Beloslav Riečan

Upper and lower limits of sequences of observables in D-posets of fuzzy sets

Mathematica Slovaca, Vol. 46 (1996), No. 4, 419--431

Persistent URL: http://dml.cz/dmlcz/136680

Mathematica Slovaca
©1996
Mathematical Institute
Slovak Academy of Sciences

Math. Slovaca, **46** (1996), No. 4, 419–431

Dedicated to the memory of Professor Milan Kolibiar

UPPER AND LOWER LIMITS OF SEQUENCES OF OBSERVABLES IN D-POSETS OF FUZZY SETS

BELOSLAV RIEČAN

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. A convergence theory is developed for sequences of observables. A probability space is constructed related to the sequence.

0. Introduction

The theory of D-posets (introduced in [3] and studied in many papers, e.g., in [2], [4], [5], [6], [7], [10], [11]) represents a very general structure containing many important models appearing in the quantum theory. Besides of quantum logics (= orthomodular posets), some families of fuzzy subsets of a set with the Lukasziewicz operations can be regarded as typical examples of D-posets. The model was suggested by Pykacz ([8]), the notion of an observable in the framework has been defined in [9].

Also the almost everywhere convergence of observables (to the zero observable) has been defined ([10]), and the strong law of large numbers has been proved ([4]). The main tool is a translation formula between the Kolmogorov theory and the D-poset theory. So the D-poset law of large numbers is an almost immediate consequence of the classical law of large numbers.

Of course, in the law of large numbers, the limit function x is known a priori, so we can say that the differences $x_n - x$ converge to 0. On the other hand, there are important results where the limit function is constructed a posteriori, so our translation formula can not be used.

AMS Subject Classification (1991): Primary 28E10.
Key words: D-poset, observable, almost everywhere convergence.

Research supported by Grants VEGA 2/1228/95 and 95/5305/471.

The aim of the paper is a modification of the almost everywhere convergence by the help of lim sup and lim inf and the construction of translations formulas for these limits. This modification gives a possibility to translate, e.g., the individual ergodic theorem. Of course, such a result will be presented in another paper.

1. Preliminaries

A partially ordered set $\mathcal{F}$ with the greatest element 1 and the least element 0 is called to be a *D-poset* if a partial binary operation $\setminus$ is defined assigning to every $a, b \in \mathcal{F}$ such that $a \le b$ an element $b \setminus a$ satisfying the following conditions:

(i) If $a \le b$, then $b \setminus a \le b$ and $b \setminus (b \setminus a) = a$.
(ii) If $a \le b \le c$, then $c \setminus b \le c \setminus a$ and $(c \setminus a) \setminus (c \setminus b) = b \setminus a$.

A *state* is a mapping $m \colon \mathcal{F} \to \langle 0, 1 \rangle$ such that

(i) $m(1) = 1$.
(ii) $m(b \setminus a) = m(b) - m(a)$ whenever $a \le b$.
(iii) $m(a) = \lim\limits_{n \to \infty} m(a_n)$ whenever $a_n \nearrow a$.

An *observable* is a mapping $x \colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ ($\mathcal{B}(\mathbb{R})$ is the family of all Borel subsets of $\mathbb{R}$) such that

(i) $x(\mathbb{R}) = 1$.
(ii) $x(\mathcal{B} \setminus A) = x(B) \setminus x(A)$ whenever $A \subset B$.
(iii) $x(A_n) \nearrow x(A)$, whenever $A_n \nearrow A$.

EXAMPLE. Let $(\Omega, \mathcal{S})$ be a measurable space, $\Omega \in \mathcal{S}$, $\mathcal{F}$ be the set of all $\mathcal{S}$-measurable functions $f \colon \Omega \to \langle 0, 1 \rangle$. By a theorem of Butnariu and Klement ([1]), every state $m \colon \mathcal{F} \to \langle 0, 1 \rangle$ can be represented by a probability measure $\mu \colon \mathcal{S} \to \langle 0, 1 \rangle$ as an integral

$$\mu(f) = \int_{\Omega} f \, \mathrm{d}\mu .$$

We shall call the family $\mathcal{F}$ described above a generated tribe.

THEOREM 1. *Let $\mathcal{F}$ be a generated tribe. Then every sequence $(x_n)_n$ of observables is compatible in the following sense: To every finite, non-empty set $J \subset \mathbb{N}$ there is a mapping $h_J \colon \mathcal{B}(\mathbb{R}^{|J|}) \to \mathcal{F}$ satisfying the following conditions:*

(i) $h_J(\mathbb{R}^{|J|}) = 1$.
(ii) $h_J(B \setminus A) = h_J(B) \setminus h_J(A)$ *whenever* $A \subset B$.
(iii) $h_J(A_i) \nearrow h_J(A)$ *whenever* $A_i \nearrow A$.

(iv) *If* $J_1 \subset J_2$ *and* π_{J_2,J_1} *is the projection, then*

$$m\big(h_{J_1}(A)\big) = m\big(h_{J_2}\big(\pi^{-1}_{J_2,J_1}(A)\big)\big) \qquad \textit{for every } A \in \mathcal{B}\big(\mathbb{R}^{|J_1|}\big).$$

(v) $m\big(h_{\{1,2,\dots,n\}}(A_1 \times A_2 \times \dots \times A_n)\big) = m\big(x_1(A_1)\cdot x_2(A_2)\cdot \ldots \cdot x_n(A_n)\big)$ *for every* $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathbb{R})$.

Proof. By [6; Theorem 1], to every n there exists a mapping $h_n\colon \mathcal{B}(\mathbb{R}^n) \to \mathcal{F}$ satisfying (i), (ii), (iii) and

$$h_n(A \times A_2 \times \dots \times A_n) = x_1(A_1)\cdot x_2(A_2)\cdot \ldots \cdot x_n(A_n)$$

for every $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathbb{R})$. If $J = \{t_1, \dots, t_k\}$, put $h_J = h_{t_k} \circ \pi^{-1}_{I,J}$, where $I = \{1, 2, \dots, t_k\}$. □

Recall that to the notion of an observable x in the quantum theory, there corresponds the notion of a random variable ξ, where $x(E) = \chi_{\xi^{-1}(E)}$. Similarly, the mapping h_J corresponds to the notion of a random vector, e.g., $h_{\{1,2\}}(A) = \chi_{T^{-1}(A)}$, where $T = (\xi_1, \xi_2)\colon \Omega \to \mathbb{R}^2$.

If $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a Borel measurable function, and $T = (\xi_1, \xi_2)\colon \Omega \to \mathbb{R}^2$ is a random vector, then $\eta = g \circ T = g(\xi_1, \xi_2)$ is a random variable. For its pre-image we obtain

$$\eta^{-1}(A) = (g \circ T)^{-1}(A) = T^{-1}\big(g^{-1}(A)\big).$$

This relation leads to the following definition of the observable $g(x_1, x_2)$:

$$g(x_1, x_2)(A) = h_{\{1,2\}}\big(g^{-1}(A)\big).$$

A generalization for the n-dimensional case is evident.

2. Upper and lower limits of sequences of observables

Since we want to define $\limsup\limits_{n\to\infty} x_n$ as an observable, we shall be inspired by the upper limit of a sequence $(\xi_n)_n$ of random variables. It is easy to see that

$$\limsup_{n\to\infty} \xi_n(\omega) < t \iff \omega \in \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \xi_n^{-1}\big((-\infty, t-\tfrac{1}{p})\big).$$

DEFINITION 1. We shall say that a sequence $(x_n)_n$ of observables has a *limes superior* if there exists an observable $\overline{x}$ such that

$$m\big(\overline{x}((-\infty, t))\big) = \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n\big((-\infty, t-\tfrac{1}{p})\big)\right)$$

for every $t \in \mathbb{R}$. If $\limsup\limits_{n\to\infty}$ exists, we shall denote

$$\overline{x} = \limsup_{n\to\infty} x_n \,.$$

If $\limsup\limits_{n\to\infty} x_n$ exists, then

$$m\Big(\limsup_{n\to\infty} x_n\big((-\infty,t)\big)\Big) = m\left(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\big((-\infty,t-\tfrac{1}{p}\rangle\big)\right)$$

for every $t \in \mathbb{R}$. We shall give a sufficient condition for the existence of $\limsup\limits_{n\to\infty} x_n$.

We shall say that a sequence $(x_n)_n$ of observables is *bounded* if there are observables y, z such that

$$y\big((-\infty,t)\big) \le x_n\big((-\infty,t)\big) \le z\big((-\infty,t)\big)$$

for every $t \in \mathbb{R}$ and every $n \in \mathbb{N}$.

PROPOSITION 1. *If a sequence $(x_n)_n$ of observables is bounded, then there exists $\limsup\limits_{n\to\infty} x_n$.*

P r o o f . For a fixed $\omega \in \Omega$ and arbitrary $t \in \mathbb{R}$ put

$$F_\omega(t) = \sup_{p\ge 1}\sup_{k\ge 1}\inf_{n\ge k} x_n\big((-\infty,t-\tfrac{1}{p}\rangle\big)(\omega)\,.$$

Evidently, F_ω is a non-decreasing function, $F_\omega\colon \mathbb{R} \to \langle 0,1\rangle$. Since $F_\omega(t) \le z\big((-\infty,t)\big)(\omega)$, we obtain

$$0 \le \lim_{t\to-\infty} F_\omega(t) \le \lim_{t\to-\infty} z\big((-\infty,t)\big)(\omega) = 0\,,$$

hence

$$\lim_{t\to-\infty} F_\omega(t) = 0\,. \tag{2.1}$$

Further,

$$\inf_{n\ge k} x_n\big((-\infty,t-\tfrac{1}{p}\rangle\big)(\omega) \ge y\big((-\infty,t-\tfrac{1}{p}\rangle\big)(\omega)\,,$$

hence

$$\begin{aligned} F_\omega(t) &= \sup_{p\ge 1}\sup_{k\ge 1}\inf_{n\ge k} x_n\big((-\infty,t-\tfrac{1}{p}\rangle\big)(\omega) \\ &\ge \sup_{p\ge 1} y\big((-\infty,t-\tfrac{1}{p}\rangle\big)(\omega) = y\left(\bigcup_{p=1}^{\infty}\big((-\infty,t-\tfrac{1}{p}\rangle\big)\right)(\omega) \\ &= y\big((-\infty,t)\big)(\omega)\,. \end{aligned}$$

Therefore

$$1 \geq \lim_{t\to\infty} F_\omega(t) \geq \lim_{t\to\infty} y\big((-\infty,t)\big)(\omega) = 1\,,$$

hence

$$\lim_{t\to\infty} F_\omega(t) = 1\,. \tag{2.2}$$

Finally, we shall prove that F_ω is left continuous in every $t \in \mathbb{R}$. Let $t_j \nearrow t$. Then there are $j,q \in \mathbb{N}$ such that $\big(-\infty, t-\frac{1}{p}\big\rangle \subset \big(-\infty, t_j-\frac{1}{q}\big\rangle$, hence

$$\begin{aligned}\inf_{n\geq k} x_n\big(\big(-\infty, t-\tfrac{1}{p}\big\rangle\big)(\omega) &\leq \inf_{n\geq k} x_n\big(\big(-\infty, t_j-\tfrac{1}{q}\big\rangle\big)(\omega)\\ &\leq F_\omega(t_j) \leq \lim_{j\to\infty} F_\omega(t_j)\,.\end{aligned}$$

Therefore

$$F_\omega(t) \leq \lim_{j\to\infty} F_\omega(t_j)\,.$$

Since evidently $F_\omega(t_j) \leq F_\omega(t)$, we obtain $F_\omega(t) = \lim\limits_{j\to\infty} F_\omega(t_j)$, hence

$$\lim_{s\to t-} F_\omega(s) = F_\omega(t)\,. \tag{2.3}$$

The relations (2.1)–(2.3) imply that the mapping $F_\omega \colon \mathbb{R} \to \langle 0,1\rangle$ is a distribution function. Denote by μ_ω the corresponding Stieltjes probability measure $\lambda_{F_\omega} : \mathcal{B}(\mathbb{R}) \to \langle 0,1\rangle$ determined by the equality

$$\mu_\omega\big(\langle a,b)\big) = F_\omega(b) - F_\omega(a)\,.$$

Finally, define $\overline{x} \colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ by the equality

$$\overline{x}(A)(\omega) = \mu_\omega(A)\,.$$

Now $\overline{x}(\mathbb{R})(\omega) = \mu_\omega(\mathbb{R}) = 1$ for every $\omega \in \Omega$, hence

$$\overline{x}(\mathbb{R}) = 1_\Omega\,.$$

If $A, B \in \mathcal{B}(\mathbb{R})$, $A \cap B = \emptyset$, then

$$\overline{x}(A\cup B)(\omega) = \mu_\omega(A\cup B) = \mu_\omega(A) + \mu_\omega(B) = \overline{x}(A)(\omega) + \overline{x}(B)(\omega)$$

for every $\omega \in \Omega$, hence

$$\overline{x}(A\cup B) = \overline{x}(A) + \overline{x}(B)\,.$$

Similarly, it can be proved the implication

$$A_n \nearrow A \implies \overline{x}(A_n) \nearrow \overline{x}(A)\,.$$

Hence, we have constructed an observable $\overline{x}$. Moreover

$$\overline{x}\big((-\infty,t)\big)(\omega) = \mu_\omega\big((-\infty,t)\big) = F_\omega(t)$$
$$= \sup_{p\geq 1}\sup_{k\geq 1}\inf_{n\geq k} x_n\big((-\infty, t-\tfrac{1}{p})\big)(\omega)\,,$$

hence

$$m\big(\overline{x}\big((-\infty,t)\big)\big) = m\left(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\big((-\infty, t-\tfrac{1}{p})\big)\right)$$
$$= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n\big((-\infty, t-\tfrac{1}{p})\big)\right).$$

□

DEFINITION 2. We shall say that a sequence $(x_n)_n$ of observables has a *limes inferior* if there exists an observable $\underline{x}$ such that

$$m\big(\underline{x}\big((-\infty,t)\big)\big) = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigvee_{n=k}^{k+i} x_n\big((-\infty, t-\tfrac{1}{p})\big)\right)$$

for every $t\in\mathbb{R}$. If $\liminf\limits_{n\to\infty}$ exists, we shall denote

$$\underline{x} = \liminf_{n\to\infty} x_n\,.$$

PROPOSITION 2. *If $(x_n)_n$ is bounded, then there exists $\liminf\limits_{n\to\infty} x_n$. Moreover,*

$$m\Big(\liminf_{n\to\infty} x_n\big((-\infty,t)\big)\Big) \geq m\Big(\limsup_{n\to\infty} x_n\big((-\infty,t)\big)\Big)$$

for every $t\in\mathbb{R}$.

Proof. The first assertion can be proved similarly as Proposition 1. Fix now $t\in\mathbb{R}$ and $\omega\in\Omega$. Evidently,

$$a_{kp} = \inf_{n\geq k} x_n\big((-\infty,\, t-\tfrac{1}{p})\big)(\omega) \leq \sup_{n\geq k} x_n\big((-\infty,\, t-\tfrac{1}{p})\big)(\omega) = b_{kp}\,.$$

For fixed p we have

$$a_{kp} \leq a_{k+1,p} \leq b_{k+1,p} \leq b_{k,p} \qquad \text{for every} \quad k\,.$$

Therefore,

$$\sup_k a_{kp} \leq \inf_k b_{kp} \qquad \text{for every} \quad p\,,$$

hence

$$\limsup_{n\to\infty} x_n\big((-\infty,t)\big)(\omega) = \sup_p\sup_k a_{kp} \leq \sup_p\inf_k b_{kp}$$
$$= \liminf_{n\to\infty} x_n\big((-\infty,t)\big)(\omega)\,.$$

□

DEFINITION 3. We shall say that a sequence $(x_n)_n$ of observables *converges m-almost everywhere* to an observable x if

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n\big((-\infty, t-\tfrac{1}{p})\big)\right) = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigvee_{n=k}^{k+i} x_n\big((-\infty, t-\tfrac{1}{p})\big)\right) = m\big(x\big((-\infty, t)\big)\big)$$

for every $t \in \mathbb{R}$.

Recall that in [10], the almost everywhere convergence has been defined by another form, of course, only in the case that the sequence $(x_n)_n$ converges to the zero observable O_F. Here

$$O_F(-\infty, t) = \begin{cases} 0_\Omega & \text{if } t \le 0\,, \\ 1_\Omega & \text{if } t > 0\,. \end{cases}$$

In [10], the almost everywhere convergence of a sequence $(x_n)_n$ to the zero observable is defined by the formula

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n\big((-\tfrac{1}{p}, \tfrac{1}{p})\big)\right) = 1\,,$$

or equivalently, by the formula

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigvee_{n=k}^{k+i} x_n\big(\mathbb{R} \setminus (-\tfrac{1}{p}, \tfrac{1}{p})\big)\right) = 0\,.$$

This definition has been inspired by the following characterization of almost everywhere convergence of a sequence $(\xi_n)_n$ of random variables to the zero variable:

$$P\left(\bigcap_{p=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{n=k}^{\infty} \xi_n^{-1}\big((-\tfrac{1}{p}, \tfrac{1}{p})\big)\right) = 1\,.$$

We shall show now that the convergence determined by Definition 3 is equivalent (in the case $x = O_F$) to the convergence determined in [10].

PROPOSITION 3. *A sequence $(x_n)_n$ of observables converges m-almost everywhere to O_F if and only if*

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigvee_{n=k}^{k+i} x_n\big(\mathbb{R} \setminus (-\tfrac{1}{p}, \tfrac{1}{p})\big)\right) = 0\,.$$

P r o o f .

$\Longrightarrow$:
Evidently,

$$x_n\big(\mathbb{R}\setminus(-\tfrac{1}{p},\tfrac{1}{p})\big)=x_n\big((-\infty,-\tfrac{1}{p}\rangle\big)+x_n\big(\langle\tfrac{1}{p},\infty)\big).$$

Therefore

$$\bigvee_{n=k}^{k+i}x_n\big(\mathbb{R}\setminus(-\tfrac{1}{p},\tfrac{1}{p})\big)\le\bigvee_{n=k}^{k+i}x_n\big((-\infty,-\tfrac{1}{p}\rangle\big)+\bigvee_{n=k}^{k+i}x_n\big(\langle\tfrac{1}{p},\infty)\big).$$

But

$$\begin{aligned}\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}m\left(\bigvee_{n=k}^{k+i}x_n\big((-\infty,-\tfrac{1}{p}\rangle\big)\right)&=m\Big(\liminf_{n\to\infty}x_n(-\infty,0)\Big)\\&=m\big(O_F((-\infty,0))\big)=0.\end{aligned}$$

and

$$\begin{aligned}\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}m\left(\bigvee_{n=k}^{k+i}x_n\big(\langle\tfrac{1}{p},\infty)\big)\right)&=1-\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}m\left(\bigwedge_{n=k}^{k+i}x_n\big((-\infty,\tfrac{1}{p})\big)\right)\\&=1-m\Big(\limsup_{n\to\infty}x_n\big((-\infty,0\rangle\big)\Big)\\&=1-m\big(O_F((-\infty,0\rangle)\big)=1-1=0.\end{aligned}$$

$\Longleftarrow$:
Since

$$m\left(\bigvee_{n=k}^{k+i}x_n\big((-\infty,t-\tfrac{1}{p}\rangle\big)\right)\le m\left(\bigvee_{n=k}^{k+i}x_n\big(\mathbb{R}\setminus(-\tfrac{1}{p},\tfrac{1}{p})\big)\right)$$

for $t\le0$, we obtain

$$\begin{aligned}0&\le\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}m\left(\bigwedge_{n=k}^{k+i}x_n\big((-\infty,-\tfrac{1}{p}\rangle\big)\right)\\&\le\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}m\left(\bigvee_{n=k}^{k+i}x_n\big((-\infty,-\tfrac{1}{p}\rangle\big)\right)\\&\le\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\left(\bigvee_{n=k}^{k+i}x_n\big(\mathbb{R}\setminus(-\tfrac{1}{p},\tfrac{1}{p})\big)\right)\\&=0=m\big(0_F((-\infty,t))\big).\end{aligned}$$

On the other hand, if $t > 0$, then there is p such that $\left(-\frac{1}{p}, \frac{1}{p}\right) \subset \left(-\infty, t-\frac{1}{p}\right\rangle$. Therefore

$$m\big(0_F((-\infty,t))\big) = 1 = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n\big(\big(-\tfrac{1}{p},\tfrac{1}{p}\big\rangle\big)\right)$$
$$\le \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n\big(\big(-\infty,t-\tfrac{1}{p}\big\rangle\big)\right)$$
$$\le \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n\big(\big(-\infty,t-\tfrac{1}{p}\big\rangle\big)\right).$$

□

3. The Kolmogorov model

Now we shall construct a probability space related to a sequence $(x_n)_n$ of observables. As a support the set $\mathbb{R}^{\mathbb{N}} = \{(u_i)_{i=1}^{\infty};\ u_i \in \mathbb{R}\}$ will be taken. If $J \subset \mathbb{N}$ is a non-empty finite set, then $\pi_J : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{|J|}$ is the projection, i.e.,

$$\pi_J\big((u_i)_{i=1}^{\infty}\big) = (u_{i_1}, \dots, u_{i_k}),$$

where $J = \{i_1, \dots, i_k\}$. By ξ_n $(n \in \mathbb{N})$, we denote the coordinate function $\xi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ defined by

$$\xi_n\big((u_i)_{i=1}^{\infty}\big) = u_n .$$

THEOREM 2. *There is a σ-algebra Σ of subsets of $\mathbb{R}^{\mathbb{N}}$ and a probability measure $P: \Sigma \to \langle 0,1\rangle$ satisfying the following conditions:*

(i) *$\pi_J^{-1}(A) \in \Sigma$ for every finite $J \subset \mathbb{N}$ and every $A \in \mathcal{B}(\mathbb{R}^{|J|})$.*
(ii) *$P(\pi_J^{-1}(A)) = m(h_J(A))$ for every $J \subset \mathbb{N}$ and $A \in \mathcal{B}(\mathbb{R}^{|J|})$.*
(iii) *If $\mu: \Sigma \to \langle 0,1\rangle$ is a probability measure satisfying* (ii), *then $\mu = P$.*

P r o o f. For finite $J \subset \mathbb{N}$ put $P_J = m \circ h_J : \mathcal{B}(\mathbb{R}^{|J|}) \to \langle 0,1\rangle$ (see Theorem 1). If $J_1 \subset J_2$, then

$$P_{J_2}\big(\pi_{J_2,J_1}(A)\big) = P_{J_1}(A)$$

by Theorem 1 (iv), hence the Kolmogorov consistency theorem is satisfied. Therefore the Kolmogorov extension theorem can be used. □

PROPOSITION 4. *Let $g_n \colon \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function ($n = 1, 2, \dots$), $h_n = h_{\{1,2,\dots,n\}}$. Then*

$$P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}}\,;\ \limsup_{n\to\infty} g_n\big(\xi_1(u), \dots, \xi_n(n)\big) < t\Big\}\Big)$$
$$\le \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} g_n(x_1, \dots, x_n)\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right),$$

$$P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}}\,;\ \liminf_{n\to\infty} g_n\big(\xi_1(u), \dots, \xi_n(u)\big) < t\Big\}\Big)$$
$$\ge \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigvee_{n=k}^{k+i} g_n(x_1, \dots, x_n)\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right).$$

Proof. We have

$$P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}}\,;\ \limsup_{n\to\infty} g_n\big(\xi_1(u), \dots, \xi_n(n)\big) < t\Big\}\Big)$$
$$= P\left(\bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{u \in \mathbb{R}^{\mathbb{N}}\,;\ g_n(u_1, \dots, u_n) \le t - \tfrac{1}{p}\} \right)$$
$$= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \{u \in \mathbb{R}^{\mathbb{N}}\,;\ g_n(u_1, \dots, u_n) \le t - \tfrac{1}{p}\} \right)$$
$$= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \pi_J^{-1}\Big(g_n^{-1}\big((-\infty, t - \tfrac{1}{p}\rangle\big)\Big) \right)$$
$$= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} P\left(\pi_J^{-1}\left(\bigcap_{n=k}^{k+i} g_n^{-1}\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right) \right)$$
$$= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(h_{k+i}\left(\bigcap_{n=k}^{k+i} g_n^{-1}\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right) \right)$$
$$\le \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} h_{k+i} \circ g_n^{-1}\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right)$$
$$= \lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} g_n(x_1, \dots, x_n)\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right).$$

The second assertion can be proved similarly. □

THEOREM 3. *Let $(x_n)_n$ be a sequence of observables, $(\xi_n)_n$ be the sequence of corresponding coordinate functions, $(g_n)_n$ be a sequence of Borel functions $g_n\colon \mathbb{R}^n \to \mathbb{R}$. If $(g_n(\xi_1,\dots,\xi_n))_n$ converges P-almost everywhere, then $(g_n(x_1,\dots,x_n))_n$ converges m-almost everywhere, i.e., there exists $\overline{x} = \limsup\limits_{n\to\infty} g_n(x_1,\dots,x_n)$ and $\underline{x} = \liminf\limits_{n\to\infty} g_n(x_1,\dots,x_n)$ and*

$$m\Big(\limsup_{n\to\infty} g_n(x_1,\dots,x_n)(-\infty,t)\Big) = m\Big(\liminf_{n\to\infty} g_n(x_1,\dots,x_n)(-\infty,t)\Big)$$

for every t. Moreover, $P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}};\ \limsup\limits_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\Big\}\Big) = \overline{x}\big((-\infty,t)\big)$ for every $t \in \mathbb{R}$.

P r o o f. Since $(g_n(\xi_1,\dots,\xi_n))_n$ converges P-almost everywhere,

$$\begin{aligned} &P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}};\ \limsup_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\Big\}\Big) \\ &= P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}};\ \liminf_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\Big\}\Big) \end{aligned} \tag{3.1}$$

for every $t \in \mathbb{R}$. From this fact and Proposition 3, we obtain

$$\begin{aligned} &\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} g_n(x_1,\dots,x_n)\big((-\infty,t-\tfrac{1}{p})\big)\right) \\ &= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigvee_{n=k}^{k+i} g_n(x_1,\dots,x_n)\big((-\infty,t-\tfrac{1}{p})\big)\right). \end{aligned} \tag{3.2}$$

Put

$$\varphi\big((-\infty,t)\big) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} g_n(x_1,\dots,x_n)\big((-\infty,t-\tfrac{1}{p})\big), \qquad t \in \mathbb{R}.$$

By (3.1), (3.2) and Proposition 3

$$m\big(\varphi((-\infty,t))\big) = P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}};\ \limsup_{n\to\infty} g_n(\xi_1,\dots,\xi_n) < t\Big\}\Big). \tag{3.3}$$

By (3.3), we obtain that the function $F\colon \mathbb{R} \to \langle 0,1\rangle$ defined by the equality

$$F(t) = m\big(\varphi((-\infty,t))\big)$$

is a distribution function.

By a theorem of D. B u t n a r i u and E. P. K l e m e n t ([1]), there exists a probability measure μ on $(\Omega,\mathcal{S})$ such that

$$F(t) = m\big(\varphi((-\infty,t))\big) = \int_{\Omega} \varphi\big((-\infty,t)\big)\,\mathrm{d}\mu$$

for every $t \in \mathbb{R}$. Therefore, by the Beppo Levi theorem,

$$0 = \lim_{t \to -\infty} F(t) = \int_\Omega \lim_{t \to -\infty} \varphi\big((-\infty, t)\big) \, \mathrm{d}\mu .$$

Since $\varphi\big((-\infty, t)\big) \geq 0$, we obtain

$$\lim_{t \to -\infty} \varphi\big((-\infty, t)\big) = 0 \qquad \text{a.e. } [\mu] ,$$

hence,

$$\lim_{t \to -\infty} \varphi\big((-\infty, t)\big)(\omega) = 0 \tag{3.4}$$

for μ-almost all $\omega \in \Omega$. Similarly, by $\lim_{t \to \infty} F(t) = 1$, we obtain

$$\lim_{t \to \infty} \varphi\big((-\infty, t)\big)(\omega) = 1 \tag{3.5}$$

for μ-almost all $\omega \in \Omega$. Finally,

$$0 = \lim_{t \to s-} \big(F(s) - F(t)\big) = \int_\Omega \lim_{t \to s-} \Big(\varphi\big((-\infty, s)\big) - \varphi\big((-\infty, t)\big)\Big) \, \mathrm{d}\mu ,$$

hence,

$$\lim_{t \to s-} \varphi\big((-\infty, t)\big)(\omega) = \varphi\big((-\infty, s)\big)(\omega) \tag{3.6}$$

for μ-almost all $\omega \in \Omega$. By (3.4)–(3.6), there exists a set $A \in \mathcal{S}$ such that $\mu(A) = 1$, and the function $F_\omega \colon \mathbb{R} \to \langle 0, 1\rangle$ defined by

$$F_\omega(t) = \varphi\big((-\infty, t)\big)(\omega)$$

is a distribution function for all $\omega \in A$. Let $F_0 \colon \mathbb{R} \to \langle 0, 1\rangle$ be a fixed distribution function. Let λ_{F_ω}, λ_{F_0} be the corresponding Lebesgue-Stieltjes probability measures. Put for any $E \in \mathcal{B}(\mathbb{R})$ and $\omega \in \Omega$

$$\overline{x}(E)(\omega) = \begin{cases} \lambda_{F_\omega}(E) & \text{if } \omega \in A , \\ \lambda_{F_0}(E) & \text{if } \omega \notin A . \end{cases}$$

The mapping $\overline{x} \colon \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ is an observable. Moreover,

$$\overline{x}\big((-\infty, t)\big)(\omega) = \lambda_{F_\omega}\big((-\infty, t)\big) = F_\omega(t) = \varphi\big((-\infty, t)\big)(\omega)$$

for every $\omega \in A$. Therefore

$$\begin{aligned} m\big(\overline{x}\big((-\infty, t)\big)\big) &= \int_\Omega \overline{x}\big((-\infty, t)\big) \, \mathrm{d}\mu = \int_\Omega \varphi\big((-\infty, t)\big) \, \mathrm{d}\mu \\ &= F(t) = m\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} g_n(x_1, \dots, x_n)\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right) \\ &= \lim_{p \to \infty} \lim_{k \to \infty} \lim_{i \to \infty} m\left(\bigwedge_{n=k}^{k+i} g_n(x_1, \dots, x_n)\big((-\infty, t - \tfrac{1}{p}\rangle\big) \right) . \end{aligned}$$

We have proved that $\overline{x}$ is the $\limsup\limits_{n\to\infty} x_n$. Similarly, it can be proved the existence of $\liminf\limits_{n\to\infty} x_n$. Then (3.2) implies the equality $\overline{x}\big((-\infty,t)\big) = \underline{x}\big((-\infty,t)\big)$, $t \in \mathbb{R}$, hence, $\big(g(x_1,\dots,x_n)\big)_n$ converges m-a.e. to $\overline{x}$. Moreover, by (3.1)–(3.2) and Proposition 3,

$$
\begin{aligned}
&P\Big(\Big\{u \in \mathbb{R}^{\mathbb{N}}\,;\ \limsup_{n\to\infty} g_n\big(\xi_1(u),\dots,\xi_n(u)\big) < t\Big\}\Big)\\
&= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} g_n(x_1,\dots,x_n)\big((-\infty,t-\tfrac{1}{p})\big)\right)\\
&= m\big(\overline{x}\big((-\infty,t)\big)\big)\,.
\end{aligned}
$$

□

REFERENCES

[1] BUTNARIU, D.—KLEMENT, E. P.: *Triangular norm-based measures and their Markov kernel representation*, J. Math. Anal. Appl. **169** (1991), 111–143.

[2] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *Difference posets, effects and quantum measurements*, Internat. J. Theoret. Phys. **33** (1994), 819–850.

[3] CHOVANEC, F.—KÔPKA, F.: *On a representation of observables in D-posets of fuzzy sets*, Tatra Mt. Math. Publ. **1** (1992), 19–24.

[4] JUREČKOVÁ, M.—RIEČAN, B.: *On the strong law of large numbers in D-posets*, Internat J. Theoret. Phys. **34** (1995), 1495–1500.

[5] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[6] MESIAR, R.: *Fuzzy observables*, J. Math. Anal. Appl. **174** (1993), 178–193.

[7] MESIAR, R.—RIEČAN, B.: *On the joint observables in some quantum structures*, Tatra Mt. Math. Publ. **3** (1993), 183–190.

[8] NAVARA, M.—PTÁK, P.: *Difference posets and orthoalgebras.*

[9] PYKACZ, J.: *Quantum logics as families of fuzzy subsets of the set of physical states.* In: Proc. 2nd Int. Fuzzy Systems Assoc. Congress, Tokyo, Vol. 2, Tokyo, 1987, pp. 437–440.

[10] RIEČAN, B.: *Fuzzy connectives and quantum models.* In: Cybernetics and Systems Research '92. Vol. 1 (R. Trappl, ed.), 1992, pp. 335–338.

[11] RIEČAN, B.: *On the almost everywhere convergence of observables in some algebraic structures*, Atti Sem. Mat. Fis. Univ. Modena **44** (1996), 95–104.

[12] RIEČANOVÁ, Z.—BRŠEL, D.: *Contraexamples in difference posets and orthoalgebras*, Internat. J. Theoret. Phys. **23** (1994), 133–141.

Received April 3, 1995
Revised September 11, 1995

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK-814 73 Bratislava
SLOVAKIA
E-mail: riecan@mau.savba.sk