{i-1}Q_n}{p_{i-1}q_n}\Delta_{11}T_{i-2,n-1}-\sum_{i=2}^{m}iE_{m-i}^{\alpha-1}\frac{P_iQ_n}{p_iq_n}\Delta_{11}T_{i-1,n-1}\right)\right]$$

$$=\frac{1}{E_m^\alpha E_n^\beta}\left[2E_{n-2}^{\beta-1}\left(2E_{m-2}^{\alpha-1}\frac{P_1Q_1}{p_1q_1}\Delta_{11}T_{00}-m\frac{P_mQ_1}{p_mq_1}\Delta_{11}T_{m-1,0}\right.\right.$$

$$\left.+\sum_{i=2}^{m-1}\left((i+1)E_{m-i-1}^{\alpha-1}-iE_{m-i}^{\alpha-1}\right)\frac{P_iQ_1}{p_iq_1}\Delta_{11}T_{i-1,0}\right)$$

$$+\sum_{j=2}^{n-1}\left((j+1)E_{n-j-1}^{\beta-1}-jE_{n-j}^{\beta-1}\right)\times$$

$$\times\left(2E_{m-2}^{\alpha-1}\frac{P_1Q_j}{p_1q_j}\Delta_{11}T_{0,j-1}-m\frac{P_mQ_j}{p_mq_j}\Delta_{11}T_{m-1,j-1}\right.$$

$$\left.+\sum_{i=2}^{m-1}\left((i+1)E_{m-i-1}^{\alpha-1}-iE_{m-i}^{\alpha-1}\right)\frac{P_iQ_j}{p_iq_j}\Delta_{11}T_{i-1,j-1}\right)$$

$$-n\left(2E_{m-2}^{\alpha-1}\frac{P_1Q_n}{p_1q_n}\Delta_{11}T_{0,n-1}-m\frac{P_mQ_n}{p_mq_n}\Delta_{11}T_{m-1,n-1}\right.$$

$$\left.\left.+\sum_{i=2}^{m-1}\left((i+1)E_{m-i-1}^{\alpha-1}-iE_{m-i}^{\alpha-1}\right)\frac{P_iQ_n}{p_iq_n}\Delta_{11}T_{i-1,n-1}\right)\right]$$

$$=w_{411}+w_{412}+w_{413}+w_{414}+w_{415}+w_{416}+w_{417}+w_{418}+w_{419},$$

say.

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{411}|^k=O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{n-2}^{\beta-1}E_{m-2}^{\alpha-1}}{E_m^\alpha E_n^\beta}\right|^k$$

$$=O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(mn)^{-k-1}=O(1).$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{412}|^k$$

$$=O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{n-2}^{\beta-1}}{E_m^\alpha E_n^\beta}m\frac{P_m}{p_m}\Delta_{11}T_{m-1,0}\right|^k$$

$$=O(1)\sum_{n=2}^{\infty} n^{-k-1} \sum_{m=2}^{\infty} m^{-1}\left(\frac{P_m}{m^{\alpha}p_m}\right)^k |\Delta_{11}T_{m-1,0}|^k$$

$$=O(1)\sum_{m=2}^{\infty} m^{-1}|\Delta_{11}T_{m-1,0}|^k = O(1).$$

Using the identity $(i+1)E_{m-i-1}^{\alpha-1} - iE_{m-i}^{\alpha-1} = -iE_{m-i}^{\alpha-2} + E_{m-i-1}^{\alpha-1}$, we may write $w_{413} = w_{4131} + w_{4132}$.

Using Hölder's inequality and the results on the last line of page 322 and on page 323 of [2],

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4131}|^k$$

$$=O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{n-2}^{\beta-1}}{E_m^{\alpha}E_n^{\beta}}\sum_{i=2}^{m-1}(-iE_{m-i}^{\alpha-2})\frac{P_i}{p_i}\Delta_{11}T_{i-1,0}\right|^k$$

$$=O(1)\sum_{n=2}^{\infty} n^{-k-1}\sum_{m=2}^{\infty} m^{-1}\left[\frac{1}{E_m^{\alpha}}\sum_{i=2}^{m-1}|E_{m-i}^{\alpha-2}|\left(\frac{iP_i}{p_i}\right)^k|\Delta_{11}T_{i-1,0}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^{\alpha}}\sum_{i=2}^{m-1}|E_{m-i}^{\alpha-2}|\right]^{k-1}$$

$$=O(1)\sum_{i=2}^{\infty}\left(\frac{iP_i}{p_i}\right)^k|\Delta_{11}T_{i-1,0}|^k\frac{1}{(E_{i+1}^{\alpha})^{k-1}}\sum_{m=i+1}^{\infty}\frac{|E_{m-i}^{\alpha-2}|}{mE_m^{\alpha}}$$

$$=O(1)\sum_{i=2}^{\infty} i^{k+\alpha-2}|\Delta_{11}T_{i-1,0}|^k = O(1).$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4132}|^k$$

$$=O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{n-2}^{\beta-1}}{E_m^{\alpha}E_n^{\beta}}\sum_{i=2}^{m-1}E_{m-i-1}^{\alpha-1}\frac{P_i}{p_i}\Delta_{11}T_{i-1,0}\right|^k$$

$$=O(1)\sum_{n=2}^{\infty} n^{-k-1}\sum_{m=2}^{\infty} m^{-1}\left[\frac{1}{E_m^{\alpha}}\sum_{i=2}^{m-1}E_{m-i-1}^{\alpha-1}\left(\frac{P_i}{p_i}\right)^k|\Delta_{11}T_{i-1,0}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^{\alpha}}\sum_{i=2}^{m-1}E_{m-i-1}^{\alpha-1}\right]^{k-1}$$

$$=O(1)\sum_{i=2}^{\infty}\left(\frac{P_i}{p_i}\right)^k|\Delta_{11}T_{i-1,0}|^k\sum_{m=i+1}^{\infty}\frac{E_{m-i-1}^{\alpha-1}}{mE_m^{\alpha}}$$

$$= O(1)\sum_{i=2}^{\infty} i^{\alpha k-1}|\Delta_{11}T_{i-1,0}|^k = O(1).$$

Writing $w_{414} = w_{4141} + w_{4142}$, we have

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4141}|^k$$
$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{m-2}^{\alpha-1}}{E_m^{\alpha}E_n^{\beta}}\sum_{j=2}^{n-1}(-jE_{n-j}^{\beta-2})\frac{Q_j}{q_j}\Delta_{11}T_{0,j-1}\right|^k$$
$$= O(1)\sum_{m=2}^{\infty} m^{-k-1}\sum_{n=2}^{\infty} n^{-1}\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}|E_{n-j}^{\beta-2}|\left(\frac{jQ_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\right]\times$$
$$\times\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}|E_{n-j}^{\beta-2}|\right]^{k-1}$$
$$= O(1)\sum_{j=2}^{\infty}\left(\frac{jQ_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\frac{1}{(E_{j+1}^{\beta})^{k-1}}\sum_{n=j+1}^{\infty}\frac{|E_{n-j}^{\beta-2}|}{nE_n^{\beta}}$$
$$= O(1)\sum_{j=2}^{\infty} j^{k+\beta-2}|\Delta_{11}T_{0,j-1}|^k = O(1).$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4142}|^k$$
$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{E_{m-2}^{\alpha-1}}{E_m^{\alpha}E_n^{\beta}}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\frac{Q_j}{q_j}\Delta_{11}T_{0,j-1}\right|^k$$
$$= O(1)\sum_{m=2}^{\infty} m^{-k-1}\sum_{n=2}^{\infty} n^{-1}\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\left(\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\right]\times$$
$$\times\left[\frac{1}{E_n^{\beta}}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\right]^{k-1}$$
$$= O(1)\sum_{j=2}^{\infty}\left(\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{0,j-1}|^k\sum_{n=j+1}^{\infty}\frac{E_{n-j-1}^{\beta-1}}{nE_n^{\beta}}$$
$$= O(1)\sum_{j=2}^{\infty} j^{k\beta-1}|\Delta_{11}T_{0,j-1}|^k = O(1).$$

Writing $w_{415} = w_{4151} + w_{4152}$,

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4151}|^k$$
$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{j=2}^{n-1}(-jE_{n-j}^{\beta-2})\frac{mP_mQ_j}{p_mq_j}\Delta_{11}T_{m-1,j-1}\right|^k$$
$$= O(1)\sum_{m=2}^{\infty}m^{k-1}\sum_{n=2}^{\infty}n^{-1}\left[\frac{1}{E_n^\beta}\sum_{j=2}^{n-1}|E_{n-j}^{\beta-2}|\left(\frac{jQ_j}{q_j}\right)^k|\Delta_{11}T_{m-1,j-1}|^k\right]\times$$
$$\times\left[\frac{1}{E_n^\beta}\sum_{j=2}^{n-1}|E_{n-j}^{\beta-2}|\right]^{k-1}$$
$$= O(1)\sum_{m=2}^{\infty}m^{k-1}\sum_{j=2}^{\infty}\left(\frac{jQ_j}{q_j}\right)^k|\Delta_{11}T_{m-1,j-1}|^k\frac{1}{(E_{j+1}^\beta)^{k-1}}\sum_{n=j+1}^{\infty}\frac{|E_{n-j}^{\beta-2}|}{nE_n^\beta}$$
$$= O(1)\sum_{m=2}^{\infty}m^{k-1}\sum_{j=2}^{\infty}j^{k+\beta-2}|\Delta_{11}T_{m-1,j-1}|^k = O(1).$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4152}|^k$$
$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\frac{mP_mQ_j}{p_mq_j}\Delta_{11}T_{m-1,j-1}\right|^k$$
$$= O(1)\sum_{m=2}^{\infty}m^{k-1}\sum_{n=2}^{\infty}n^{-1}\left[\frac{1}{E_n^\beta}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\left(\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{m-1,j-1}|^k\right]\times$$
$$\times\left[\frac{1}{E_n^\beta}\sum_{j=2}^{n-1}E_{n-j-1}^{\beta-1}\right]^{k-1}$$
$$= O(1)\sum_{m=2}^{\infty}m^{k-1}\sum_{j=2}^{\infty}\left(\frac{Q_j}{q_j}\right)^k|\Delta_{11}T_{m-1,j-1}|^k\sum_{n=j+1}^{\infty}\frac{E_{n-j-1}^{\beta-1}}{nE_n^\beta}$$
$$= O(1)\sum_{j=2}^{\infty}j^{k\beta-1}|\Delta_{11}T_{m-1,j-1}|^k = O(1).$$

Writing $w_{416} = w_{4161} + w_{4162} + w_{4163} + w_{4164}$,

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4161}|^k$$
$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1} ijE_{m-i}^{\alpha-2}E_{n-j}^{\beta-2}\frac{P_iQ_j}{p_jq_j}\Delta_{11}T_{i-1,j-1}\right|^k$$
$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}|E_{m-i}^{\alpha-2}E_{n-j}^{\beta-2}|\left(\frac{ijP_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\right]\times$$
$$\times\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}|E_{m-i}^{\alpha-2}E_{n-j}^{\beta-2}|\right]^{k-1}$$
$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{ijP_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\times$$
$$\times\frac{1}{(E_{i+1}^\alpha E_{j+1}^\beta)^{k-1}}\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\frac{|E_{m-i}^{\alpha-2}E_{n-j}^{\beta-2}|}{mnE_m^\alpha E_n^\beta}$$
$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty} i^{(1+\alpha)k}j^{(1+\beta)k}|\Delta_{11}T_{i-1,j-1}|^k\frac{(ij)^{-2}}{(i^\alpha j^\beta)^{k-1}}$$
$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}(ij)^{k-1}|\Delta_{11}T_{i-1,j-1}|^k=O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4162}|^k$$
$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}(-iE_{m-i}^{\alpha-2})E_{n-j-1}^{\beta-1}\left(\frac{P_iQ_j}{p_jq_j}\right)\Delta_{11}T_{i-1,j-1}\right|^k$$
$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}|E_{m-i}^{\alpha-2}|E_{n-j-1}^{\beta-1}\left(\frac{iP_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\right]\times$$
$$\times\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}|E_{m-i}^{\alpha-2}|E_{n-j-1}^{\beta-1}\right]^{k-1}$$
$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{iP_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\frac{1}{(E_{i+1}^\alpha)^{k-1}}\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\frac{|E_{m-i}^{\alpha-2}|E_{n-j-1}^{\beta-1}}{mnE_m^\alpha E_n^\beta}$$
$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty} i^{k+\alpha-2}j^{k\beta-1}|\Delta_{11}T_{i-1,j-1}|^k=O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4163}|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}E_{m-i-1}^{\alpha-1}(-jE_{n-j}^{\beta-2})\left(\frac{P_iQ_j}{p_iq_j}\right)\Delta_{11}T_{i-1,j-1}\right|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}E_{m-i-1}^{\alpha-1}|E_{n-j}^{\beta-2}|\left(\frac{jP_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}E_{m-i-1}^{\alpha-1}|E_{n-j}^{\beta-2}|\right]^{k-1}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{jP_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\frac{1}{(E_{j+1}^\beta)^{k-1}}\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\frac{E_{m-i-1}^{\alpha-1}|E_{n-j}^{\beta-2}|}{mnE_m^\alpha E_n^\beta}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}j^{k+\beta-2}i^{k\alpha-1}|\Delta_{11}T_{i-1,j-1}|^k=O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4164}|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}E_{m-i-1}^{\alpha-1}E_{n-j-1}^{\beta-1}\left(\frac{P_iQ_j}{p_iq_j}\right)\Delta_{11}T_{i-1,j-1}\right|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}E_{m-i-1}^{\alpha-1}E_{n-j-1}^{\beta-1}\left(\frac{P_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}\sum_{j=2}^{n-1}E_{m-i-1}^{\alpha-1}E_{n-j-1}^{\beta-1}\right]^{k-1}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{P_iQ_j}{p_iq_j}\right)^k|\Delta_{11}T_{i-1,j-1}|^k\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\frac{E_{m-i-1}^{\alpha-1}E_{n-j-1}^{\beta-1}}{mnE_m^\alpha E_n^\beta}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}i^{k\alpha-1}j^{k\beta-1}|\Delta_{11}T_{i-1,j-1}|^k=O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{417}|^k$$

$$=O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}nE_{m-2}^{\alpha-1}\left(\frac{Q_n}{q_n}\right)\Delta_{11}T_{0,n-1}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty} m^{-k-1}\sum_{n=2}^{\infty} n^{k-1}|\Delta_{11}T_{0,n-1}|^k = O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{418}|^k$$

$$= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\frac{mnP_mQ_n}{p_mq_n}\Delta_{11}T_{m-1,n-1}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(mn)^{k-1}|\Delta_{11}T_{m-1,n-1}|^k = O(1)\,.$$

Writing $w_{419} = w_{4191} + w_{4192}$,

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4191}|^k$$

$$= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}(-iE_{m-i}^{\alpha-2})\frac{P_iQ_n}{p_iq_n}\Delta_{11}T_{i-1,n-1}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha}\sum_{i=2}^{m-1}|E_{m-i}^{\alpha-2}|\left(\frac{iP_i}{p_i}\right)^k|\Delta_{11}T_{i-1,n-1}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^\alpha}\sum_{i=2}^{m-1}|E_{m-i}^{\alpha-2}|\right]^{k-1}$$

$$= O(1)\sum_{n=2}^{\infty}n^{-1}\sum_{i=2}^{\infty}\left(\frac{iP_i}{p_i}\right)^k|\Delta_{11}T_{i-1,n-1}|^k\frac{1}{(E_{i+1}^\alpha)^{k-1}}\sum_{m=i+1}^{\infty}\frac{|E_{m-i}^{\alpha-2}|}{mE_m^\alpha}$$

$$= O(1)\sum_{n=2}^{\infty}n^{-1}\sum_{i=2}^{\infty}i^{k+\alpha-2}|\Delta_{11}T_{i-1,n-1}|^k = O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{4192}|^k$$

$$= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m-1}E_{m-i-1}^{\alpha-1}\frac{P_iQ_n}{p_iq_n}\Delta_{11}T_{i-1,n-1}\right|^k$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha}\sum_{i=2}^{m-1}E_{m-i-1}^{\alpha-1}\left(\frac{P_i}{p_i}\right)^k|\Delta_{11}T_{i-1,n-1}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^\alpha}\sum_{i=2}^{m-1}|E_{m-i-1}^{\alpha-1}\right]^{k-1}$$

$$= O(1) \sum_{n=2}^{\infty} n^{-1} \sum_{i=2}^{\infty} \left(\frac{P_i}{p_i}\right)^k |\Delta_{11} T_{i-1,n-1}|^k \sum_{m=i+1}^{\infty} \frac{E_{m-i-1}^{\alpha-1}}{m E_m^{\alpha}}$$

$$= O(1) \sum_{n=2}^{\infty} n^{-1} \sum_{i=2}^{\infty} i^{k\alpha-1} |\Delta_{11} T_{i-1,n-1}|^k = O(1)\,.$$

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{mn} |w_{42}|^k$$

$$= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{mn} \left| \frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} \frac{Q_j}{q_j} \Delta_{11} T_{i-2,j-1} \right|^k$$

$$= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{mn} \left[\frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} \left(\frac{Q_j}{q_j}\right)^k |\Delta_{11} T_{i-2,j-1}|^k \right] \times$$

$$\times \left[\frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} \right]^{k-1}$$

$$= O(1) \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left(\frac{Q_j}{q_j}\right)^k |\Delta_{11} T_{i-2,j-1}|^k \sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} \frac{E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1}}{mn E_m^{\alpha} E_n^{\beta}}$$

$$= O(1) \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} i^{-1} j^{k\beta-1} |\Delta_{11} T_{i-2,j-1}|^k = O(1)\,.$$

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{mn} |w_{43}|^k$$

$$= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{mn} \left| \frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} \frac{Q_{j-1}}{q_{j-1}} \Delta_{11} T_{i-2,j-2} \right|^k$$

$$= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{mn} \left[\frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} \left(\frac{Q_{j-1}}{q_{j-1}}\right)^k |\Delta_{11} T_{i-2,j-2}|^k \right] \times$$

$$\times \left[\frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} \right]^{k-1}$$

$$= O(1) \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left(\frac{Q_{j-1}}{q_{j-1}}\right)^k |\Delta_{11} T_{i-2,j-2}|^k \sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} \frac{E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1}}{mn E_m^{\alpha} E_n^{\beta}}$$

$$= O(1) \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} i^{-1} j^{k\beta-1} |\Delta_{11} T_{i-2,j-2}|^k = O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{44}|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\frac{P_i}{p_i}\Delta_{11}T_{i-1,j-2}\right|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\left(\frac{P_i}{p_i}\right)^k|\Delta_{11}T_{i-1,j-2}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\right]^{k-1}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{P_i}{p_i}\right)^k|\Delta_{11}T_{i-1,j-2}|^k\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\frac{E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}}{mnE_m^\alpha E_n^\beta}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}i^{k\beta-1}j^{-1}|\Delta_{11}T_{i-1,j-2}|^k=O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{45}|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\frac{P_{i-1}}{p_{i-1}}\Delta_{11}T_{i-2,j-2}\right|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\left(\frac{P_{i-1}}{p_{i-1}}\right)^k|\Delta_{11}T_{i-2,j-2}|^k\right]\times$$

$$\times\left[\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\right]^{k-1}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(\frac{P_{i-1}}{p_{i-1}}\right)^k|\Delta_{11}T_{i-2,j-2}|^k\sum_{m=i+1}^{\infty}\sum_{n=j+1}^{\infty}\frac{E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}}{mnE_m^\alpha E_n^\beta}$$

$$=O(1)\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}i^{k\beta-1}j^{-1}|\Delta_{11}T_{i-2,j-2}|^k=O(1)\,.$$

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}|w_{46}|^k$$

$$=\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{mn}\left|\frac{1}{E_m^\alpha E_n^\beta}\sum_{i=2}^{m}\sum_{j=2}^{n}E_{m-i}^{\alpha-1}E_{n-j}^{\beta-1}\Delta_{11}T_{i-2,j-2}\right|^k$$

$$= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{mn} \left[\frac{1}{E_m^\alpha E_n^\beta} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} |\Delta_{11} T_{i-2,j-2}|^k \right] \times$$

$$\times \left[\frac{1}{E_m^\alpha E_n^\beta} \sum_{i=2}^{m} \sum_{j=2}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} \right]^{k-1}$$

$$= O(1) \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} |\Delta_{11} T_{i-1,j-1}|^k \sum_{m=i+1}^{\infty} \sum_{n=j+1}^{\infty} \frac{E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1}}{mn E_m^\alpha E_n^\beta}$$

$$= O(1) \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} (ij)^{-1} |\Delta_{11} T_{i-1,j-1}|^k = O(1).$$

There is no need to consider the comparison for either α or $\beta > 1$, since those cases come as consequences of the translativity of inclusion and the fact that the well-known result of Flett comparing absolutely the Cesáro matrices of orders γ and δ readily extends to double summability. □

REFERENCES

[1] MORICZ, F.—RHOADES, B. E.: *Double weighted mean methods equivalent to* $(C,1,1)$, Publ. Math. Debrecen **47** (1995), 29–64.

[2] SARIGÖL, M. A.—BOR, H.: *On two summability methods*, Math. Slovaca **43** (1993), 317–325.

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