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*Dedicated to the memory*
*of Professor Milan Kolibiar*

# GENERALIZED HERGLOTZ THEOREM IN VECTOR LATTICES

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ABSTRACT. We present a Herglotz theorem in the context of vector lattices.

## Introduction

It is well known that Fourier-Stieltjes coefficients of positive measures can be characterized as positive definite sequences. Recall that a numerical sequence $(a_n)_{n=-\infty}^{\infty}$ is said to be positive definite if for any (complex) sequence $(z_n)$ having only a finite number of terms different from zero we have

$$\sum_{n,m} a_{n-m} z_n \bar{z}_m \geq 0 .$$

Now, according to the Herglotz theorem [5; Theorem I.7.6], a numerical sequence $(a_n)_{n=-\infty}^{\infty}$ is positive definite if and only if there exists a positive Borel measure $\mu$ on $[-\pi,\pi]$ with $\mu(\{-\pi\}) = \mu(\{\pi\})$, such that

$$a_n = \int\limits_{[-\pi,\pi)} \mathrm{e}^{-ins} \, \mathrm{d}\mu(s)$$

for all $n = 0, \pm 1, \dots$ (cf. also [1] and [4]).

In this paper, we give a generalization of the Herglotz theorem for $a_n$ being elements of a vector lattice. As for terminology and some results from vector lattices we shall use as reference the book [2].

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## 1. Preliminaries

Let $Y$ be a (Dedekind) complete vector lattice. Denote by $L^o(X,Y)$ the vector space of all $o$-bounded operators on the normed space $X$ into $Y$, that is, if $U \in L^o(X,Y)$, then $\{U(x);\ \|x\| \le 1\}$ is an $o$-bounded subset of $Y$. For $U \in L^o(X,Y)$ we put

$$\|U\| = \sup\{|U(x)|;\ \|x\| \le 1\}.$$

In the following, let $\mathbf{T}$ denote the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ ($\mathbb{R}$ and $\mathbb{Z}$ denoting the additive group of reals, integers, respectively), as a model we may think of the interval $[0,2\pi)$, and let $C(\mathbf{T})$ denote the space of all scalar continuous functions on $\mathbf{T}$ with the usual sup norm. If $U \in L^o(C(\mathbf{T}),Y)$, then an element of $Y$ of the form

$$\hat{U}(n) = U(\mathrm{e}^{-\mathrm{i}nt})$$

is called the $n$th Fourier coefficient of $U$. The (formal) series

$$\sum_{n\in\mathbb{Z}} \hat{U}(n)\,\mathrm{e}^{\mathrm{i}nx}$$

is called the Fourier series of $U$. It is clear that there exists an element $0 \le C \in Y$ such that

$$|\hat{U}(n)| \le C, \qquad n \in \mathbb{Z}.$$

We shall investigate some properties of such Fourier series.

A trigonometric polynomial on $\mathbf{T}$ is a function $a = a(t)$ defined on $\mathbf{T}$ by $a(t) = \sum_{-n}^{n} a_j\,\mathrm{e}^{\mathrm{i}jt}$. Denote by $p(\mathbf{T})$ the set of all trigonometric polynomials on $\mathbf{T}$. We shall need the following theorem ([5; Theorem 2.12]) asserting that trigonometric polynomials are dense in $C(\mathbf{T})$.

**THEOREM A.** *For every $f \in C(\mathbf{T})$ we have $\sigma_n(f) \to f$, $n \to \infty$, in the $C(\mathbf{T})$ norm.*

Recall that

$$\sigma_n(f,t) = \sum_{-n}^{n}\Big(1 - \frac{|j|}{n+1}\Big)\hat{f}(j)\,\mathrm{e}^{\mathrm{i}jt},$$

where $\hat{f}(j)$ is the $j$th Fourier-Lebesgue coefficient of $f$ defined by

$$\hat{f}(j) = \frac{1}{2\pi}\int f(t)\,\mathrm{e}^{-\mathrm{i}jt}\,\mathrm{d}t.$$

(The integration is taken over $\mathbf{T}$.)

The following simple lemma will be useful for us.

**LEMMA.** *Let $U: C(\mathbf{T}) \to Y$ be an $(o)$-bounded linear operator. For every $a = \sum_{-n}^{n} a_j e^{ijt}$ we have $U(a) = \sum_{-n}^{n} a_j \hat{U}(-j)$ and $|U(a)| \le \|a\| \|U\|$, where*

$$\|a\| = \sup_t |a(t)|.$$

We have the following result.

**THEOREM 1.** (**PARSEVAL'S FORMULA**) *Let $f \in C(\mathbf{T})$ and $U \in L^o(C(\mathbf{T}), Y)$. Then*

$$U(f) = \lim_{N\to\infty} \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j)\hat{U}(-j).$$

Proof. Since $f = \lim_{n\to\infty} \sigma_n(f)$ in the $C(\mathbf{T})$ norm, it follows from lemma and the fact that $U$ is $(o)$-bounded (hence $(o)$-continuous) that

$$\begin{aligned} U(f) = U\Big(\lim_{n\to\infty} \sigma_n(f)\Big) &= \lim_{n\to\infty} U\big(\sigma_n(f)\big) \\ &= \lim_{n\to\infty} \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j)\hat{U}(-j). \end{aligned}$$

□

**Remark.** The fact that the preceding limit exists is an implicit part of the theorem. It is equivalent to the C-1 (Cesàro) summability of the series $\sum \hat{f}(j)\hat{U}(-j)$, the members of which are elements of the space $Y$. If this last series converges, then clearly,

$$U(f) = \sum_{-\infty}^{\infty} \hat{f}(j)\hat{U}(-j).$$

**COROLLARY.** (**UNIQUENESS THEOREM**) *If $\hat{U}(j) = 0$ for all $j \in \mathbb{Z}$, then $U = 0$.*

Parseval's formula enables us to characterize sequences of Fourier coefficients of $(o)$-bounded linear operators on $C(\mathbf{T})$ similarly as in the case of linear functionals ([5; 7.3])

**THEOREM 2.** *Let $(y_j)$ be a two-way sequence of elements of $Y$. Then the following two conditions are equivalent:*

(a) *There is an operator $U \in L^o(C(\mathbf{T}), Y)$ with $\|U\| \le C \in Y$ such that $\hat{U}(j) = y_j$ for all $j \in \mathbb{Z}$.*

(b) *For all trigonometric polynomials* $a = \sum_{-l}^{l} a_j \,\mathrm{e}^{\mathrm{i}jt}$ *there holds*

$$\Big|\sum_{-l}^{l} a_{-j} y_j\Big| \le \|a\| C \qquad \textit{with} \quad 0 \le C \in Y\,.$$

P r o o f . Clearly, (a) implies (b) since

$$\begin{aligned}\Big|\sum_{-l}^{l} a_{-j} y_j\Big| &= \Big|\sum_{-l}^{l} a_{-j} \hat{U}(j)\Big| \\ &= \Big|\sum_{-l}^{l} a_{-j} U\big(\mathrm{e}^{-\mathrm{i}jt}\big)\Big| \le \|U\| \cdot \sup_t \Big|\sum_{-l}^{l} a_{-j}\,\mathrm{e}^{-\mathrm{i}jt}\Big| \le C\|a\|\,.\end{aligned}$$

Conversely, let for $\{y_j\} \subset Y$ and for some $C \in Y$

$$\Big|\sum_{-l}^{l} a_{-j} y_j\Big| \le C \sup_t \Big|\sum_{-l}^{l} a_{-j}\,\mathrm{e}^{-\mathrm{i}jt}\Big|\,.$$

Put

$$U\Big(\sum_{-l}^{l} a_j\,\mathrm{e}^{\mathrm{i}jt}\Big) = \sum_{-l}^{l} a_{-j} y_j\,.$$

Then

$$\Big|U\Big(\sum_{-l}^{l} a_{-j}\,\mathrm{e}^{-\mathrm{i}jt}\Big)\Big| \le C \sup_t \Big|\sum_{-l}^{l} a_{-j}\,\mathrm{e}^{-\mathrm{i}jt}\Big|\,.$$

It follows that $U$ is an $o$-bounded operator on trigonometric polynomials, these are dense in $C(\mathbf{T})$, hence $U$ has an $o$-bounded extension to $C(\mathbf{T})$. Also we obtain $\hat{U}(j) = y_j$. □

Let $(y_j)$ be a two-way sequence of elements of $Y$. Put

$$\sigma_N(Y,t) = \sum_{-N}^{N} \Big(1 - \frac{|j|}{N+1}\Big) y_{-j}\,\mathrm{e}^{-\mathrm{i}jt}\,, \qquad N = 1, 2, \dots\,,$$

and denote by $S_N(Y)$ the $(o)$-bounded linear operator on $C(\mathbf{T})$ defined by

$$S_N(Y)(f) = \frac{1}{2\pi} \int_{\mathbf{T}} f(t) \sigma_N(Y,t)\,\mathrm{d}t\,, \qquad f \in C(\mathbf{T})\,, \quad N = 1, 2, \dots\,.$$

If $U \in L^o(C(\mathbf{T}), Y)$ and if $y_j = \hat{U}(j)$, we shall write

$$\sigma_N(Y,t) = \sigma_N(U,t) \quad \text{and} \quad S_N(Y) = S_N(U).$$

We have

$$S_N(Y)(f) = \frac{1}{2\pi}\int_{\mathbf{T}} f(t)\sigma_N(Y,t)\,\mathrm{d}t = \sum_{-N}^{N}\Big(1 - \frac{|j|}{N+1}\Big)\hat{f}(j)y_{-j},$$
$$N = 1,2,\dots, \quad f \in C(\mathbf{T}).$$

We may now prove the following.

**THEOREM 3.** *The members of a two-way sequence $(y_j)$ in $Y$ are the Fourier coefficients of some $U \in L^o(C(\mathbf{T}), Y)$ with $\|U\| \le C \in Y$ if and only if $\|S_N(Y)\| \le C$, $N = 1,2,\dots$.*

Proof.

*The necessity:* Let $y_j = \hat{U}(j)$ for some $U \in L^o(C(\mathbf{T}), Y)$ with $\|U\| \le C$. Then $S_N(Y) = S_N(U)$, $N = 1,2,\dots$. Recall that $\|\sigma_N(f)\| \le \|f\|$ for all $f \in C(\mathbf{T})$. Since, for $f \in C(\mathbf{T})$, $S_N(U)(f) = U(\sigma_N(f))$, we have

$$\begin{aligned}\|S_N(Y)\| = \|S_N(U)\| &= \sup\{|S_N(U)(f)| : f \in C(\mathbf{T}),\ \|f\| \le 1\}\\ &= \sup\Big\{\big|U(\sigma_N(f))\big| : f \in C(\mathbf{T}),\ \|f\| \le 1\Big\}\\ &\le \sup\{|U(f)| : f \in C(\mathbf{T}),\ \|f\| \le 1\}\\ &= \|U\| \le C\end{aligned}$$

for $N = 1,2,\dots$.

*The sufficiency:* Take $a = \sum_{-l}^{l} a_j\,\mathrm{e}^{\mathrm{i}jt}$. Then we have

$$\sum_{-l}^{l} y_{-j}a_j = \lim_{N\to\infty}\sum_{-N}^{N}\Big(1 - \frac{|j|}{N+1}\Big)y_{-j}a_j = \lim_{N\to\infty} S_N(Y)(a).$$

Thus

$$\Big|\sum_{-l}^{l} y_{-j}a_j\Big| = \lim_{N\to\infty}|S_N(Y)(a)| \le \|a\|\sup_N\|S_N(Y)\| \le \|a\|C.$$

According to the preceding theorem, there exists $U \in L^o(C(\mathbf{T}), Y)$ such that $y_j = \hat{U}(j)$ and $\|U\| \le C$.

□

## 2. Fourier-Stieltjes coefficients of vector measures of $(o)$-bounded variation

Let $Y$ be a complete vector lattice. Recall that $\mathbf{T}$ is a compact Hausdorff space, and let $B(\mathbf{T})$ be the sigma algebra of Borelian subsets of $\mathbf{T}$. Let $\boldsymbol{m}: B(\mathbf{T}) \to Y$ be an additive set function which satisfies the condition that for any $E \in B(\mathbf{T})$ the set

$$G(E) = \left\{ \sum_{i=1}^{k} |\boldsymbol{m}(A_i)| ; \ (A_1, \dots, A_k) \text{ is } B(\mathbf{T})\text{-partition of } E \right\}$$

is $(o)$-bounded. We shall say that $\boldsymbol{m}$ is a vector measure of the $(bv)$-type or of $(o)$-bounded variation, and we shall denote

$$v_{\boldsymbol{m}}(E) = \sup G(E) .$$

If $f$ is a $B(\mathbf{T})$-simple function, $f(t) = \sum_{i=1}^{k} c_i \chi_{A_i}(t)$, we define

$$\int f(t) \, \mathrm{d}\boldsymbol{m}(t) = \sum_{i=1}^{k} c_i \boldsymbol{m}(A_i) ,$$

and then we extend this integral for bounded Borel functions on $\mathbf{T}$ ([3]).

Denote by $BV^o(\mathbf{T}, Y)$ the vector space of all measures on $\mathbf{T}$ with values in $Y$ of $o$-bounded variation.

Further, if $\boldsymbol{m} \in BV^o(\mathbf{T}, Y)$, then an element of $Y$ of the form

$$\hat{\boldsymbol{m}}(n) = \frac{1}{2\pi} \int_{\mathbf{T}} \mathrm{e}^{-\mathrm{i}nt} \, \mathrm{d}\boldsymbol{m}(t)$$

is called the $n$th Fourier-Stieltjes coefficient of $\boldsymbol{m}$.

We shall make use of the following result.

The general form of the $(o)$-bounded linear operator $U: C(\mathbf{T}) \to Y$ is given by the formula

$$U(f) = \int f(t) \, \mathrm{d}\boldsymbol{m}(t) ,$$

where $\boldsymbol{m}: B(\mathbf{T}) \to Y$ is a measure of $(o)$-bounded variation ([3]).

Now we can prove the following.

**THEOREM 4.** *Let $Y$ be a complete vector lattice. Let $(y_k)$ be a two-way sequence of elements of $Y$. Then the following two conditions are equivalent:*

(a) *There is a measure $\boldsymbol{m}: B(\mathbf{T}) \to Y$ of $(o)$-bounded variation with $v_{\boldsymbol{m}}(\mathbf{T}) \le C \in Y$ such that $y_j$ are Fourier-Stieltjes coefficients of $\boldsymbol{m}$.*

*i.e.*,

$$y_j = \hat{\boldsymbol{m}}(j) = \frac{1}{2\pi} \int\limits_{\mathbf{T}} \mathrm{e}^{-\mathrm{i}jt} \, \mathrm{d}\boldsymbol{m}(t) \qquad \textit{for all} \quad j \in \mathbb{Z}.$$

(b) *For all trigonometric polynomials* $a = \sum_{-l}^{l} a_j \,\mathrm{e}^{\mathrm{i}jt} \in p(\mathbf{T})$ *there holds*

$$\Big| \sum_{-l}^{l} a_{-j} y_j \Big| \le \|a\| C$$

*for some* $C \in Y$.

Proof. Clearly, (a) implies (b) since

$$\begin{aligned} \Big| \sum_{-l}^{l} a_{-j} y_j \Big| &= \Big| \sum_{-l}^{l} a_{-j} \frac{1}{2\pi} \int \mathrm{e}^{-\mathrm{i}jt} \, \mathrm{d}\boldsymbol{m}(t) \Big| \\ &= \Big| \frac{1}{2\pi} \int \Big( \sum_{-l}^{l} a_{-j} \,\mathrm{e}^{-\mathrm{i}jt} \Big) \, \mathrm{d}\boldsymbol{m}(t) \Big| \le \|a\| v_{\boldsymbol{m}}(\mathbf{T}) \end{aligned}$$

by using ([3; p. 407]).

If we assume (b), then the linear operator $U \colon p(\mathbf{T}) \to Y$ from the proof of Theorem 2 is an $(o)$-bounded linear operator that admits an extension that is an $(o)$-bounded linear operator on $C(\mathbf{T})$ with $\|U\| \le C$. But according to ([3; Corollary]), there exists a measure $\boldsymbol{m}$ of $(o)$-bounded variation such that

$$U(f) = \int f(t) \, \mathrm{d}\boldsymbol{m}(t) , \qquad f \in C(\mathbf{T}) .$$

Clearly, $\hat{U}(j) = \hat{\boldsymbol{m}}(j) = y_j$. □

If $\boldsymbol{m} \in BV^o(\mathbf{T}, Y)$, then the (formal) series

$$\sum_{n \in \mathbb{Z}} \hat{\boldsymbol{m}}(n) \,\mathrm{e}^{\mathrm{i}nx}$$

is called the Fourier-Stieltjes series of $\boldsymbol{m}$.

If the measure $\boldsymbol{m}$ is of the $(o)$-bounded variation, and $y_j = \hat{\boldsymbol{m}}(j)$, $j \in \mathbb{Z}$, we shall write

$$\sigma_N(Y, t) = \sigma_N(\boldsymbol{m}, t) \quad \text{and} \quad S_N(Y) = S_N(\boldsymbol{m}) .$$

We can now prove the following.

**THEOREM 5.** *Let $Y$ be a complete vector lattice. The trigonometric series*

$$\sum_{n\in\mathbb{Z}} y_j \, \mathrm{e}^{\mathrm{i}nx}, \qquad y_j \in Y,$$

*is the Fourier-Stieltjes series of the measure $\boldsymbol{m}\colon B(\mathbf{T}) \to Y$ of the $(o)$-bounded variation, i.e., $y_j = \hat{\boldsymbol{m}}(j)$, $j \in \mathbb{Z}$, if and only if there exists an element $0 \le C \in Y$ such that*

$$\|S_N(Y)\| \le C, \qquad N = 1, 2, \dots .$$

P r o o f. If there exists a measure $\boldsymbol{m}$ of $(o)$-bounded variation, $\boldsymbol{m} \in BV^o(\mathbf{T}, Y)$ such that $y_j = \hat{\boldsymbol{m}}(j)$, $j \in \mathbb{Z}$, then, as we know, the equation

$$U(f) = \int f(t)\, \mathrm{d}\boldsymbol{m}(t), \qquad f \in C(\mathbf{T}),$$

defines an $(o)$-bounded linear operator $U\colon C(\mathbf{T}) \to Y$ with $\|U\| \le C$ for some $0 \le C \in Y$. Hence, according to Theorem 3, we have

$$\|S_N(Y)\| = \|S_N(U)\| = \|S_N(\boldsymbol{m})\| \le C, \qquad N = 1, 2, \dots .$$

Conversely, if $\|S_N(Y)\| \le C$, $N = 1, 2, \dots$, for some $0 \le C \in Y$, then, according to Theorem 3, there exists an $(o)$-bounded linear operator $U\colon C(\mathbf{T}) \to Y$ such that $\hat{U}(j) = y_j$. But then there exists a measure $\boldsymbol{m}$ of $(o)$-bounded variation such that

$$U(f) = \int f(t)\, \mathrm{d}\boldsymbol{m}(t), \qquad f \in C(\mathbf{T}).$$

But $\|U\| = v_{\boldsymbol{m}}(\mathbf{T}) \le C$. Clearly, $\hat{U}(j) = \hat{\boldsymbol{m}}(j) = y_j$, $j \in \mathbb{Z}$. □

It is useful to establish the Parseval formula explicitly also for the Fourier-Stieltjes series of the measure $\boldsymbol{m}$ of $(o)$-bounded variation.

**THEOREM 6.** *Let $Y$ be a complete vector lattice, and let $f \in C(\mathbf{T})$. Then we have*

$$\int f(t)\, \mathrm{d}\boldsymbol{m}(t) = \lim_{N\to\infty} \sum_{-N}^{N} \left(1 - \frac{|y|}{N+1}\right) \hat{f}(j) \hat{\boldsymbol{m}}(-j).$$

P r o o f. By the Parseval formula from Theorem 1, the last equality holds for $f \in C(\mathbf{T})$. □

It is a very important fact that we have established not only a characterization of the Fourier-Stieltjes series of the measure of $(o)$-bounded variation but also a method how to recapture the measure by means of its Fourier-Stieltjes series. Theorem 6 gives a recipe how to recover the measure $\boldsymbol{m}$. In this sense, we may, by abuse of notation, write

$$\mathrm{d}\boldsymbol{m}(t) \sim \sum_{j\in\mathbb{Z}} \hat{\boldsymbol{m}}(j)\, \mathrm{e}^{\mathrm{i}jx}$$

for $\boldsymbol{m} \in BV^o(\mathbf{T}, Y)$.

It is easy to see that if the measure $\boldsymbol{m}\colon B(\mathbf{T}) \to Y$ is positive, then $\boldsymbol{m}$ is of the $(o)$-bounded variation. Hence we may establish the following.

**THEOREM 7.** *Let $Y$ be a complete vector lattice. The necessary and sufficient condition for*

$$\sum_{k\in\mathbb{Z}} y_k\,\mathrm{e}^{\mathrm{i}kx}$$

*to be the Fourier-Stieltjes series of a positive measure $\boldsymbol{m}$ with values in $Y$ is that $\sigma_N(Y,t) \geq 0$ for all $N$ on $\mathbf{T}$.*

Proof.

*The necessity:* If $y_k = \hat{\boldsymbol{m}}(k)$ for a positive measure $\boldsymbol{m}$, we have

$$\sigma_N(Y,t) = \sum_{-N}^{N}\Big(1 - \frac{|j|}{N+1}\Big) y_{-j}\,\mathrm{e}^{-\mathrm{i}jt} = \sum_{-N}^{N}\Big(1 - \frac{|j|}{N+1}\Big) \hat{\boldsymbol{m}}(-j)\,\mathrm{e}^{-\mathrm{i}jt}$$

$$= \frac{1}{2\pi}\int \sum_{-N}^{N}\Big(1 - \frac{|j|}{N+1}\Big)\,\mathrm{e}^{-\mathrm{i}j(t-s)}\,\mathrm{d}\boldsymbol{m}(t) = \int K_N(s-t)\,\mathrm{d}\boldsymbol{m}(t) \geq 0$$

since $\boldsymbol{m}$ is positive, and Féjer's kernel $K_n$ is nonnegative. So we have $\sigma_N(Y,t) \geq 0$ on $\mathbf{T}$.

Assuming $\sigma_N(Y,t) \geq 0$ we obtain

$$\|S_N(Y)\| = \sup_{\|f\|\leq 1}\left|\int f(t)\sigma_N(Y,t)\,\mathrm{d}t\right| = \frac{1}{2\pi}\int \sigma_N(Y,t)\,\mathrm{d}t = y_0\,,$$

and by Theorem 5,

$$\sum_{j\in\mathbb{Z}} y_j\,\mathrm{e}^{\mathrm{i}jx}$$

is the Fourier-Stieltjes series for some $\boldsymbol{m} \in BV^o(\mathbf{T}, Y)$. For arbitrary nonnegative $f \in C(\mathbf{T})$

$$\int f(t)\,\mathrm{d}\boldsymbol{m}(t) = \lim_{N\to\infty}\frac{1}{2\pi}\int f(t)\sigma_N(Y,t)\,\mathrm{d}t \geq 0\,,$$

hence

$$U\colon f \to \int f(t)\,\mathrm{d}\boldsymbol{m}(t)$$

defines a positive linear operator on $C(\mathbf{T})$ into $Y$ which can be extended ([2; 5.1.2, Theorem]) to the positive linear operator (denoted again by) $U$ defined on the complete vector lattice containing characteristic functions $c_A$ of Borel sets $A$ in $\mathbf{T}$. From the definition ([3; Theorem]), $\boldsymbol{m}(A) = U(c_A)$, and it follows that $\boldsymbol{m}$ is positive. □

It is not unexpected that Theorem 7 gives rise to a representation of positive definite functions defined in a suitable sense, analogous to those known for complex-valued positive definite functions.

Suppose that $(y_n)$, $n = 0, \pm 1, \pm 2, \dots$, is a two-way sequence of elements in a vector lattice $Y$. Then it is called positive definite if for any sequence $(c_n)$ of complex numbers having only a finite number of terms different from zero we have

$$\sum_{m,n} c_n \overline{c_m} y_{n-m} \geq 0 .$$

**THEOREM 8.** *Let $Y$ be a complete vector lattice. A necessary and sufficient condition for a sequence $(y_n)_{n=-\infty}^{\infty} \subset Y$ to be positive definite is that there exists a positive measure $\boldsymbol{m}\colon B(\mathbf{T}) \to Y$ such that $y_n = \hat{\boldsymbol{m}}(n)$ for all $n$.*

Proof. Assume $y_j = \hat{\boldsymbol{m}}(j)$ with $\boldsymbol{m}\colon B(\mathbf{T}) \to Y$ positive, then

$$\begin{aligned} \sum_{m,n} c_n \overline{c_m} y_{n-m} &= \int \Big( \sum_{m,n} c_n \overline{c_m}\, \mathrm{e}^{\mathrm{i}(n-m)t} \Big)\, \mathrm{d}\boldsymbol{m}(t) \\ &= \int \Big| \sum_{n} c_n\, \mathrm{e}^{\mathrm{i}nt} \Big|^2 \,\mathrm{d}\boldsymbol{m}(t) \geq 0 . \end{aligned}$$

Conversely, if the sequence $y_j$ is positive definite, and we take $c_l = \mathrm{e}^{\mathrm{i}lt}$, then

$$\sum_{m,n}^{N} c_n \overline{c_m} y_{n-m} = (N+1)\sigma_N(Y,t) \geq 0 ,$$

and it is enough to apply Theorem 7. □

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