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Dedicated to the memory
of Professor Milan Kolibiar

M-SOLID VARIETIES AND Q-FREE CLONES

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(*Communicated by Tibor Katriňák*)

ABSTRACT. A variety of algebras is called solid if every identity is satisfied as a hyperidentity. The clone of a solid variety is free with respect to itself. M-solid varieties generalize the concept of solidity. In this paper, we describe the clone of an arbitrary M-solid variety.

Introduction

An identity $t \approx t'$ is called a *hyperidentity* in a variety V if whenever the operation symbols occurring in t and t' are replaced by any terms of the appropriate arity, the identity which results holds in V. Hyperidentities are particular sentences in a second order language and were considered at first by Belousov ([2]), Aczél ([1]), and Taylor ([19]). For a survey on these topics, see [17] and [5].

An easy example is the identity

$$(x+x)+(y+y) \approx (x+y)+(x+y)$$

satisfied in any abelian group. Replacing the group operation by a binary operation symbol F we get

$$F\big(F(x,x),F(y,y)\big) \approx F\big(F(x,y),F(x,y)\big)\,.$$

If we substitute for F any binary term $f(x,y) = ax+by$ (a, b integers) of the variety of all abelian groups, we get identities. The commutative law $F(x,y) \approx$

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$F(y,x)$ shows that the concept of a hyperidentity is very strong since we have to take into account any mapping which assigns to the binary operation symbol F a binary term. Under these mappings, there is also the mapping with $F \mapsto x$. The resulting identity $x \approx y$ is only satisfied in a trivial variety. Therefore, in [10], we generalized hyperidentities to so called *M-hyperidentities*, where M is a submonoid of the monoid of all such substitutions. If every identity in the variety V is an M-hyperidentity, then the variety V is called *M-solid*, or *solid* if M consists of all possible substitutions of n-ary terms for n-ary operation symbols.

Clones are sets of operations defined on the same set, closed under superposition, and containing all projections. Hyperidentities in the variety V correspond to identities in the clone of V ([18], [16]). By clone(V), the clone of a variety V, we mean a heterogeneous algebra with carrier sets $\mathcal{F}_V^n(X)$ (the sets of all n-ary term operations of the V-free algebra freely generated by the n-element alphabet X_n) with operations describing the superposition of term operations and containing all projections e_i^n. If $\{f_i \mid i \in I\}$ are the operation symbols of V, then the family $\{f_i^{\mathcal{F}_V(X)} \mid i \in I\}$ of all fundamental operations of the free algebra $\mathcal{F}_V(X)$ forms a generating system of clone(V). In [9], we proved that the variety V is solid if and only if the heterogeneous algebra clone(V) is free relative to itself. In this paper, we will generalize this result to M-solid varieties.

Preliminaries

Hyperidentities can be defined more precisely using the concept of a hypersubstitution ([7]). We fix a type $\tau = \{n_i \mid i \in I\}$, $n_i \geq 1$ for all $i \in I$, and operation symbols $\{f_i \mid i \in I\}$, where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet $X = \{x_1, x_2, \dots\}$. Terms in $W_\tau(X_n)$ with $X_n = \{x_1, x_2, \dots, x_n\}$, $n \geq 1$, are called n-ary. Let Alg(τ) be the class of all algebras of type τ.

A mapping

$$\sigma\colon \{f_i \mid i \in I\} \to W_\tau(X)$$

which assigns to every n_i-ary operation symbol f_i an n_i-ary term of type τ will be called a *hypersubstitution of type* τ (for short, a hypersubstitution). The mapping σ can be extended to all terms in $W_\tau(X)$. The result of applying a hypersubstitution σ to a term $t \in W_\tau(X)$ will be denoted by $\hat{\sigma}[t]$. More precisely, $\hat{\sigma}[t]$ can be defined inductively by:

(i) $\hat{\sigma}[x] := x$ for any variable x in the alphabet X, and
(ii) $\hat{\sigma}\big[f_i(t_1, \dots, t_{n_i})\big] := \sigma(f_i)^{\mathcal{F}_\tau(X)}\big(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}]\big)$.

Here, $\sigma(f_i)^{\mathcal{F}_\tau(X)}$ on the right hand side of (ii) denotes the term operation induced by the term $\sigma(f_i)$ on the term algebra $\mathcal{F}_\tau(X) = \left(W_\tau(X); \left(f_i^{\mathcal{F}_\tau(X)}\right)_{i\in I}\right)$, $f_i^{\mathcal{F}_\tau(X)}: (t_1,\dots,t_{n_i}) \mapsto f_i(t_1,\dots,t_{n_i})$.

Let t, t' be terms of type τ. The identity $t \approx t'$ is called a *hyperidentity of type* τ (for short, a hyperidentity) in an algebra $\mathcal{A} \in \mathrm{Alg}(\tau)$ if $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ are identities in $\mathcal{A}$ for every hypersubstitution σ. In this case, we say $\mathcal{A}$ *hypersatisfies* the equation $t \approx t'$. For two hypersubstitutions σ_1, σ_2 of type τ the product $\sigma_1 \circ_h \sigma_2$ defined by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ is again a hypersubstitution of type τ. It is easy to show ([10]) that all hypersubstitutions of type τ form a monoid $(\mathrm{Hyp}(\tau); \circ_h, \sigma_{\mathrm{id}})$, where σ_{id} is defined by $\sigma_{\mathrm{id}}(f_i)(x_1,\dots,x_{n_i}) = f_i(x_1,\dots,x_{n_i})$ for all $i \in I$. Let M be a submonoid of $(\mathrm{Hyp}(\tau); \circ_h, \sigma_{\mathrm{id}})$. The elements of M are called *M-hypersubstitutions*. Then an equation $s \approx t$, $s,t \in W_\tau(X)$, is an *M-hyperidentity* of type τ in an algebra $\mathcal{A} \in \mathrm{Alg}(\tau)$ if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in $\mathcal{A}$ for every M-hypersubstitution σ. If every identity of a variety V is an M-hyperidentity in any algebra of V, then V is said to be *M-solid*.

Q-free clones

A *clone* as a set of operations defined on the same set, closed under superposition, and containing all projections can be equipped with an algebraic structure which gives a *heterogeneous* (*many-sorted, multibased*) algebra ([12], [3])

$$\mathcal{C} := \left((C^{(n)})_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (e^n_i)_{n\in\mathbb{N}^+,\,1\le i\le n}\right) \qquad (\mathbb{N}^+ := \{1,2,\dots\}),$$

where $C^{(n)}$ is a set of n-ary operations defined on the set A, and where S^n_m are the operations defined by

$$S^n_m := C^{(n)} \times \left(C^{(m)}\right)^n \to C^{(m)}$$

with

$$S^n_m(f, g_1, \dots, g_n) := f[g_1,\dots,g_n],$$

and $f[g_1,\dots,g_n](a_1,\dots,a_m) := f\big(g_1(a_1,\dots,a_m),\dots,g_n(a_1,\dots,a_m)\big)$ for all $a_1,\dots,a_m \in A$. The e^n_i, $1 \le i \le n$, are the n-ary projections with $e^n_i(a_1,\dots,a_n) := a_i$ for all $a_1,\dots,a_n \in A$.

To every one-based algebra $\mathcal{A} = \left(A; (f_i^{\mathcal{A}})_{i\in I}\right)$ of type τ it belongs a clone, the clone of all *term operations* of $\mathcal{A}$. Let $O_A^{(n)}$ be the set of all n-ary operations $f^A: A^n \to A$, and put $O_A := \bigcup\limits_{n=1}^{\infty} O_A^{(n)}$. We set $F^A := \{f_i^A \mid i \in I\}$ and $F^{A(n)} := F^A \cap O_A^{(n)}$. Let $\mathcal{O}_A$ be the heterogeneous clone where the carrier sets are the sets $O_A^{(n)}$ for every $n \in \mathbb{N}^+$. Then the clone $\mathcal{T}(\mathcal{A})$ of all term operations of $\mathcal{A}$

is the subclone of $\mathcal{O}_A$ generated by $(F^{A(n)})_{n\in\mathbb{N}^+}$: $\mathcal{T}(\mathcal{A}) := \langle (F^{A(n)})_{n\in\mathbb{N}^+}\rangle_{\mathcal{O}_A}$. The carrier sets of $\mathcal{T}(\mathcal{A})$ are the sets $T^{(n)}(\mathcal{A})$ of all n-ary term operations of $\mathcal{A}$ $(n \geq 1)$. For $\mathcal{A} = \mathcal{F}_\tau(X)$ (the absolutely free algebra of type τ, for short written as $\mathcal{F}_\tau$), instead of $\mathcal{T}(\mathcal{A})$, we will write clone(τ), and if $\mathcal{F}_V(X)$ is the free algebra with respect to V, we write clone(V) instead of $\mathcal{T}(\mathcal{F}_V(X))$.

We remark further that all clones are elements of the variety K_0 of heterogeneous algebras which is defined by the following identities ([18]).

(C1) $S^p_m(z, S^n_m(y_1, x_1, \dots, x_n), \dots, S^n_m(y_p, x_1, \dots, x_n))$
$\approx S^n_m(S^p_n(z, y_1, \dots, y_p), x_1, \dots, x_n) \quad (m, n, p \in \mathbb{N}^+)$,

(C2) $S^n_m(e^n_i, x_1, \dots, x_n) \approx x_i \quad (m \in \mathbb{N}^+,\ 1 \leq i \leq n)$,

(C3) $S^n_n(y, e^n_1, \dots, e^n_n) \approx y \quad (n \in \mathbb{N}^+)$

(here S^n_m, e^n_i are operation symbols corresponding to the type of clone(τ)).

An arbitrary element of the variety K_0 is called an *abstract clone*. It should be pointed out that every abstract clone is isomorphic to a clone of operations, i.e., to a concrete one. Note that a concrete clone is the dual category of an *algebraic theory* in the sense of F. W. Lawvere ([14]).

Definition 3.1. Let $\mathcal{C} := ((C^{(n)})_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (e^n_i)_{n\in\mathbb{N}^+, 1\leq i\leq n})$ be a clone, and let $(X_n)_{n\in\mathbb{N}^+}$, $X_n \subseteq C^{(n)}$, be a generating system of the clone $\mathcal{C}$. Then a system $\varphi = (\varphi_n)_{n\in\mathbb{N}^+}$ of mappings $\varphi_n \colon X_n \to C^{(n)}$ with $\varphi_n(e^n_i) = e^n_i$, $n \in \mathbb{N}^+$, for projections is called a *clone substitution*. By $\mathrm{Subst}_{\langle (X_n)_{n\in\mathbb{N}^+}\rangle}$, we denote the set of all clone substitutions.

Definition 3.2. ([15]) A set $I := (I_n)_{n\in\mathbb{N}^+}$, $I_n \subseteq C^{(n)}$ for every $n \in \mathbb{N}^+$, is said to be *independent with respect to a family* Q of mappings $\psi = (\psi_n)_{n\in\mathbb{N}^+}$, $\psi_n \colon I_n \to C^{(n)}$, ($Q$-*independent*) if every ψ can be extended to a homomorphism $\overline{\psi}$ of the subclone $\langle I\rangle_{\mathcal{C}}$ of $\mathcal{C}$ generated by I into $\mathcal{C}$, i.e., $\overline{\psi} \colon \langle I\rangle_{\mathcal{C}} \to \mathcal{C}$.

Properties of Q-independent sets are discussed in [11].

Definition 3.3. Let $\mathcal{C}$ be a clone, let $Q \subseteq \mathrm{Subst}_{\langle (X_n)_{n\in\mathbb{N}^+}\rangle}$, where $(X_n)_{n\in\mathbb{N}^+}$ is a generating system of $\mathcal{C}$. Then $\mathcal{C}$ is called Q-*free with respect to itself* if $(X_n)_{n\in\mathbb{N}^+}$ is Q-independent (i.e., $(X_n)_{n\in N^+}$ is a Q-basis, see [11]).

If $Q = \mathrm{Subst}_{\langle (X_n)_{n\in\mathbb{N}^+}\rangle}$, we have the usual concept of freeness with respect to itself.

The extensions $\hat{\varphi}$ of elements $\varphi \in \mathrm{Subst}_{\langle (X_n)_{n\in\mathbb{N}^+}\rangle}$ to arbitrary elements of $(C^{(n)})_{n\in\mathbb{N}^+}$ are defined in the usual inductive way. If $\varphi_1, \varphi_2 \in \mathrm{Subst}_{\langle (X_n)_{n\in\mathbb{N}^+}\rangle}$, we define a product $\varphi_1 \circ_s \varphi_2$ of substitutions by $\hat{\varphi}_1 \circ \varphi_2$. This is again a substitution from $\mathrm{Subst}_{\langle (X_n)_{n\in\mathbb{N}^+}\rangle}$. Since this product is associative, and since the identity φ_{id} belongs to $\mathrm{Subst}_{\langle (X_n)_{n\in\mathbb{N}^+}\rangle}$, we obtain a monoid.

PROPOSITION 3.4.

(i) *There is a bijection between the set* $\mathrm{Hyp}(\tau)$ *of all hypersubstitutions of type* τ, *and the set* $\mathrm{Subst}_{\langle (F^{\mathcal{F}_\tau(n)})_{n\in\mathbb{N}^+}\rangle}$ *of all clone substitutions of* $\mathrm{clone}(\tau)$.

(ii) *For every variety* V *of type* τ *every hypersubstitution of type* τ *defines a clone homomorphism* $\mathrm{clone}(\tau) \to \mathrm{clone}(V)$.

Proof.

(i): Let $\sigma\colon \{f_i \mid i \in I\} \to W_\tau(X)$ be a hypersubstitution of type τ. We put $F := \{f_i \mid i \in I\}$, and let $F^{(n)}$ be the set of all n-ary operation symbols from F. Then σ defines a family $\sigma := (\sigma_n)_{n\in\mathbb{N}^+}$ of mappings such that $\sigma_n\colon F^{(n)} \to W_\tau(X_n)$.

Note that $\mathrm{clone}(\tau)$ is generated by $\left(F^{\mathcal{F}_\tau(n)}\right)_{n\in\mathbb{N}^+}$.

For every $\sigma := (\sigma_{n_i})_{n_i\in\mathbb{N}^+}$ we define a family $\varphi := (\varphi_{n_i})_{n_i\in\mathbb{N}^+}$ of mappings $\varphi_{n_i}\colon F^{\mathcal{F}_\tau(n_i)} \to \mathrm{clone}^{(n_i)}(\tau)$ (here $\mathrm{clone}^{(n_i)}(\tau)$ is the n_ith carrier set of $\mathrm{clone}(\tau)$) by

$$\varphi_{n_i}\left(f_i^{\mathcal{F}_\tau(X)}\right) = \sigma_{n_i}\left(f_i^{\mathcal{F}_\tau(X)}\right).$$

Note that $\sigma_{n_i}(f_i)^{\mathcal{F}_\tau(X)}$ is the term operation of $\mathcal{F}_\tau(X)$ induced by the term $\sigma(f_i)$, i.e., $\sigma_{n_i}(f_i)^{\mathcal{F}_\tau(X)}$ is an element of $clone^{(n_i)}(\tau)$. Remember that any n-ary term of $W_\tau(X_n)$ induces an n-ary element of $\mathrm{clone}^{(n)}(\tau)$ in the following inductive way:

(1) if x_i is an element of X_n, (an n-ary variable), then $x_i^{\mathcal{F}_\tau(X)} := e_i^{n,\mathcal{F}} \in \mathrm{clone}^{(n)}(\tau)$,

(2) if $f_i(t_1,\dots,t_{n_i})$ is a composed term, and if $t_i^{\mathcal{F}_\tau(X)}$, $i = 1,\dots,n_i$ are the n-ary term operations induced by t_i, then we define

$$[f_i(t_1,\dots,t_{n_i})]^{\mathcal{F}_\tau(X)} = S_n^{n_i}\left(f_i^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)},\dots,t_{n_i}^{\mathcal{F}_\tau(X)}\right) \in \mathrm{clone}^{(n)}(\tau).$$

Therefore $\varphi\colon F^{\mathcal{F}_\tau(X)} \to \mathrm{clone}(\tau)$ is really a clone substitution. By definition, σ defines φ uniquely.

Conversely, assume that $\varphi\colon F^{\mathcal{F}_\tau(X)} \to \mathrm{clone}(\tau)$ is a clone substitution. Then for each f_i we choose a term $\sigma(f_i) \in W_\tau(X)$ such that $\sigma(f_i)^{\mathcal{F}_\tau(X)} = \varphi\left(f_i^{\mathcal{F}_\tau(X)}\right)$. It is clear that $\sigma\colon \{f_i \mid i \in I\} \to W_\tau(X)$ is a hypersubstitution, and the image of this hypersubstitution is the clone substitution φ. Clearly, φ defines σ uniquely.

(ii): We are going to show that the mapping $\hat{\varphi}\colon \mathrm{clone}(\tau) \to \mathrm{clone}(V)$ defined by $t^{\mathcal{F}_\tau(X)} \mapsto \hat{\sigma}[t]^{\mathcal{F}_V(X)}$ is an homomorphism $\hat{\varphi}$ of $\mathrm{clone}(\tau)$. Because of the bijection $t \mapsto t^{\mathcal{F}_V(X)}$ for every $t \in W_\tau(X)$ mentioned above the mapping φ is well-defined.

Since $e_i^{n,\mathcal{F}_\tau(X)} = t^{\mathcal{F}_\tau(X)}$ for $t = x_i \in W_\tau(X_n)$, we have

$$\varphi\left(e_i^{n,\mathcal{F}_\tau(X)}\right) = \varphi_n\left(x_i^{\mathcal{F}_\tau(X)}\right) = \hat{\sigma}(x_i)^{\mathcal{F}_V(X)} = x_i^{\mathcal{F}_V(X)} = e_i^{\mathcal{F}_V(n)}.$$

Thus projections are mapped to projections. Now, for $t \in W_\tau(X_n)$, $t_1, \dots, t_n \in W_\tau(X_m)$, it is easy to prove by induction on the complexity of term definition of t and by the axioms (C1) and (C2), that

$$\hat{\varphi}\big(S^n_m\big(t^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, \dots, t_n^{\mathcal{F}_\tau(X)}\big)\big) = S^n_m\big(\hat{\varphi}\big(t^{\mathcal{F}_\tau(X)}\big), \hat{\varphi}\big(t_1^{\mathcal{F}_\tau(X)}\big), \dots, \hat{\varphi}\big(t_n^{\mathcal{F}_\tau(X)}\big)\big). \tag{$*$}$$

□

Note that Proposition 3.4. (i) expresses the well-known fact that hypersubstitutions of type τ and clone substitutions of clone(τ) are essentially the same thing if the generating family of clone(τ) consists of the basic operations of the free algebra $\mathcal{F}_\tau(X)$. The reason for that is the natural bijection between terms of type τ and the term operations of the absolutely free algebra $\mathcal{F}_\tau(X)$ on countably many generators.

Since clone(V) is the quotient algebra clone$(\tau)/\operatorname{Id} V$, where $\operatorname{Id} V$ has to be regarded as a heterogeneous fully invariant congruence on clone(τ), there is a natural homomorphism

$$\mathrm{nat}_V \colon \mathrm{clone}(\tau) \to \mathrm{clone}(V).$$

The homomorphisms from Proposition 3.4. (ii) are compositions of the extensions of clone substitutions corresponding to hypersubstitutions (which exist since clone(τ) is free with $\big(F^{\mathcal{F}_\tau(X)(n)}\big)_{n\in\mathbb{N}^+}$ as free generating system) and nat_V.

As a consequence of Proposition 3.4, we have:

Corollary 3.5. *The monoid* $(\mathrm{Hyp}(\tau); \circ_h, \sigma_{\mathrm{id}})$ *is isomorphic to the monoid* $\big(\mathrm{Subst}_{\langle (F^{\mathcal{F}_\tau(n)})_{n\in\mathbb{N}^+}\rangle}; \circ_s, \varphi_{\mathrm{id}}\big)$, *where* $\circ_s$ *is defined by* $\varphi_1 \circ_s \varphi_2 := \hat{\varphi}_1 \circ_s \hat{\varphi}_2$, *and where* φ_{id} *is the identical clone substitution of* clone(τ).

P r o o f. By Lemma 2.3. (i), we have a bijection between $\mathrm{Hyp}(\tau)$ and $\mathrm{Subst}_{\langle (F^{\mathcal{F}_\tau(n)})_{n\in\mathbb{N}^+}\rangle}$. Further we have $\varphi_{\mathrm{id}}\big(f_i^{\mathcal{F}_\tau(X)}\big) = f_i^{\mathcal{F}_\tau(X)} = f_i(x_1, \dots, x_{n_i})^{\mathcal{F}_\tau(X)} = \sigma_{\mathrm{id}}(f_i)^{\mathcal{F}_\tau(X)}$, and if $\sigma_1, \sigma_2 \in \mathrm{Hyp}(\tau)$, then $(\sigma_1 \circ_h \sigma_2)(f_i)^{\mathcal{F}_\tau(X)} = \hat{\sigma}_1\big[\sigma_2(f_i)\big]^{\mathcal{F}_\tau(X)} = \hat{\varphi}_1\big(\sigma_2(f_i)^{\mathcal{F}_\tau(X)}\big) = \hat{\varphi}_1\big(\varphi_2\big(f_i^{\mathcal{F}_\tau(X)}\big)\big) = (\varphi_1 \circ_s \varphi_2)\big(f_i^{\mathcal{F}_\tau(X)}\big)$. □

If $M \subseteq \mathrm{Hyp}(\tau)$ is a submonoid of the monoid of all hypersubstitutions of type τ, then by Proposition 3.4, there is a subset $Q \subseteq \mathrm{Subst}_{\mathrm{clone}(\tau)}$ corresponding to M. Now we are asking whether a similar proposition is true for clone(V) if V is an M-solid variety of type τ.

Lemma 3.6. *Let V be an M-solid variety of type τ, and let* clone(V) *be the clone of all term operations of the V-free algebra $\mathcal{F}_V(X)$. Then to M it corresponds a set of clone substitutions of* clone(V).

Proof. $\{f_i^{\mathcal{F}_V(X)} \mid i \in I\}$ is a generating system of clone(V). For any $\sigma \in M$ we define a mapping

$$\varphi_V^\sigma \colon \{f_i^{\mathcal{F}_V(X)} \mid i \in I\} \to \mathrm{clone}(V)$$

by $\varphi_V^\sigma(f_i^{\mathcal{F}_V(X)}) = \sigma(f_i)^{\mathcal{F}_V(X)}$ ($\sigma(f_i)^{\mathcal{F}_V(X)}$ is the term induced by $\sigma(f_i)$ on the V-free algebra $\mathcal{F}_V(X)$). We show that φ_V^σ is well-defined: Assume that $f_i^{\mathcal{F}_V(X)} = f_j^{\mathcal{F}_V(X)}$, then $f_i(x_1,\dots,x_{n_i}) \approx f_j(x_1,\dots,x_{n_j}) \in \mathrm{Id}(V)$ (here $\mathrm{Id}(V)$ denotes the set of all identities satisfied in V.) Since V is M-solid for every $\sigma \in M$, we have

$$\hat\sigma[f_i(x_1,\dots,x_{n_i})] \approx \hat\sigma[f_j(x_1,\dots,x_{n_j})] \in \mathrm{Id}(V),$$

and by definition of the extension $\hat\sigma$, we have

$$\sigma(f_i)^{\mathcal{F}_V(X)}(x_1^{\mathcal{F}_V(X)},\dots,x_{n_i}^{\mathcal{F}_V(X)}) = \sigma(f_j)^{\mathcal{F}_V(X)}(x_1^{\mathcal{F}_V(X)},\dots,x_{n_j}^{\mathcal{F}_V(X)}),$$

and thus

$$\begin{aligned} &S_n^{n_i}(\sigma(f_i)^{\mathcal{F}_V(X)}, e_1^{n_i,\mathcal{F}_V(X)},\dots,e_{n_i}^{n_i,\mathcal{F}_V(X)}) \\ &= S_n^{n_j}(\sigma(f_j)^{\mathcal{F}_V(X)}, e_1^{n_j,\mathcal{F}_V(X)},\dots,e_{n_j}^{n_i,\mathcal{F}_V(X)}). \end{aligned}$$

By axiom (C3), it follows $\sigma(f_i)^{\mathcal{F}_V(X)} = \sigma(f_j)^{\mathcal{F}_V(X)}$, and by definition of φ_V^σ, we have $\varphi_V^\sigma(f_i^{\mathcal{F}_V(X)}) = \varphi_V^\sigma(f_j^{\mathcal{F}_V(X)})$. Since $\sigma(f_i)^{\mathcal{F}_V(X)}$ is an n_i-ary operation from clone(V), the mapping φ_V^σ can be regarded as a family $\varphi_V^\sigma = ((\varphi_V^\sigma)_n)_{n\in\mathbb{N}^+}$. For projections in $\{f_i^{\mathcal{F}_V(X)} \mid i \in I\}$ we have

$$\begin{aligned} \varphi_V^\sigma(e_i^{n_i,\mathcal{F}_V(X)}) &= \sigma(e_i^{n_i})^{\mathcal{F}_V(X)} = \hat\sigma(e_i^{n_i}(x_1,\dots,x_{n_i}))^{\mathcal{F}_V(X)} \\ &= \hat\sigma(x_i)^{\mathcal{F}_V(X)} = x_i^{\mathcal{F}_V(X)} = e_i^{n_i,\mathcal{F}_V(X)}. \end{aligned}$$

This shows that φ_V^σ is a clone substitution of clone(V). If, conversely, φ_V^σ is a clone substitution of clone(V), then it defines a hypersubstitution σ with $\sigma(f_i)^{\mathcal{F}_V(X)} = \varphi_V^\sigma(f_i^{\mathcal{F}_V(X)})$ for every $i \in I$. □

To prove that φ_V^σ is well-defined, we needed that if two operation symbols induce the same term operations of $\mathcal{F}_V(X)$, then their images under a hypersubstitution σ also have these properties. We define:

Definition 3.7. A hypersubstitution σ of type τ is called *meaningful* for the variety V of type τ from $f_i^{\mathcal{F}_V(X)} = f_j^{\mathcal{F}_V(X)}$, it follows that $\sigma(f_i)^{\mathcal{F}_V(X)} = \sigma(f_j)^{\mathcal{F}_V(X)}$.

Now let Q_M be the set of clone substitutions of clone(V) corresponding to the submonoid M of Hyp(τ) by Proposition 3.4. (i). Then we obtain the following characterization of M-solidity:

THEOREM 3.8. *For a submonoid $M \subseteq \mathrm{Hyp}(\tau)$ the variety V of type τ is M-solid if and only if each $\sigma \in M$ is meaningful for V, and for $Q_M = \{\varphi_V \mid \sigma \in M\}$ the algebra* clone(V) *is Q_M-free with respect to itself with Q_M-basis $F^{\mathcal{F}_V(X)}$.*

P r o o f . Assume that V is M-solid. By Lemma 3.6, every $\sigma \in M$ is meaningful for V. Let $\varphi\colon \{f_i^{\mathcal{F}_V(X)} \mid i \in I\} \to \mathrm{clone}(V)$ be an element of Q_M (the set of all clone substitutions of clone(V) corresponding by Lemma 3.6 to M). By definition of φ, there is a hypersubstitution $\sigma \in M$ such that for every $i \in I$ we have $\varphi(f_i^{\mathcal{F}_V(X)}) = \sigma(f_i)^{\mathcal{F}_V(X)}$. We are going to show that φ can be extended to a clone endomorphism of clone(V). Clearly, $\{f_i^{\mathcal{F}_V(X)} \mid i \in I\}$ is a generating system of clone(V). The mapping $\mathrm{clone}(\tau) \to \mathrm{clone}(V)\colon t \mapsto t^{\mathcal{F}_V(X)}$ is obviously a surjective homomorphism with the kernel $\mathrm{Id}(V)$. For any $\sigma \in M$, $\sigma[\mathrm{Id}(V)]$ is the kernel of the homomorphism $\mathrm{clone}(\tau) \to \mathrm{clone}(V)\colon t \mapsto \hat{\sigma}[t]^{\mathcal{F}_V(X)}$ considered in Proposition 3.4. (ii). Since V is M-solid, every identity of V is an M-hyperidentity, that means, $\mathrm{Id}(V) \subseteq \sigma[\mathrm{Id}(V)]$. By the general homomorphism theorem, there exists an homomorphism $\mathrm{clone}(V) \to \mathrm{clone}(V)\colon t^{\mathcal{F}_V(X)} \mapsto \hat{\sigma}[t]^{\mathcal{F}_V(X)}$, and this homomorphism extends φ. So, clone(V) is Q_M-free with respect to itself, and $\left(F^{\mathcal{F}_V(X)(n)}\right)_{n\in\mathbb{N}^+}$ is a Q_M-free independent generating system.

Conversely, we assume that clone(V) is Q_M-free freely generated by the Q_M-independent set $\left(F^{\mathcal{F}_V(X)(n)}\right)_{n\in\mathbb{N}^+}$. That means, every $\varphi \in Q_M$ can be extended to a clone endomorphism of clone(V). Since every $\sigma \in M$ is meaningful for V from $f_i^{\mathcal{F}_V(X)} = f_j^{\mathcal{F}_V(X)}$, we obtain $\varphi(f_i^{\mathcal{F}_V(X)}) = \sigma(f_i)^{\mathcal{F}_V(X)} = \sigma(f_j)^{\mathcal{F}_V(X)} = \varphi(f_j^{\mathcal{F}_V(X)})$.

If $t \approx t' \in \mathrm{Id}(V)$, then $t^{\mathcal{F}_V(X)} = t'^{\mathcal{F}_V(X)}$, and applying the extension of φ we get $\hat{\varphi}(t^{\mathcal{F}_V(X)}) = \hat{\varphi}(t'^{\mathcal{F}_V(X)})$, and thus $\hat{\sigma}[t']^{\mathcal{F}_V(X)}$, i.e., $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in \mathrm{Id}(V)$. This is true for any $\sigma \in M$ and $t \approx t'$ is an M-hyperidentity. □

Examples

In [8], all clones generated by a single unary operation $f^{\mathcal{A}}$ which are free with respect to itself, i.e., Q-free with respect to itself for $Q = \mathrm{Hyp}(1)$ were determined. By Theorem 3.8, these clones can be regarded as clones of term operations of algebras of type $\tau = (1)$ which generate solid varieties. An algebra $\mathcal{A} = (A; f^{\mathcal{A}})$ of type $\tau = (1)$ is called a *mono-unary* algebra (or 1-*unoid*). Instead of $f^{\mathcal{A}}$, we will write f. As usual, we define powers f^k of f by $f^0(x) := x$ and $f^k(x) = f(f^{k-1}(x))$, $k \geq 1$. Every variety of mono-unary algebras is defined either by an identity of the form

$$f^k(x) \approx f^l(x) \quad (k, l \in \{0, 1, 2, \dots\})$$

or by an identity of the form

$$f^k(x) \approx f^k(y) \quad (k \geq 1) \qquad \text{(see, e.g., [13])}.$$

Identities of the second form cannot be hyperidentities since, by the substitution $f \mapsto \mathrm{id}_A$, we get $x \approx y$ (id_A denotes the identity function on A).

For $f\colon A \to A$ let $\operatorname{Im} f := \{f(a) \mid a \in A\}$ be the image of f, and let $\lambda(f)$ denote the least non-negative integer m such that $\operatorname{Im} f^m = \operatorname{Im} f^{m+1}$.

In [8], it was proved:

LEMMA 4.1. ([8]) *The clone $\langle f\rangle_{\mathcal{O}_A}$ generated by a single unary function defined on A is free with respect to itself if and only if $|\operatorname{Im} f^{\lambda(f)}| > 1$ (i.e., if $\langle f\rangle_{\mathcal{O}_A}$ contains no constant operation).*

Let σ_x be the hypersubstitution $\sigma_x\colon f \mapsto x$. Then every hypersubstitution different from σ_x is called a *pre-hypersubstitution of type* $\tau = (1)$. All pre-hypersubstitutions of type $\tau = (1)$ form a monoid M, and we can consider pre-hyperidentities and presolid varieties ([4]). By Theorem 3.8, the clone of a presolid variety is Q_M-free with the set $\{f\}$ consisting of the only (unary) fundamental operation f as Q_M-independent generating set. In this case, we will speak of pre-free clones and pre-independent sets. Now we have:

PROPOSITION 4.2. *Every clone $\langle f\rangle_{\mathcal{O}_A}$ generated by a single unary operation defined on A is pre-free relative to itself with $\{f\}$ as pre-independent generating set.*

P r o o f. We show that every algebra $\mathcal{A} = (A; f^A)$ with one unary fundamental operation is presolid (generates a presolid variety). Obviously, $(A; \mathrm{id}^A)$ is solid and thus, presolid. Assume that $f^A \neq \mathrm{id}^A$. If $\mathcal{A}$ satisfies an identity of the form $f^k(x) \approx f^l(x)$, then by the hypersubstitution $f \mapsto f^m$, $m \geq 1$, we obtain $(f^m)^k \approx (f^m)^l \in \operatorname{Id} \mathcal{A}$, and if $\mathcal{A}$ satisfies an identity $f^k(x) \approx f^k(y)$, we get $(f^m)^k(x) \approx (f^m)^k(y) \in \operatorname{Id} \mathcal{A}$. □

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