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# $L^2$ DISCREPANCY

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ABSTRACT. Let $\mathbf{X}$ be a space of elements $X$ with a measure $\mathrm{d}X$, $\omega = (x_n)_{n=1}^{\infty}$ be an infinite sequence of points $x_n \in \mathbf{Y}$, $\omega_N = (x_n)_{n=1}^{N}$ be the initial segment of $\omega$. Furthermore, let $A(X, \omega_N)$ be a general counting function (i.e. any mapping $\mathbf{X} \times \mathbf{Y}^N \to \mathbb{R}$), $\mathbf{X}^* \subset \mathbf{X}$, and $g$ be a real-valued function defined on $\mathbf{X}$. In this paper we derive

$$\lim_{N\to\infty} \int_{\mathbf{X}} \left( \frac{A(X,\omega_N)}{N} - g(X) \right)^2 \mathrm{d}X = 0 \iff \forall_{X\in\mathbf{X}^*} \lim_{N\to\infty} \frac{A(X,\omega_N)}{N} = g(X)$$

in terms of certain relations between the above objects and additional structures on $\mathbf{X}$ (measure, topology, and partial ordering). A general method of computing $L^2$ discrepancy $\int_{\mathbf{X}} \left( \frac{A(X,\omega_N)}{N} - g(X) \right)^2 \mathrm{d}X$ is presented. Under certain conditions, we show the expression

$$\int_{\mathbf{X}} \left( \frac{A(X,\omega_N)}{N} - g(X) \right)^2 \mathrm{d}X = \frac{1}{N^2} \sum_{m,n=1}^{N} F_{A,g}(x_m, x_n),$$

where a symmetric function $F_{A,g}(x_m, x_n)$ is determined uniquely. These results are applied calculating the $L^2$ discrepancy for special counting functions $A(X, \omega_N)$. This yields many old and new results.

Motivated by the above expression we study a generalized $L^2$ discrepancy $D_F$ defined as

$$D_F(x_1, \dots, x_N) := \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n)$$

for any $F\colon \mathbf{Y}^2 \to \mathbb{R}$. We also discuss necessary conditions for classes of sequences having discrepancy $D_F$. We obtain, for such a class, that all distribution functions of $\omega$ belong to a corresponding fixed set. Some result on the Baire properties is added.

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## I. Introduction

This paper is concerned with discrepancies of a class of sequences. We give a brief review.

### 1. Definitions and basic results.

Let $\omega_N = (x_n)_{n=1}^N$ be a given finite sequence of real numbers from the unit interval $[0,1]$. For a subinterval $[0,x)$ of $[0,1]$, let the counting function $A\big([0,x),\omega_N\big)$ be defined as the number $x_n$, $1 \le n \le N$, for which $x_n \in [0,x)$. The infinite sequence $\omega = (x_n)_{n=1}^\infty$ in $[0,1]$ is said to be *uniformly distributed* if for every $x \in [0,1]$ we have

$$\lim_{N\to\infty} \frac{A\big([0,x),\omega_N\big)}{N} = x\,,$$

where $\omega_N = (x_n)_{n=1}^N$, $N = 1,2,\dots$. The sequence of numbers

$$D_N = \sup_{x\in[0,1]} \left| \frac{A\big([0,x),\omega_N\big)}{N} - x \right|$$

is called the *discrepancy* of $\omega$.

Systematic studies of uniformly distributed sequences were initiated by H. Weyl [14]. He proved the following qualitative result.

*The sequence $\omega$ is uniformly distributed if and only if for every continuous function $f\colon [0,1] \to \mathbb{R}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_0^1 f(x)\,\mathrm{d}x\,.$$

In the theory of uniform distribution, discrepancy is used to quantify the distribution behaviour of a given point sequence. For example J. F. Koksma [19] proved

*Let $f\colon [0,1] \to \mathbb{R}$ be a function of bounded variation $V(f)$. Then*

$$\left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_0^1 f(x)\,\mathrm{d}x \right| \le V(f) D_N\,.$$

Thus the quantity of the approximation of the integral by arithmetic means is linked directly to the discrepancy of the sequence $\omega$. But first of all the pertinence of the concept of discrepancy $D_N$ is revealed by the following criterion:

*The sequence* $\omega$ *is uniformly distributed if and only if* $\lim_{N\to\infty} D_N = 0$ ,

which also comes from H. Weyl [14]. In this paper, we study discrepancy from this purely qualitative point of view.

Note that the first extensive study of discrepancy was undertaken by van der Corput and Pisot [15]. A detailed survey of the results on discrepancy can be found in the classical monograph by L. Kuipers and H. Niederreiter [2].

## 2. Generalization.

Strictly speaking, we have to deal not only with one, but with several concepts of discrepancy (cf. [2; Chapter 2]). To unify various definitions of discrepancy, we start from the following general concept:

Let $\mathbf{Y}$ be a set of elements $x$. Suppose we are given a class $\boldsymbol{\Omega}$ of sequences $\omega = (x_n)_{n=1}^{\infty}$ of elements in $\mathbf{Y}$, viewed as a subset of $\mathbf{Y}^{\infty}$. A sequence $D(x_1,\dots,x_N)$ of real-valued functions defined on $\mathbf{Y}^N$, $N = 1,2,\dots$, is said to be a *discrepancy* [1)] of the given class $\boldsymbol{\Omega}$ if

$$\omega \in \boldsymbol{\Omega} \iff \lim_{N\to\infty} D(x_1,\dots,x_N) = 0$$

for all $\omega \in \mathbf{Y}^{\infty}$, where $x_1,\dots,x_N$ are the first $N$ terms of $\omega$. [2)]

For our present purpose (in main part) we restrict the attention to the case where the class $\boldsymbol{\Omega}$ of sequences is defined with respect to a counting function: Let $\mathbf{X}$ be a space [3)] of objects $X$. For a positive integer $N$ and an object $X$, the *counting function* $A(X,\omega_N)$ be any real-valued mapping $A\colon \mathbf{X}\times\mathbf{Y}^N \to \mathbb{R}$. In most cases $A(X,\omega_N) = \#\{n \le N;\ X\,\mathbf{R}\,x_n\}$, where $\mathbf{R}$ is a relation on $\mathbf{X}\times\mathbf{Y}$ and $\omega_N = (x_n)_{n=1}^{N} \in \mathbf{Y}^N$. Now consider the class $\boldsymbol{\Omega}_{A,g}$ of all sequences $\omega \in \mathbf{Y}^{\infty}$ for which

$$\lim_{N\to\infty} \frac{A(X,\omega_N)}{N} = g(X) \tag{1}$$

for $X \in \mathbf{X}^*$. Here $g$ is a given real-valued function defined on $\mathbf{X}$, $\mathbf{X}^*$ is a given subset of $\mathbf{X}$, and $\omega_N$ is the initial segment of $\omega$ formed by the first $N$ terms of $\omega$. This function $g$ may be called the *asymptotic distribution function* or *limit law* of $\omega$. Sequences $\omega$ having this property are called $g$*-distributed* on $\mathbf{X}$. This definition is a generalization of the familiar definition of the $g$-distribution [6] as extended to $\mathbf{X}$.

1) More adequate $\boldsymbol{\Omega}$-testing sequence of functions. An interesting more general notion of a test selecting certain class $\boldsymbol{\Omega}$ of sequences was introduced by R. Winkler [10].

2) The first and most natural question to ask is of course whether discrepancy exists at all for an arbitrary class $\boldsymbol{\Omega}$. A negative answer is due to M. Goldstern [12].

3) set, class of sets, etc.

The discrepancy test for $\omega \in \mathbf{\Omega}_{A,g}$ does not require determining the counting function $A(X, \omega_N)$ at all objects $X$. To obtain an explicit representation of a discrepancy of the class $\mathbf{\Omega}_{A,g}$, the object $X$ must be eliminated from the given definition (1).

One elimination technique of getting a discrepancy of $\mathbf{\Omega}_{A,g}$ is to evaluate the following supremum[4)]

$$\sup_{X \in \mathbf{X}^*} \left| \frac{A(X, \omega_N)}{N} - g(X) \right| .$$

We shall present here other method whose result is known as the $L^2$ discrepancy[5)]

$$\int_{\mathbf{X}} \left( \frac{A(X, \omega_N)}{N} - g(X) \right)^2 \mathrm{d}X . \tag{2}$$

The $L^2$ discrepancy has ben studied in some detail by H. Niederreiter [18].

Of course, to compute (2) we first need an integration theory on $\mathbf{X}$. On account of the known results we see that the study of the $L^2$ discrepancy also depends on the definition of a topology and an ordering on the space $\mathbf{X}$. The main purpose of this article is to establish a relation between these structures which shows that $L^2$ discrepancy (2) is a discrepancy of $\mathbf{\Omega}_{A,g}$. Our results are further applied to obtain discrepancies for special classes.

We disregard a question whether $\mathbf{\Omega}_{A,g} \neq \emptyset$.

**3. Outline of paper.**

Thus, when applying the method of $L^2$ discrepancy presented here we proceed from the definition of a suitable topology, measure and partial ordering on given $\mathbf{X}$. By analyzing all classical $L^2$ discrepancies it will even be possible to prove, in Part II (Theorem 1), that under additional assumptions for these structures on $\mathbf{X}$, (2) is really a discrepancy of $\mathbf{\Omega}_{A,g}$. From the metrical point of view (Theorem 2), when $L^2$ discrepancy (2) ranges to zero, by selecting a suitable sequence of indices $N$, one can only guarantee the existence of the limit of (1) almost everywhere.

The rule (Theorem 3) for computing the $L^2$ discrepancy is extremely simple, it expresses (2) in the form

$$\int_{\mathbf{X}} \left( \frac{A(X, \omega_N)}{N} - g(X) \right)^2 \mathrm{d}X = \frac{1}{N^2} \sum_{m,n=1}^{N} F_{A,g}(x_m, x_n) ,$$

4) This supremum norm is sometimes referred to as the *extreme* discrepancy.

5) The integral (2) is said to be the $L^2$ discrepancy of given $\omega_N$ and $g$, with respect to $A(X, \omega_N)$.

for a general counting function $A(X, \omega_N)$.

In accordance with this expression, the notion of $L^2$ discrepancy can be viewed as a special case of the following discrepancy. Define

$$D_F(x_1, \dots, x_N) := \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n),$$

where $F(x, y)$ is any real-valued function on $\mathbf{Y}^2$.

In Part III, we shall describe characteristic properties of a class $\mathbf{\Omega}$ with such a discrepancy. We shall see (Remark 3), under additional assumptions on $\mathbf{X}$ and $F$, there is a fixed set $S$ of distribution functions on $\mathbf{X}$ such that

$$\omega \in \mathbf{\Omega} \iff G(\omega) \subset S,$$

for all $\omega \in \mathbf{Y}^\infty$. Here $G(\omega)$ is the set of all distribution functions of $\omega$.

We shall note (Theorem 5), for continuous and bounded $F$, that $D_F(\omega_N)$ tends to zero independently on finitely many terms of a given sequence $\omega$. This condition indicates when the class $\mathbf{\Omega}$ with discrepancy $D_F$ is a $F_{\sigma\delta}$-set of the first Baire category (Theorem 6).

In Part IV, we have applied Theorems 1 and 3 to lists of $L^2$ discrepancies available to us. Our results contain those already known for the usual notion of counting functions and give new ones for a general $A(X, \omega_N)$.

Let us begin with the exact formulation of Theorems 1–7.

## II. Main results

In connection with measure theory we can consider essentially four convergence concepts for the limit $\dfrac{A(X, \omega_N)}{N} \to g(X)$, so far: pointwise convergence, convergence a.e., $L^2$-norm convergence, and stochastic convergence. Our next aim is to formulate conditions that the $L^2$-norm convergence and pointwise convergence are equivalent. The idea is to start from a proof known for uniformly distributed sequences and reformulate it into the language of measure theory, topology and ordering.

**THEOREM 1.** *Let $\mathbf{X}$ be a space of objects $X$ with a topology and a finite $\sigma$-measure $\mathrm{d}X$ with the Lebesgue integral,[6] partially ordered by $\prec$. Furthermore, let $\omega = (x_n)_{n=1}^\infty$ be a fixed infinite sequence of points $x_n \in \mathbf{Y}$, with an*

[6] In the sequel, if we speak of the Lebesgue integration theory, it will be tacitly assumed that the classical theorems of the Lebesgue integral in the $n$-dimensional Euclidean space are satisfied. For reference, see K. Jacobs [1].

*initial segment* $\omega_N = (x_n)_{n=1}^N$. *Finally, let* $A(X, \omega_N)$ *be a given counting function,* $g$ *a real valued function defined on* $\mathbf{X}$, *and* $\mathbf{X}^* \subset \mathbf{X}$. *Suppose that the following assumptions are satisfied:*

(i) $\dfrac{A(X,\omega_N)}{N} - g(X)$, $N = 1, 2, \dots$, *are square-integrable and uniformly bounded functions a.e. on* $\mathbf{X}$;

(ii) *for every* $X \in \mathbf{X}^*$, *the sequence* $\dfrac{A(X,\omega_N)}{N}$, $N = 1, 2, \dots$, *is bounded;*

(iii) $\mathbf{X}^*$ *is a subset of the set of points of continuity of* $g$;

(iv) $\mathbf{X} - \mathbf{X}^*$ *is a null set;*

(v) *for any* $X_0 \in \mathbf{X}^*$, *every open neighbourhood* $O$ *of* $X_0$ *has measurable intersections* $O \cap \{X \in \mathbf{X};\ X \prec X_0\}$ *and* $O \cap \{X \in \mathbf{X};\ X_0 \prec X\}$ *of positive measure;*

(vi) (*monotonicity*) $X \prec X' \implies A(X,\omega_N) \le A(X',\omega_N)$ *for* $X, X' \in \mathbf{X}$ *and* $\omega_N \in \mathbf{Y}^N$.

*Then*

$$\lim_{N\to\infty} \int_{\mathbf{X}} \left( \frac{A(X,\omega_N)}{N} - g(X) \right)^2 \mathrm{d}X = 0 \tag{3}$$

*if and only if*

$$\lim_{N\to\infty} \frac{A(X,\omega_N)}{N} = g(X) \qquad \textit{for each} \quad X \in \mathbf{X}^* . \tag{4}$$

P r o o f. The implication (4) $\implies$ (3) follows by using (i) and applying Lebesgue's theorem on dominated convergence.

In order to show the implication (3) $\implies$ (4), consider

$$\lim_{N\to\infty} \frac{A(X_0,\omega_N)}{N} \ne g(X_0) \qquad \text{for some} \quad X_0 \in \mathbf{X}^* ,$$

and put $\beta = g(X_0)$. Selecting a suitable subsequence $(N_k)_{k=1}^\infty$ of positive integers, one can guarantee the existence of a limit

$$\lim_{k\to\infty} \frac{A(X_0,\omega_{N_k})}{N_k} = \alpha \ne \beta .$$

We consider only the case $\alpha > \beta$, the case $\alpha < \beta$ being analogous. Then choosing a neighbourhood $O$ of $X_0$ and a positive integer $k_0$ such that

$$g(X) \le \beta + \frac{\alpha - \beta}{4} \qquad \text{for} \quad X \in O ,$$

and

$$\frac{A(X_0, \omega_{N_k})}{N_k} \geq \alpha - \frac{\alpha - \beta}{4} \qquad \text{for} \quad k \geq k_0 \,,$$

we conclude, from the monotonicity (vi), that

$$\left( \frac{A(X, \omega_{N_k})}{N_k} - g(X) \right)^2 \geq \left( \frac{\alpha - \beta}{2} \right)^2$$

for $k \geq k_0$ and $X \in O \cap \{X \in \mathbf{X}\,;\ X_0 \prec X\}$. Thus, by (v), one concludes that

$$\limsup_{N \to \infty} \int_{\mathbf{X}} \left( \frac{A(X, \omega_N)}{N} - g(X) \right)^2 \mathrm{d}X > 0 \,.$$

□

R e m a r k 1. The equivalence (4) $\iff$ (3) remains valid if $\mathbf{X}^*$ is extended to a large set $\mathbf{X}^{**}$ of elements $X_0 \in \mathbf{X}$ which can be described by the following:

(vii) $X_0$ is a point of continuity of $g$;

(viii) there exist two sequences $(X_n)_{n=1}^{\infty}$ and $(X_n')_{n=1}^{\infty}$ in $\mathbf{X}^*$ such that [7]

$$X_n \prec X_0 \prec X_n' \quad \text{for } n = 1, 2, \dots \,, \qquad \text{and} \qquad \lim_{n \to \infty} X_n = \lim_{n \to \infty} X_n' = X_0 \,.$$

Thus the class of sequences $\omega \in \mathbf{Y}^{\infty}$ defined by the limit of (1) for $X \in \mathbf{X}^{**}$ is the same as the class $\mathbf{\Omega}_{A,g}$ and hence, $L^2$ discrepancy (2) is a discrepancy of this class, assuming (i) – (vi).

R e m a r k 2. It should be noted that Theorem 1 works also in the case if we replace the assumptions (v) and (vi) by the following

(v') *for any $X_0 \in \mathbf{X}^*$, every open neighbourhood $O$ of $X_0$ has a positive measure;*

(vi') $\dfrac{A(X, \omega_N)}{N}$, $N = 1, 2, \dots$, *are uniformly continuous on $\mathbf{X}^*$;*

without requiring the partial ordering $\prec$. By *uniform continuity* we mean that given any $X_0 \in \mathbf{X}^*$ and any $\varepsilon > 0$ there is an open neighbourhood $O \subset \mathbf{X}$ of $X_0$ such that for all $X \in O$ and $N = 1, 2, \dots$ the inequality

$$\left| \frac{A(X, \omega_N)}{N} - \frac{A(X_0, \omega_N)}{N} \right| < \varepsilon$$

[7] If $g(X_0) = \min_{X \in \mathbf{X}} g(X)$, then (viii) may be replaced by the following conditions: $g(X_0) \leq \liminf_{N \to \infty} \dfrac{A(X_0, \omega_N)}{N}$, $X_0 \prec X_n'$ for $n = 1, 2, \dots$, and $\lim_{n \to \infty} X_n' = X_0$; similarly for maximum.

is fulfilled. Here $\omega_N = (x_n)_{n=1}^N$ and the infinite sequence $\omega = (x_n)_{n=1}^\infty \in \mathbf{Y}^\infty$ is fixed.[8)]

If we cannot find an ordering and a topology on $\mathbf{X}$ for which both of the assumptions (v) and (vi) of Theorem 1 (or alternatively (v') and (vi') of Remark 2) would be satisfied, then we have only the following weak theorem:

**THEOREM 2.** *Let $\mathbf{X}$, $\mathrm{d}X$, $\mathbf{Y}$, $\omega$, $\omega_N$, $A(X,\omega_N)$, $g$ be defined as above, and suppose that*

(i) $\dfrac{A(X,\omega_N)}{N} - g(X)$, $N = 1, 2, \dots$, *are square-integrable and uniformly bounded functions a.e. on* $\mathbf{X}$.

*Then, from the zero-limit of $L^2$ discrepancy*

$$\lim_{N\to\infty} \int_{\mathbf{X}} \left( \frac{A(X,\omega_N)}{N} - g(X) \right)^2 \mathrm{d}X = 0\,, \tag{$*$}$$

*it follows the existence of an increasing sequence $(N_k)_{k=1}^\infty$ of natural numbers such that*

$$\lim_{k\to\infty} \frac{A(X,\omega_{N_k})}{N_k} = g(X) \qquad \text{for almost all} \quad X \in \mathbf{X}\,, \tag{$**$}$$

*and vice-versa, almost convergence $(**)$ implies $L^2$-norm convergence $(*)$ for the index set $(N_k)_{k=1}^\infty$.*

P r o o f. $L^2$-norm convergence implies stochastic convergence, and [1; p. 143, 12.7. Corollary] every stochastically convergent sequence has a subsequence that converges a.e. to the same limit. Conversely, Lebesgue's theorem on dominated convergence can be applied. □

There is a still simpler method of determining the $L^2$ discrepancy.

**THEOREM 3.** *Let $\omega_N = (x_n)_{n=1}^N$ be a finite sequence in $\mathbf{Y}$. For $1 \le n \le N$, let $(x_n)$ be a sequence of one-elements from $\omega_N$, and put $\mathbf{X}_n = \{X \in \mathbf{X}$;*

[8)] Assuming (v') and (vi'), in an alternative proof of (3) $\Longrightarrow$ (4), one uses a neighbourhood $O$ of $X_0$ with a positive measure for which $g(X) \le \beta + \dfrac{\alpha-\beta}{4}$ and $\dfrac{A(X,\omega_{N_k})}{N_k} \ge \alpha - \dfrac{\alpha-\beta}{4}$ holds for $X \in O$ and $k \ge k_0$. (4) $\Longrightarrow$ (3) follows as in the previous proof of Theorem 1.

$A(X,(x_n)) = 1\}$. *Suppose the counting function* $A(X,\omega_N)$ *and an* $\sigma$*-measure* $\mathrm{d}X$ *satisfy the following conditions:*

(i) *(the additivity)* $A(X,\omega_N) = \sum_{n=1}^{N} A(X,(x_n))$, *and* $A(X,(x_n)) = 0$ *or* $1$ *for* $1 \le n \le N$,

(ii) *the set* $\mathbf{X}_n$ *is* $\mathrm{d}X$*-measurable for* $1 \le n \le N$.

*Then the* $L^2$ *discrepancy of* $\omega_N$ *with respect to* $A(X,\omega_N)$ *satisfies*

$$\int_{\mathbf{X}} \left( \frac{A(X,\omega_N)}{N} - g(X) \right)^2 \mathrm{d}X$$

$$= \int_{\mathbf{X}} g^2(X)\,\mathrm{d}X - \frac{2}{N}\sum_{n=1}^{N} \int_{\mathbf{X}_n} g(X)\,\mathrm{d}X + \frac{1}{N^2}\sum_{m,n=1}^{N} \int_{\mathbf{X}_m \cap \mathbf{X}_n} 1 \cdot \mathrm{d}X . \tag{5}$$

*The above expressing of the* $L^2$ *discrepancy immediately yields the following alternative expression*

$$\int_{\mathbf{X}} \left( \frac{A(X,\omega_N)}{N} - g(X) \right)^2 \mathrm{d}X = \frac{1}{N^2}\sum_{m,n=1}^{N} F_{A,g}(x_m,x_n), \tag{6}$$

*where*

$$F_{A,g}(x_m,x_n) = \int_{\mathbf{X}} g^2(X)\,\mathrm{d}X - \int_{\mathbf{X}_m} g(X)\,\mathrm{d}X - \int_{\mathbf{X}_n} g(X)\,\mathrm{d}X + \int_{\mathbf{X}_m \cap \mathbf{X}_n} 1 \cdot \mathrm{d}X . \tag{7}$$

*Moreover, if the* $L^2$ *discrepancy can be expanded into the sum*

$$\frac{1}{N^2}\sum_{m,n=1}^{N} G(x_m,x_n)$$

*and* $G(x,y) = G(y,x)$ *for all* $x,y$, *then* $G$ *is uniquely determined as* $G \equiv F_{A,g}$.

P r o o f. Let $c_{\mathbf{X}_n}(X)$ be a characteristic function of the set $\mathbf{X}_n$. Here it is sufficient to express the counting functions $A(X,\omega_N)$ as

$$A(X,\omega_N) = \sum_{n=1}^{N} c_{\mathbf{X}_n}(X),$$

and, when calculated the $L^2$ discrepancy of $\omega_N$, the desired equality (5) follows immediately. To obtain the uniqueness of $F_{A,g}$, we can look for the values $N = 1, 2$ in (6). $\square$

As we have already initiated in our Introduction, consider now the new concept of $L^2$ discrepancy putting:

$$D_F(x_1, \dots, x_N) := \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n),$$

where $F(x, y)$ is any given real-valued function on $\mathbf{Y}^2$.

We will prove certain necessary conditions for the existence of discrepancy $D_F$ for a given class $\mathbf{\Omega}$.

## III. Necessary conditions

To obtain a characterization of a class $\mathbf{\Omega}$ with the discrepancy $D_F$ we shall need a theory of distribution functions. On multidimensional unit cube $\mathbf{Y} = [0, 1]^s$ we know such a theory, and thus any class $\mathbf{\Omega} \subset \left([0, 1]^s\right)^\infty$ with discrepancy $D_F$ with bounded continuous $F$ can be characterized by the set-inclusion (which we shall prove in Theorem 4)

$$\omega \in \mathbf{\Omega} \iff G(\omega) \subset G(F)$$

for $\omega \in \left([0, 1]^s\right)^\infty$. Here $G(\omega)$ denotes the set of all distribution functions of $\omega$, and $G(F)$ is the set of all distribution functions $g$ for which

$$\int_{[0,1]^s} \int_{[0,1]^s} F(\mathbf{x}, \mathbf{y}) \, \mathrm{d}g(\mathbf{x}) \, \mathrm{d}g(\mathbf{x}) = 0 .$$

Especially, any class $\mathbf{\Omega}_{A,g} \subset \left([0, 1]^s\right)^\infty$ with $L^2$ discrepancy has this property.

The set of distribution functions on $\mathbf{Y} = [0, 1]^s$ may be defined in the following manner (cf. [22; pp. xi–xiv]):

Starting with an auxiliary space $\mathbf{X} = \{[\mathbf{0}, \mathbf{x}) ;\ \mathbf{x} \in [0, 1]^s\}$ and an auxiliary counting function $A([\mathbf{0}, \mathbf{x}), \omega_N) = \#\{n \le N ;\ \mathbf{x}_n \in [\mathbf{0}, \mathbf{x})\}$, and for any probability measure $P$ defined on Borel sets in $[0, 1]^s$, $P([\mathbf{0}, \mathbf{x}))$ is said to be a distribution function on $\mathbf{X}$. For a given sequence $\omega = (\mathbf{x}_n)_{n=1}^\infty$ in $[0, 1]^s$, $P([\mathbf{0}, \mathbf{x}))$ is said to be a distribution function of $\omega$ if

$$\lim_{k \to \infty} \frac{A([\mathbf{0}, \mathbf{x}), \omega_{N_k})}{N_k} = P([\mathbf{0}, \mathbf{x}))$$

for a suitable sequence of indices $N_k$ and for all continuity intervals of $P$. In the end, we transmit distribution functions $P([\mathbf{0},\mathbf{x}))$ from $\mathbf{X}$ to $\mathbf{Y}$ by $P([\mathbf{0},\mathbf{x}))= g(\mathbf{x})$.

In the general case, we shall copy this method, with the following differences: We shall not transmit distribution functions from auxiliary space $\mathbf{X}$ to $\mathbf{Y}$, but we shall transmit $F(x,y)$ from $\mathbf{Y}^2$ to $\mathbf{X}^2$ using partial ordering $\prec$ on $\mathbf{X}$, assuming the existence of $X(x) := \inf\{X \in \mathbf{X};\ A(X,(x)) = 1\}$ and putting $F(X(x),X(y)) = F(x,y)$.

For various auxiliary spaces $\mathbf{X}$ and $A(X,\omega_N)$ we have various meanings of the following definitions, and thus we have various characterizations of a given fixed class $\mathbf{\Omega}$ with discrepancy $D_F$.

## 1. Distribution functions.

Let $\mathbf{Y}$ be a given set with a topology, and $\mathbf{Y}^\infty$ be a space of sequences $\omega$ with the product topology. Furthermore, let $F(x,y)$ be a given real-valued function continuous and bounded on $\mathbf{Y}^2$. Assume that we have an auxiliary topological space $\mathbf{X}$ with measure $\mathrm{d}X$ and a counting function $A(X,\omega_N)$ and partially ordered by $\prec$.

A *distribution function* in $\mathbf{X}$ will be any $g\colon \mathbf{X} \to \mathbb{R}$ satisfying

$$\lim_{N\to\infty} \frac{A(X,\omega_N)}{N} = g(X)$$

for some $\omega \in \mathbf{Y}^\infty$ and selected $N$ and for all continuity points $X$ of $g$.

Two distribution functions are identified if they have the same points of continuity and their values coincide over all such points.

By $\tilde{G}$ we mean the set of all distribution functions on $\mathbf{X}$.

For the following, we suppose that we are given, for every distribution function $g \in \tilde{G}$, a Lebesgue-Stieltjes measure $\mathrm{d}g(X)$, and we assume the validity of the following versions of theorems of Helly.

**"First Theorem of Helly".** *Given a sequence $g_n \in \tilde{G}$, then there exists a subsequence $(k_n)_{n=1}^\infty$ of natural numbers and $g \in \tilde{G}$ such that*

$$\lim_{n\to\infty} g_{k_n}(X) = g(X)$$

*for all points $X$ of continuity of $g$.*

**"Second Theorem of Helly".** *Given a sequence $g_n \in \tilde{G}$ which converges to $g \in \tilde{G}$ for all continuity points $X$ of $g$, we have*

$$\lim_{n\to\infty} \int\limits_{\mathbf{X}} \int\limits_{\mathbf{X}} H(X,Y)\ \mathrm{d}g_n(X)\,\mathrm{d}g_n(Y) = \int\limits_{\mathbf{X}} \int\limits_{\mathbf{X}} H(X,Y)\ \mathrm{d}g(X)\,\mathrm{d}g(Y)$$

*for any bounded continuous function* $H\colon \mathbf{X}^2 \to \mathbb{R}$.

Now, let $g\colon \mathbf{X} \to \mathbb{R}$ be a given distribution function, and let $\omega = (x_n)_{n=1}^{\infty}$ be a fixed infinite sequence of points $x_n \in \mathbf{Y}$, with an initial segment $\omega_N = (x_n)_{n=1}^{N}$. If there exists a subsequence $(N_n)_{n=1}^{\infty}$ of the natural numbers such that the relation

$$\lim_{n\to\infty} \frac{A(X,\omega_{N_n})}{N_n} = g(X)$$

holds for every point $X \in \mathbf{X}$ of continuity of $g$, then $g(X)$ is called a *distribution function of the sequence* $\omega$.

The set of all distribution functions of the sequence $\omega$ will be denoted by $G(\omega)$.

Let us consider the moment problem

$$\int_{\mathbf{X}}\int_{\mathbf{X}} F(X,Y)\,\mathrm{d}g(X)\,\mathrm{d}g(Y) = 0$$

in distribution functions $g\colon \mathbf{X} \to \mathbb{R}$, where $F(X,Y)$ is continuous and bounded on $\mathbf{X}$.

We denote $G(F)$ the set of all solutions $g \in \tilde{G}$ of this moment problem.

For any $x \in \mathbf{Y}$ we put $\mathbf{X}(x) = \bigl\{X \in \mathbf{X}\,;\ A\bigl(X,(x)\bigr) = 1\bigr\}$ and assume that $\mathbf{X}(x)$ is measurable and has an infimum $X(x)$ in $\mathbf{X}$ with regard to the partial ordering $\prec$. To a given real-valued function $F(x,y)$ defined on $\mathbf{Y}^2$ we shall define $F(X,Y)$ on $\mathbf{X}^2$ by $F\bigl(X(x),X(y)\bigr) = F(x,y)$.

There is a close connection between $G(\omega)$, $G(F)$, and a zero limit of $D_F(\omega_N)$. We now establish the required connection.

**THEOREM 4.** *Let* $\mathbf{Y}$, $F(x,y)$, $\omega$, $\omega_N$, $\mathbf{X}$, $\mathrm{d}X$, $A(X,\omega_N)$, $\prec$, $\tilde{G}$, $G(\omega)$, $G(F)$, $\mathbf{X}(x)$, *and* $X(x)$ *have the same meaning as in the above definitions. Suppose we are given the Lebesgue-Stieltjes integration theory on* $\mathbf{X}$ *for any* $g \in \tilde{G}$. *Moreover suppose that the following assumptions are satisfied:*

- (i) *With the above notation, the First and Second Theorems of Helly hold in* $\tilde{G}$;
- (ii) *for every* $x \in \mathbf{Y}$, *the set* $\mathbf{X}(x)$ *is measurable and has an infimum* $X(x)$ *in* $\mathbf{X}$ *with regard to the partial ordering* $\prec$;
- (iii) *assume that the mapping* $x \to X(x)$ *is onto and for any* $\bigl(X(x),X(y)\bigr) = \bigl(X(x'),X(y')\bigr)$ *let* $F(x,y) = F(x',y')$;
- (iv) *for every* $x, y \in \mathbf{Y}$ *and every bounded continuous* $H\colon \mathbf{X}^2 \to \mathbb{R}$ *the following Lebesgue-Stieltjes integral becomes*

$$\int_{\mathbf{X}}\int_{\mathbf{X}} H(X,Y)\,\mathrm{d}c_{\mathbf{X}(x)}(X)\,\mathrm{d}c_{\mathbf{X}(y)}(Y) = H\bigl(X(x),X(y)\bigr)\,;$$

(v) *for every* $\omega_N \in \mathbf{Y}^N$ *and* $X \in \mathbf{X}$ *the counting function* $A(X,\omega_N)$ *satisfies the additivity* $A(X,\omega_N) = \sum_{n=1}^{N} A\big(X,(x_n)\big)$, *and* $A\big(X,(x_n)\big) = 0$ *or* $1$.

*Then*

$$G(\omega) \subset G(F) \iff \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m,x_n) = 0\,.$$

Proof. We write

$$F_N(X) = \frac{A(X,\omega_N)}{N}$$

in the proof. Using the assumptions (iv) and (v), for the Riemann-Stieltjes integral, we have

$$\int_{\mathbf{X}}\int_{\mathbf{X}} F(X,Y)\,\mathrm{d}F_N(X)\,\mathrm{d}F_N(Y) = \frac{1}{N^2}\sum_{m,n=1}^{N} F(x_m,x_n)\,.$$

Suppose that $\lim_{k\to\infty} F_{N_k}(X) = g(X)$ for all continuity points $X$ of $g$. Then, applying the Second Theorem of Helly, we find

$$\lim_{k\to\infty}\int_{\mathbf{X}}\int_{\mathbf{X}} F(X,Y)\,\mathrm{d}F_{N_k}(X)\,\mathrm{d}F_{N_k}(Y) = \int_{\mathbf{X}}\int_{\mathbf{X}} F(X,Y)\,\mathrm{d}g(X)\,\mathrm{d}g(Y)\,,$$

and the implication "$\Longleftarrow$" follows immediately.

In order to show the implication "$\Longrightarrow$", consider

$$\lim_{k\to\infty}\frac{1}{{N_k}^2}\sum_{m,n=1}^{N_k} F(x_m,x_n) = \beta > 0\,.$$

By the First Theorem of Helly, there exists a subsequence $(N'_k)_{k=1}^{\infty}$ of $(N_k)_{k=1}^{\infty}$ such that $\lim_{k\to\infty} F_{N'_k}(X) = g(X) \in G(\omega)$. Again, by the Second Theorem we find $\int_{\mathbf{X}}\int_{\mathbf{X}} F(X,Y)\,\mathrm{d}g(X)\,\mathrm{d}g(Y) = \beta$. We conclude $g \notin G(F)$. □

In the following, the independence of $\omega \in \mathbf{\Omega}$ on finitely many terms of $\omega$ is deduced for $\mathbf{\Omega}$ having discrepancy $D_F$.

**THEOREM 5.** *Let $\mathbf{\Omega}$ be a class of sequences with discrepancy $D_F$, and suppose that $F$ is continuous and bounded on $\mathbf{Y}^2$. Then, for every $\omega = (x_n)_{n=1}^{\infty} \in \mathbf{Y}^{\infty}$,*

$$(x_1, \dots, x_n, x_{n+1}, \dots) \in \mathbf{\Omega} \implies (y_1, \dots, y_n, x_{n+1}, \dots) \in \mathbf{\Omega}$$

*for arbitrary $n = 1, 2, \dots$ and any $(y_1, \dots, y_n) \in \mathbf{Y}^n$.*[9]

Proof. This immediately follows from

$$\lim_{N \to \infty} \big( D_F(x_1, \dots, x_{M+N}) - D_F(x_{M+1}, \dots, x_{M+N}) \big) = 0$$

for $M = 1, 2, \dots$, and $\omega = (x_n)_{n=1}^{\infty} \in \mathbf{Y}^{\infty}$. □

The continuity of $F$ leads to the following result on the Baire properties.[10]

**THEOREM 6.** *Suppose $F$ is continuous and bounded on $\mathbf{Y}^2$, and $\mathbf{Y}^{\infty} - \mathbf{\Omega} \neq \emptyset$. Then the class $\mathbf{\Omega}$ with discrepancy $D_F$ is a $F_{\sigma\delta}$-set of the first category in $\mathbf{Y}^{\infty}$.*

Proof. $D_F(x_1, \dots, x_N)$ is continuous on $\mathbf{Y}^N$, $N = 1, 2, \dots$, and we can extend $D_F(x_1, \dots, x_N)$ on $\mathbf{Y}^{\infty}$ by $D_N(\omega) = D_F(x_1, \dots, x_N)$, where $x_1, \dots, x_N$ is the initial segment of $\omega$. Putting

$$\mathbf{\Omega}_{m,k} = \left\{ \omega \in \mathbf{Y}^{\infty} ;\ D_m(\omega) < \frac{1}{k} \right\},$$

we shall show that the intersection $\bigcap_{m=n}^{\infty} \mathbf{\Omega}_{m,k}$ is a nowhere dense set for $k > k_0$ and $n = 1, 2, \dots$. This will prove our theorem, since[11]

$$\mathbf{\Omega} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \mathbf{\Omega}_{m,k}.$$

For the proof of nowhere-density of $\bigcap_{m=n}^{\infty} \mathbf{\Omega}_{m,k}$ we require that

$$\omega^{(0)} = \left( x_n^{(0)} \right)_{n=1}^{\infty} \in \mathbf{Y}^{\infty} - \mathbf{\Omega},$$

[9] $\mathbf{\Omega}$ is called *terminal*. This notation is from R. Winkler [10].

[10] Compare with [10; Theorem 8], [2; p. 184, Theorem 2.3], and [12; 1.3. Fact].

[11] The $F_{\sigma\delta}$-property of $\mathbf{\Omega}$ is implied by a similar expression of $\mathbf{\Omega}$ employing the closure of $\mathbf{\Omega}_{m,k}$.

and $D_N(\omega^{(0)}) \geq \frac{1}{k_0}$ for infinitely many $N$. We shall additionally suppose that

$$\omega^{(1)} = \left(x_n^{(1)}\right)_{n=1}^{\infty} \in \mathbf{G} \cap \bigcap_{m=n}^{\infty} \mathbf{\Omega}_{m,k} ,$$

where $\mathbf{G}$ is an open set in $\mathbf{Y}^\infty$ having the form

$$\mathbf{G}_1 \times \cdots \times \mathbf{G}_s \times \mathbf{Y} \times \cdots \times \mathbf{Y} \times \ldots .$$

Then, there exists $N$ with $N + s > n$ such that

$$D_{N+s}\left(\left(x_1^{(1)}, \ldots, x_s^{(1)}, x_1^{(0)}, x_2^{(0)}, \ldots\right)\right) > \frac{1}{k_0 + 1} .$$

Because of the continuity of $D_{N+s}$ the last inequality can be extended over an open $\mathbf{G}^0 \subset \mathbf{G}$, and consequently $\mathbf{G}^0 \cap \bigcap_{m=n}^{\infty} \mathbf{\Omega}_{m,k} = \emptyset$ for $k > k_0$. □

Finally, based on our concept of discrepancy $D_F$, we give a generalization of [2; Exercise 2.11].

**THEOREM 7.** *Let $\mathbf{\Omega}$ be a class of sequences $\omega \in \mathbf{Y}^\infty$ with the discrepancy $D_F$. Let $\mathbf{Y}$ be a metric space with the metric $d$ and let $F$ satisfies*

$$|F(x, y) - F(x', y')| \leq c \cdot \left(d(x, x') + d(y, y')\right)$$

*for $x, y, x', y' \in \mathbf{Y}$, where $c > 0$ is a constant. If $\omega = (x_n)_{n=1}^{\infty} \in \mathbf{\Omega}$ and $\tilde{\omega} = (y_n)_{n=1}^{\infty} \in \mathbf{Y}^\infty$ satisfies*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d(x_n, y_n) = 0 ,$$

*then $\tilde{\omega} \in \mathbf{\Omega}$.*

P r o o f. The proof is obvious. □

Our elementary method presented by Theorems 1 and 3 can be applied to the following concrete examples of counting functions $A(X, \omega_N)$ and spaces $\mathbf{X}$.

## IV. Applications

We begin by giving a few basic facts about the usual counting functions. In the following paragraphs in most cases the space $\mathbf{X}$ will be formed by intervals from one-dimensional or multidimensional unit square and for such $\mathbf{X}$ the measure and topology we shall identify with Lebesgue's measure and Euclidean topology on $[0,1]^s$. The ordering $\prec$ on $\mathbf{X}$ will be the set-inclusion, and $g\colon \mathbf{X} \to \mathbb{R}$ will be a distribution function defined by Part III (which coincide with a classical definition in [22; pp. xi – xiii]).

We shall first establish the $L^2$ discrepancy of uniformly distributed sequences (abbreviated u.d. sequences).

### 1. One-dimensional uniform distribution.

We give here four results.

**1°**. Let us take $X = [0,x) \subset [0,1]$ and $\mathbf{Y} = [0,1]$. The usual notion of $A\big([0,x),\omega_N\big)$ of a finite sequence $\omega_N = (x_n)_{n=1}^N$ in $[0,1]$ is given by

$$A\big([0,x),\omega_N\big) = \#\{n \le N\,;\;\; x_n \in [0,x)\}\,,$$

and for the classical one-dimensional $L^2$ discrepancy, the following are known [1; pp. 144–145]:

$$\int_0^1 \left(\frac{A\big([0,x),\omega_N\big)}{N} - x\right)^2 \mathrm{d}x = \frac{1}{N^2}\sum_{m,n=1}^N F(x_m,x_n)\,,$$

where

$$\begin{aligned} F(x,y) &= \frac{1}{3} + \frac{x^2+y^2}{2} - \max(x,y) \\ &= \left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right) + \int_0^1 \left(\{x+t\}-\frac{1}{2}\right)\left(\{y+t\}-\frac{1}{2}\right)\mathrm{d}t\,. \end{aligned}$$

Here $\{x\}$ denotes the fractional part of $x$. Theorems 1 and 3 also work in this case and give the same results.

**2°**. Note that the $L^2$ discrepancy can also be expressed by [1; p. 110]

$$\int_0^1 \left(\frac{A\big([0,x),\omega_N\big)}{N} - x\right)^2 \mathrm{d}x = \frac{1}{N^2}\sum_{m,n=1}^N G(x_m,x_n)\,,$$

where

$$G(x,y)=\Big(x-\frac{1}{2}\Big)\Big(y-\frac{1}{2}\Big)+\frac{1}{2\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\,\mathrm{e}^{2\pi\,\mathrm{i}\,h(x-y)}\,.$$

In this case the function $G(x,y)$ is asymmetrical, but

$$F(x,y)=\frac{G(x,y)+G(y,x)}{2}\,.$$

**3°**. An expression other than above can be obtained for the $L^2$ discrepancy, namely (cf. [18])

$$\int_0^1\left(\frac{A\big([0,x),\omega_N\big)}{N}-x\right)^2\mathrm{d}x=\frac{1}{12N^2}+\frac{1}{N}\sum_{m,n=1}^{N}\left(x_n-\frac{2n-1}{2N}\right)^2$$

provided that $x_1\le x_2\le\cdots\le x_N$.

**4°**. There is another counting function $A(X,\omega_N)$ for obtaining the $L^2$ discrepancy for the class of u.d. sequences. Let us consider the space $\mathbf{X}$ of all continuous functions $f\colon[0,1]\to\mathbb{R}$ satisfying $f(0)=0$. For a given finite sequence $\omega_N=(x_n)_{n=1}^N$ in $[0,1]$ and $f\in\mathbf{X}$ we define the counting function

$$A(f,\omega_N)=\sum_{n=1}^{N}f(x_n)\,,\qquad\text{and let}\qquad g(f)=\int_0^1 f(x)\,\mathrm{d}x\,.$$

Under the norm $\|f\|=\sup\limits_{x\in[0,1]}|f(x)|$ and with the usual Wiener measure $\mathrm{d}f$, the set $\mathbf{X}$ forms a space satisfying the conditions of Theorem 1 in the version of Remark 2. Then the following $L^2$ discrepancy

$$\int_{\mathbf{X}}\left(\frac{A(f,\omega_N)}{N}-\int_0^1 f(x)\,\mathrm{d}x\right)^2\mathrm{d}f$$

again characterizes the class of u.d. sequences.[12] But it is known (cf. [21; p. 80]) that

$$\int_{\mathbf{X}}f(x_m)f(x_n)\,\mathrm{d}f=\frac{\min(x_m,x_n)}{2}\,,$$

[12] The restriction $f(0)=0$ is insignificant, since

$$\frac{A(f-f(0),\omega_N)}{N}-\int_0^1\big(f(x)-f(0)\big)\,\mathrm{d}x=\frac{A(f,\omega_N)}{N}-\int_0^1 f(x)\,\mathrm{d}x\,.$$

and consequently

$$\int\limits_{\mathbf{X}} \left( f(x_n) \int\limits_0^1 f(x)\,\mathrm{d}x \right) \mathrm{d}f = \frac{x_n}{2} - \frac{x_n^2}{4}\,, \qquad \int\limits_{\mathbf{X}} \left( \int\limits_0^1 f(x)\,\mathrm{d}x \right)^2 \mathrm{d}f = \frac{1}{6}\,,$$

and then such the $L^2$ discrepancy is equal to one-half of the classical $L^2$ discrepancy described in $1^\circ$. □

**2. $g$-distributed sequences.**

**$1^\circ$.** Let us apply the method of Theorem 3 to the same counting function $A([0,x),\omega_N)$. We find,[13] by formulas (6) and (7),

$$\int\limits_0^1 \left( \frac{A([0,x),\omega_N)}{N} - g(x) \right)^2 \mathrm{d}x = \frac{1}{N^2} \sum_{m,n=1}^{N} F_{A,g}(x_m, x_n)\,,$$

where

$$F_{A,g}(x,y) = \int\limits_0^1 g^2(t)\,\mathrm{d}t - \int\limits_x^1 g(t)\,\mathrm{d}t - \int\limits_y^1 g(t)\,\mathrm{d}t + 1 - \max(x,y)\,.$$

Using Theorem 1, one can show that the $L^2$ discrepancy is a discrepancy of the class of all sequences with limit law $g(x)$.

**$2^\circ$.** It should be noted that if $g$ is continuous on $[0,1]$ and $F$ is the same function as in Paragraph 1 ($1^\circ$), then

$$D_{F\cdot g}(x_1, x_2, \dots, x_N) := \frac{1}{N^2} \sum_{m,n=1}^{N} F\big(g(x_m), g(x_n)\big)$$

is again a discrepancy for the class of sequences having $g$ as their limit law. This follows from the following fact (cf. [2; p. 68, Exercise 7.19]):

*If $g$ is the continuous limit law of the sequence $\omega = (x_n)_{n=1}^{\infty}$ in $[0,1]$, then the sequence $g(\omega) = \big(g(x_n)\big)_{n=1}^{\infty}$ is u.d.* □

[13] Compare with [8; Theorem 1].

**3. Statistically convergent sequences.**

**1°**. Let $\omega = (x_n)_{n=1}^{\infty}$ be a sequence of real numbers. The $\omega$ is said to be *statistically convergent* to the number $\alpha$ provided that for each $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N\,;\ \ |x_n - \alpha| \ge \varepsilon\} = 0\,.$$

The definition of statistical convergence was given by H. Fast [16] and I. J. Schoenberg [17], independently. As in Schoenberg [17] we see that

*For $\alpha \in [0,1]$ let $c_\alpha(x)$ be the one-jump function which has a jump of size 1 at $\alpha$. The sequence $\omega = (x_n)_{n=1}^{\infty} \subset [0,1]$ is statistically convergent to the number $\alpha$ if and only if the sequence $\omega$ admits the limiting distribution $c_\alpha(x)$.*

Choose $g(x) = c_\alpha(x)$ in the above function $F_{A,g}$. Then, we get

$$F_{A,g}(x,y) = \frac{|x-\alpha|}{2} + \frac{|y-\alpha|}{2} - \frac{|x-y|}{2}\,.$$

We thus have the $L^2$ discrepancy

$$D(x_1, x_2, \dots, x_N) = \frac{1}{N}\sum_{n=1}^{N} |x_n - \alpha| - \frac{1}{2N^2}\sum_{m,n=1}^{N} |x_m - x_n|$$

for the class of sequences statistically convergent to $\alpha$.

**2°**. To illustrate Theorem 4, we consider $F(x,y) = |x-y|$ and every distribution function defined on $\mathbf{X} = \{[0,x)\,;\ \ x \in [0,1]\}$ we shall identify with a nondecreasing $g\colon [0,1] \to [0,1]$. For any such $g$ we have

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 0 \iff g = c_\alpha \qquad \text{for some} \quad \alpha \in [0,1]\,.$$

Thus, the discrepancy

$$D_F(x_1, \dots, x_N) = \frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - x_n|$$

characterizes the class of all sequences $\omega$ satisfying $G(\omega) \subset \{c_\alpha(x)\,;\ \ \alpha \in [0,1]\}$.

□

The following example demonstrates that the notion of *diaphony* can be viewed as a special case of the $L^2$ discrepancy.

**4. Diaphony.**

Let us consider the case $X = [x, y)$, $\mathbf{Y} = [0, 1]$, and

$$A\big([x, y), \omega_N\big) = \#\big\{n \le N\,;\ \ x_n \in [x, y)\big\}\,.$$

Applying Theorem 3 once again,

$$\int\limits_{0 \le x \le y \le 1} \left(\frac{A\big([x, y), \omega_N\big)}{N} - g(x, y)\right)^2 \mathrm{d}x\,\mathrm{d}y = \frac{1}{N^2} \sum_{m,n=1}^{N} F_{A,g}(x_m, x_n)\,,$$

where

$$F_{A,g}(x_m, x_n) = \int\limits_{0 \le x \le y \le 1} g^2(x, y)\ \mathrm{d}x\,\mathrm{d}y - \int\limits_0^{x_m} \mathrm{d}x \int\limits_{x_m}^{1} g(x, y)\ \mathrm{d}y$$

$$- \int\limits_0^{x_n} \mathrm{d}x \int\limits_{x_n}^{1} g(x, y)\ \mathrm{d}y + \min(x_m, x_n) - x_m x_n\,.$$

The function $g(x, y)$ is supposed to be continuous on $0 \le x \le y \le 1$ and $g(0, 1) = 1$ for this moment. Theorem 1 shows that the $L^2$ norm convergence and the pointwise convergence of $A\big([x, y), \omega_N\big)/N - g(x, y)$ are equivalent for $\mathbf{X}^* = \big\{[x, y)\,,\ \ 0 < x < y < 1\big\}$. Since

$$\frac{A\big([0, \varepsilon), \omega_N\big)}{N} + \frac{A\big([\varepsilon, 1 - \varepsilon), \omega_N\big)}{N} + \frac{A\big([1 - \varepsilon, 1), \omega_N\big)}{N} \le 1$$

and $g(\varepsilon, 1 - \varepsilon) \to 1$ as $\varepsilon \to 0$, we obtain small $A\big([0, \varepsilon), \omega_N\big)/N$ for sufficiently small $\varepsilon$ and large $N$. This clearly shows that the equivalence of convergences can be extended to $\mathbf{X}^{**} = \big\{[x, y)\,,\ \ 0 \le x \le y \le 1\big\}$. Therefore, $g(x, y) = g(0, y) - g(0, x)$ and $\mathbf{\Omega}_{A,g(x,y)} = \mathbf{\Omega}_{A,g(0,x)}$, i.e. the above described $L^2$ discrepancy is also a discrepancy of the class $g(0, x)$-distributed sequences other than in Paragraph 2. Furthermore, for any measurable $\psi(x, y) = \psi(0, y) - \psi(0, x)$, we have

$$\int\limits_{0 \le x \le y \le 1} \psi^2(x, y)\ \mathrm{d}x\,\mathrm{d}y = \frac{1}{2} \int\limits_0^1 \int\limits_0^1 \big(\psi(0, y) - \psi(0, x)\big)^2\ \mathrm{d}x\,\mathrm{d}y$$

$$= \int\limits_0^1 \psi^2(0, x)\ \mathrm{d}x - \left(\int\limits_0^1 \psi(0, x)\ \mathrm{d}x\right)^2.$$

and hence the $L^2$ discrepancy is representable in the form

$$\int\limits_{0\le x\le y\le 1}\left(\frac{A([x,y),\omega_N)}{N}-g(x,y)\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$=\int_0^1\left(\frac{A([0,x),\omega_N)}{N}-g(0,x)\right)^2 \mathrm{d}x-\left(\int_0^1\left(\frac{A([0,x),\omega_N)}{N}-g(0,x)\right)\mathrm{d}x\right)^2 .$$

According to Z i n t e r h o f [11], this $L^2$ discrepancy may be called a *diaphony*. In K u i p e r s [3] the following expression is given

$$\int_0^1\left(\frac{A([0,x),\omega_N)}{N}-g(0,x)\right)^2 \mathrm{d}x-\left(\int_0^1\left(\frac{A([0,x),\omega_N)}{N}-g(0,x)\right)\mathrm{d}x\right)^2$$

$$=\frac{1}{2\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\left|\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}^{-2\pi\,\mathrm{i}\,hx_n}-\int_0^1\mathrm{e}^{-2\pi\,\mathrm{i}\,hx}\,\mathrm{d}g(0,x)\right|^2 ,$$

which, in the case $g(0,x)=x$, was given by W . J . L e V e q u e [4]. □

We arrive now at the multidimensional case in which Theorems 1 and 3 are usually applied.

## 5. Multidimensional uniform distribution.

Let us take

$$\omega_N=\big((x_{n,1},\dots,x_{n,s})\big)_{n=1}^{N}\subset[0,1]^s ,$$

and

$$A\big([0,x_1)\times\cdots\times[0,x_s),\omega_N\big)=\#\big\{n\le N;\ (x_{n,1},\dots,x_{n,s})\in[0,x_1)\times\cdots\times[0,x_s)\big\} .$$

Then

$$\int\limits_{[0,1]^s}\left(\frac{A\big([0,x_1)\times\cdots\times[0,x_s),\omega_N\big)}{N}-g(x_1,\dots,x_s)\right)^2 \mathrm{d}x_1\dots\mathrm{d}x_s$$

$$=\frac{1}{N^2}\sum_{m,n=1}^{N}F_{A,g}\big((x_{m,1},\dots,x_{m,s}),(x_{n,1},\dots,x_{n,s})\big) ,$$

where

$$F_{A,g}\big((x_{m,1},\dots,x_{m,s}),(x_{n,1},\dots,x_{n,s})\big)$$

$$= \int\limits_{[0,1]^s} g^2(x_1,\dots,x_s)\,\mathrm{d}x_1\dots\mathrm{d}x_s - \int\limits_{x_{m,1}}^{1}\mathrm{d}x_1\dots\int\limits_{x_{m,s}}^{1} g(x_1,\dots,x_s)\,\mathrm{d}x_s$$

$$- \int\limits_{x_{n,1}}^{1}\mathrm{d}x_1\dots\int\limits_{x_{n,s}}^{1} g(x_1,\dots,x_s)\,\mathrm{d}x_s + \prod_{j=1}^{s}\big(1-\max(x_{m,j},x_{n,j})\big)\,.$$

For $g(x_1,\dots,x_s)=x_1\dots x_s$ we mention the following known result (see [13]):

$$\int\limits_{[0,1]^s}\left(\frac{A\big([0,x_1)\times\dots\times[0,x_s),\omega_N\big)}{N}-x_1\dots x_s\right)^2\mathrm{d}x_1\dots\mathrm{d}x_s$$

$$=\frac{1}{3^s}+\frac{1}{N^2}\sum_{m,n=1}^{N}\prod_{j=1}^{s}\big(1-\max(x_{m,j},x_{n,j})\big)-\frac{1}{2^{s-1}N}\sum_{n=1}^{N}\prod_{j=1}^{s}(1-x_{n,j}^2)\,.$$

□

There are some interesting counting functions for which the conditions of Theorems 1 and 3 are not satisfied, among them, another variant of discrepancy of the class of u.d. sequences is the following.

## 6. A variant of the $L^2$ discrepancy for u.d. sequences.

**1°**. Let $X=\big([0,x),y\big)$, where $[0,x)\subset[0,1]$, $y\in[0,1]$, and $\omega_N=(x_n)_{n=1}^N\in[0,1]^N$. Let the counting function $A\big(([0,x),y),\omega_N\big)$ on the space $\mathbf{X}=\big\{([0,x),y);\ 0\le y\le x\le 1\big\}$ be defined by the equality

$$A\big(([0,x),y),\omega_N\big)=\#\big\{(m,n);\ m,n\le N,\ x_m,x_n\in[0,x),\ |x_m-x_n|<y\big\}\,.$$

It can be shown as in [7] that the limit

$$\lim_{N\to\infty}\frac{A\big(([0,x),y),\omega_N\big)}{N^2}=2xy-y^2\qquad\text{for}\quad 0\le y\le x\le 1\,,$$

again determines the class of all uniformly distributed sequences in $[0,1]$. In this example the additivity condition (i) of $A\big(([0,x),y),\omega_N\big)$ of Theorem 3 is violated. On the other hand, for $\omega_{N^2}=\big((\max(x_m,x_n),|x_m-x_n|)\big)_{m,n=1}^{N}$, we have

$$A\big(([0,x),y),\omega_N\big)=A\big([0,x)\times[0,y),\omega_{N^2}\big)\,,$$

where the right-hand side is defined in the preceding paragraph. It can be easily seen that $A([0,x)\times[0,y),\omega_{N^2})$ is satisfying (i). Thus, from Theorems 1 and 3 we can infer the following variant of the $L^2$ discrepancy for the class of uniformly distributed sequences other than in Paragraph 1.

$$\int\limits_{0\le y\le x\le 1}\left(\frac{A([0,x)\times[0,y),\omega_{N^2})}{N^2}-(2xy-y^2)\right)^2\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{1}{N^4}\sum_{m,n,s,r=1}^{N}F_{A,2xy-y^2}(x_m,x_n,x_r,x_s)\,.$$

A straightforward calculation shows that in this case

$$F_{A,2xy-y^2}(x_m,x_n,x_r,x_s)=\frac{23}{90}+A(u_1,v_1)+A(u_2,v_2)+B(u_1,v_1,u_2,v_2)\,,$$

where

$$A(u_1,v_1)=\frac{v_1^2}{2}-\frac{v_1^3}{3}+\frac{u_1^4}{6}-\frac{u_1^2v_1^2}{2}+\frac{u_1v_1^3}{3}\,,$$

$$B(u_1,v_1,u_2,v_2)=-\max(v_1,v_2)-\frac{\max^2(u_1,u_2)}{2}+\max(u_1,u_2)\max(v_1,v_2)\,,$$

and

$$u_1=\max(x_m,x_n)\,,\qquad v_1=|x_m-x_n|\,,\qquad u_2=\max(x_r,x_s)\,,\qquad v_2=|x_r-x_s|\,.$$

**2°**. Note that in [7] we have proved the following result:

*Let $\omega=(x_n)_{n=1}^{\infty}$ be a given infinite sequence in $[0,1]$, and let $\tilde\omega$ be the sequence consisting of all the distances $|x_m-x_n|$, $m,n=1,2,\dots$, ordered so that the first $N^2$ terms are $\tilde\omega_{N^2}=\left(|x_m-x_n|\right)_{m,n=1}^{N}$. Then $\omega$ is u.d. if and only if $\tilde\omega$ has the limit law $2x-x^2$.*

This and the $L^2$ discrepancy in Paragraph 2 (2°) also implies that $\omega$ is u.d. sequence if and only if

$$\lim_{N\to\infty}\frac{1}{N^4}\sum_{m,n,r,s=1}^{N}F\left(2|x_m-x_n|-(x_m-x_n)^2,\ 2|x_r-x_s|-(x_r-x_s)^2\right)=0\,,$$

where $F$ is the same as in Paragraph 1. □

There is an example of counting function for which the monotonicity condition (iv) of Theorem 1 (or uniform continuity (vi') of Remark 2) is not satisfied.

**7. Diophantine approximations.**

For a given finite sequence $\omega_N = \big((x_n, z_n)\big)_{n=1}^N$ in $[0,1]^2$, a point $x \in [0,1]$, let us find the $L^2$ discrepancy of the counting function $A(x,\omega_N)$ defined by the equality

$$A(x,\omega_N) = \#\{n \le N\,;\ |x - x_n| < z_n\}\,.$$

This $A(x,\omega_N)$ satisfies the additivity requirement (i) imposed in Theorem 3, thus we can again set

$$\int_0^1 \left(\frac{A(x,\omega_N)}{N} - g(x)\right)^2 \mathrm{d}x = \frac{1}{N^2} \sum_{m,n=1}^N F_{A,g}(x_m, x_n)\,,$$

where

$$F_{A,g}(x_m,x_n) = \int_0^1 g^2(x)\,\mathrm{d}x - \int\limits_{(x_m-z_m,x_m+z_m)\cap[0,1]} g(x)\,\mathrm{d}x - \int\limits_{(x_n-z_n,x_n+z_n)\cap[0,1]} g(x)\,\mathrm{d}x$$
$$+ \big|(x_m - z_m, x_m + z_m) \cap (x_n - z_n, x_n + z_n) \cap [0,1]\big|\,,$$

Putting $g(x) = \Big(2\sum\limits_{n=1}^N z_n\Big)\Big/N$, we arrive at

$$\begin{aligned}
&\int_0^1 \left(\frac{A(x,\omega_N)}{2\sum\limits_{n=1}^N z_n} - 1\right)^2 \mathrm{d}x \\
&= 1 - \frac{2}{2\sum\limits_{n=1}^N z_n} \sum_{n=1}^N \big|(x_n - z_n, x_n + z_n) \cap [0,1]\big| \qquad (8)\\
&\quad + \frac{1}{\Big(2\sum\limits_{n=1}^N z_n\Big)^2} \sum_{m,n=1}^N \big|(x_m - z_m, x_m + z_m) \cap (x_n - z_n, x_n + z_n) \cap [0,1]\big|\,.
\end{aligned}$$

Since this counting function does not satisfy the monotonicity conditions (vi) and (vi'), we cannot apply Theorem 1. Therefore, for the given $A(x,\omega_N)$ we may use only the weak Theorem 2. Applying this theorem, assuming $(x_n - z_n, x_n + z_n) \subset [0,1]$ for all $n$, we have the implication: From

$$\lim_{N\to\infty} \frac{1}{\Big(2\sum\limits_{n=1}^N z_n\Big)^2} \sum_{m,n=1}^N \big|(x_m - z_m, x_m + z_m) \cap (x_n - z_n, x_n + z_n)\big| = 1\,,$$

it follows the existence of a sequence $(N_k)_{k=1}^{\infty}$ for which

$$\lim_{k\to\infty} \frac{A(x,\omega_{N_k})}{2\sum_{n=1}^{N_k} z_n} = 1 \qquad \text{for almost all} \quad x\in[0,1]\,.$$

The right-hand side implies the divergence $\sum_{n=1}^{\infty} z_n = +\infty$, and immediately $\lim_{N\to\infty} A(x,\omega_N) = +\infty$ for almost all $x\in[0,1]$. The sequence $\omega = (x_n)_{n=1}^{\infty} \subset [0,1]$ for which $\lim_{N\to\infty} A(x,\omega_N) = +\infty$ for almost all $x\in[0,1]$ and every nonincreasing $(z_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} z_n = +\infty$ is called *eutaxic* [5]. The sequence $\omega = (x_n)_{n=1}^{\infty} \subset [0,1]$ for which $\lim_{N\to\infty} \frac{A(x,\omega_N)}{2\sum_{n=1}^{N} z_n} = 1$ for almost all $x\in[0,1]$ and every nonincreasing $(z_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} z_n = +\infty$ is called *strongly eutaxic* [5].

An interesting consequence of (8) is that

$$\left(2\sum_{n=1}^{N} z_n\right)^2 \le \sum_{m,n=1}^{N} \left|(x_m - z_m, x_m + z_m)\cap(x_n - z_n, x_n + z_n)\right|,$$

provided that $(x_n - z_n, x_n + z_n)\subset[0,1]$ for $n=1,2,\dots,N$.

There is an alternative representation of the measure of intersection

$$(x_m - z_m, x_m + z_m)\cap(x_n - z_n, x_n + z_n)$$

as

$$\min\bigl(2\min(z_m, z_n),\ \max(0,\ z_m+z_n-|x_m-x_n|)\bigr)\,.$$

□

The most interesting problem that could not be solved in this paper is whether there exists the $L^2$ discrepancy of a so called *uniformly quick* sequence. This class of sequences has a role in the still unproved Duffin-Schaeffere conjecture. In the following paragraph we shall find a point of difficulty in this problem.[14)]

## 8. Uniformly quick sequences.

An important space $\mathbf{X}$ of objects $X$ is formed by the open sets $X\subset[0,1]$. Each set $X$ can be represented by an infinite union of pairwise disjoint open

[14)] The definition may be found in O. Strauch [9].

subintervals (possibly empty) of $[0,1]$, say $X = \bigcup_{n=1}^{\infty} I_n$. The Lebesgue measure of any open set $X$ is denoted by $|X|$, i.e. $|X| = \sum_{n=1}^{\infty} |I_n|$. For $\omega_N = (x_n)_{n=1}^N \subset [0,1]$, define the counting function

$$A(X,\omega_N) = \#\{i = 1,2,\dots\,;\ \text{there exists an } n \le N \text{ such that } x_n \in I_i\} + \#\{n \le N\,;\ x_n \notin X\}\,.$$

An infinite sequence $\omega = (x_n)_{n=1}^{\infty} \subset [0,1]$ is said to be *uniformly quick* if

$$\lim_{N\to\infty} \frac{A(X,\omega_N)}{N} = 1 - |X|\,,$$

for every open set $X \subset [0,1]$. In this case, $A(X,\omega_N)$ does not have the additivity properties (i) from Theorem 3. One possibility to find the $L^2$ discrepancy is to express the counting function in terms of another counting function (see Paragraph 6) which at once satisfies (i), and then to use Theorem 3.

Indeed, let $\Omega_N = (J_n)_{n=1}^N$ be a finite sequence of closed (not necessarily disjoint) subintervals $J_n \subset [0,1]$. For given $X \subset [0,1]$, we introduce the counting function

$$A(X,\Omega_N) = \#\{n \le N\,;\ J_n \subset X\}\,.$$

It follows immediately from the definition that $A(X,\Omega_N)$ is satisfying (i).

We turn now to the case $A(X,\omega_N)$. We shall order the numbers of $\omega_N$ according to their magnitude

$$\omega_N = (x_1 \le x_2 \le \cdots \le x_N)\,.$$

In the new ordering, define for $n = 1,2,\dots,N-1$, $J_n = [x_n, x_{n+1}]$, and $\Omega_{N-1} = (J_n)_{n=1}^{N-1}$. Directly from the definition we have

$$A(X,\omega_N) = N - A(X,\Omega_{N-1})\,.$$

In order to compute the $L^2$ discrepancy of $A(X,\Omega_{N-1})$, we must know a measure on $\mathbf{X}$ with the property (ii) required by Theorem 3. Bearing this in mind, the $L^2$ discrepancy of $A(X,\omega_N)$ with respect to $g(X) = 1 - |X|$ can be written as

$$\begin{aligned}
&\int\limits_{\mathbf{X}} \left(\frac{A(X,\omega_N)}{N} - \bigl(1 - |X|\bigr)\right)^2 \mathrm{d}X \\
&= \int\limits_{\mathbf{X}} \left(\frac{A(X,\Omega_{N-1})}{N} - |X|\right)^2 \mathrm{d}X \\
&= \int\limits_{\mathbf{X}} |X|^2 \,\mathrm{d}X - \frac{2}{N}\sum_{n=1}^{N-1} \int\limits_{\mathbf{X}_n} |X| \,\mathrm{d}X + \frac{1}{N^2}\sum_{m,n=1}^{N-1} \int\limits_{\mathbf{X}_m\cap\mathbf{X}_n} 1\cdot \mathrm{d}X\,,
\end{aligned}$$

where $\mathbf{X}_n = \{X \in \mathbf{X};\ J_n \subset X\}$. By making use of the $L^2$ discrepancy, we want to use Theorem 1 to characterize the class of uniformly quick sequences. To this end, we again have to describe a suitable topology, ordering and measure on the space $\mathbf{X}$. The topology can be constructed by the pseudometric

$$d(X, X') = |(X - X') \cup (X' - X)|,$$

and for the ordering we again can make use of the set-inclusion

$$X \prec X' \Longleftrightarrow X \supset X'.$$

Note that one does not know whether or not the suitable measure on $\mathbf{X}$ under the requirement (v) imposed in Theorem 1 is possible. □

To illustrate Theorem 4, we determine, for special functions $F$, the set $G(F)$.

**9. Transformation of sequences.**

Let $\omega = (x_n)_{n=1}^{\infty}$ be a sequence in $[0,1]$. For a continuous function $f$: $[0,1] \to [0,1]$, let us take $f(\omega) := \big(f(x_n)\big)_{n=1}^{\infty}$, and for any distribution function $g$: $[0,1] \to [0,1]$ we define

$$g_f(x) = \int\limits_{f^{-1}([0,x))} 1 \cdot \mathrm{d}g(x).$$

Since $f^{-1}\big([0,x)\big)$ is Jordan-measurable, distribution function $g_f(x)$ is defined a.e.[15] Clearly, for the counting function $A\big([0,x), \omega_N\big)$ defined in Paragraph 1 (1°), we have

$$A\big([0,x), f(\omega)_N\big) = A\big(f^{-1}([0,x)), \omega_N\big),$$

and applying Theorem 3, for a given distribution function $g_0$: $[0,1] \to [0,1]$, once again

$$\int\limits_0^1 \left(\frac{A\big(f^{-1}([0,x)), \omega_N\big)}{N} - g_0(x)\right)^2 \mathrm{d}x = \frac{1}{N^2} \sum_{m,n=1}^{N} F_1(x_m, x_n),$$

[15] Consider e.g. $f(x) = 4x(1-x)$. Then we have

$$g_f(x) = g\big(f_1^{-1}(x)\big) + 1 - g\big(f_2^{-1}(x)\big),$$

where $f_1^{-1}(x) = \big(1 - \sqrt{1-x}\ \big)/2$ and $f_2^{-1}(x) = \big(1 + \sqrt{1-x}\ \big)/2$ are the inverse functions to $f$.

where $F_1(x,y) = F_{A,g_0}\big(f(x), f(y)\big)$ and $F_{A,g_0}$ is defined as in Paragraph 2.

Similarly, for continuous function $h\colon [0,1] \to [0,1]$, we have

$$\int_0^1 \left( \frac{A\big(f^{-1}([0,x)), \omega_N\big)}{N} - \frac{A\big(h^{-1}([0,x)), \omega_N\big)}{N} \right)^2 \mathrm{d}x = \frac{1}{N^2} \sum_{m,n=1}^{N} F_2(x_m, x_n)\,,$$

where

$$F_2(x,y) = \max\big(f(x), h(y)\big) + \max\big(f(y), h(x)\big) - \max\big(f(x), f(y)\big) - \max\big(h(x), h(y)\big).$$

Limiting the above identities, we find

$$\int_0^1 \int_0^1 F_1(x,y)\, \mathrm{d}g(x)\, \mathrm{d}g(y) = \int_0^1 \big(g_0(x) - g_f(x)\big)^2\, \mathrm{d}x\,,$$

and

$$\int_0^1 \int_0^1 F_2(x,y)\, \mathrm{d}g(x)\, \mathrm{d}g(y) = \int_0^1 \big(g_f(x) - g_h(x)\big)^2\, \mathrm{d}x$$

for any distribution function $g$. $\square$

Analogously to the $A(X, \omega_N)$ we can study a counting function $A(X, \omega_T)$ with a continuous parameter $T$.

## 10. Continuously distributed functions.

Let $\omega\colon [0,+\infty) \to [0,1]$ be a Lebesgue-measurable function. For $T > 0$ we denote the partial function $\omega_T := \omega/[0,T]$. For $X = [0,x) \subset [0,1]$, we define the counting function $A\big([0,x), \omega_T\big)$ by

$$A\big([0,x), \omega_T\big) = \big|\omega^{-1}\big([0,x)\big) \cap [0,T]\big|\,,$$

where $|\cdot|$ is Lebesgue's measure in $[0,1]$. If for all continuity points $x \in [0,1]$ of a given distribution function $g\colon [0,1] \to [0,1]$ we have

$$\lim_{T\to\infty} \frac{A\big([0,x), \omega_T\big)}{T} = g(x)\,,$$

then the function $\omega$ is said to be *g-continuously distributed* (cf. [2; p. 78]). Along the same lines as Theorems 1, 3 it can be shown that

$$\lim_{T\to\infty} \int_0^1 \left( \frac{A\big([0,x), \omega_T\big)}{N} - g(x) \right)^2 \mathrm{d}x = 0 \iff \omega(t) \text{ is } g\text{-continuously distributed,}$$

and

$$\int_0^1 \left( \frac{A\big([0,x),\omega_T\big)}{N} - g(x) \right)^2 \mathrm{d}x = \frac{1}{T^2} \int_0^T \int_0^T F_{A,g}\big(\omega(t_1), \omega(t_2)\big) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \,,$$

where $F_{A,g}$ is the same function as in Paragraph 2. □

In our final paragraph, we shall discuss a result from a joint paper by P. J. Grabner and R. F. Tichy [20], in the way of Theorem 1. They show that an adequate quantitative measure for statistical independence of sequences is the $L^2$ discrepancy, whereas the usual extremal is not suitable for this purpose.

**11. Statistically independent sequences.**

Two sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in the unit interval $[0,1]$ are called *statistically independent if*

$$\lim_{N\to\infty} \left( \frac{1}{N} \sum_{n=1}^{N} f(x_n) g(y_n) - \frac{1}{N^2} \sum_{n=1}^{N} f(x_n) \sum_{n=1}^{N} g(y_n) \right) = 0$$

for all continuous real functions $f$, $g$.

Let the class $\mathbf{\Omega}$ be defined as the set of all two-dimensional sequences in the unit square for which the sequences of the first and second coordinates are statistically independent. For $\omega_N = \big((x_n, y_n)\big)_{n=1}^{N}$, denote $\tilde{\omega}_{N^2} = \big((x_m, y_n)\big)_{m,n=1}^{N}$, and define

$$A^*\big([0,x) \times [0,y), \omega_N\big) = A\big([0,x) \times [0,y), \omega_N\big) - \frac{A\big([0,x) \times [0,y), \tilde{\omega}_{N^2}\big)}{N} \,,$$

where the counting function $A$ have the same meaning as in Paragraph 5.

Using this notation we shall reformulate the results of [20] in the following way:

- **1°**. The class $\mathbf{\Omega}_{A^*,0}$ defined by (1) is a proper subclass of $\mathbf{\Omega}$;
- **2°**. The $L^2$ discrepancy associated with the counting function $A^*$ and the function $g(x,y) \equiv 0$ by formula (2) is a discrepancy of $\mathbf{\Omega}$.

The first result is a consequence of the fact that Theorem 1 is inapplicable to $A^*$, since we cannot find an ordering on $\mathbf{X} = \big\{[0,x) \times [0,y) \,;\; x, y \in [0,1)\big\}$ satisfying the monotonicity condition (vi).

To obtain an alternative proof of 2° we compute the associated $L^2$ discrepancy. To do this, we denote $\mathbf{X}_{m,n} = \big\{[0,x) \times [0,y) \,;\; x,y \in [0,1] \,,\; (x_m, y_n) \in$

$[0,x)\times[0,y)\}$ and the measure and topology on $\mathbf{X}=\{[0,x)\times[0,y)\,;\ x,y\in[0,1]\}$ we identify with the Lebesgue measure and Euclidean topology on the unit square. With the help of the expression

$$\frac{A^*\big([0,x)\times[0,y),\,\omega_N\big)}{N}$$
$$=\frac{1}{N}\sum_{n=1}^{N}c_{\mathbf{X}_{n,n}}\big([0,x)\times[0,y)\big)-\frac{1}{N^2}\sum_{m,n=1}^{N}c_{\mathbf{X}_{m,n}}\big([0,x)\times[0,y)\big)\,,$$

we find

$$\int_0^1\int_0^1\left(\frac{A^*\big([0,x)\times[0,y),\,\omega_N\big)}{N}-0\right)^2\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{1}{N^2}\sum_{m,n}^{N}|\mathbf{X}_{m,m}\cap\mathbf{X}_{n,n}|+\frac{1}{N^4}\sum_{m,n,k,l=1}^{N}|\mathbf{X}_{m,n}\cap\mathbf{X}_{k,l}|-\frac{2}{N^3}\sum_{m,k,l=1}^{N}|\mathbf{X}_{m,m}\cap\mathbf{X}_{k,l}|\,.$$

We complete the expression by

$$|\mathbf{X}_{m,n}\cap\mathbf{X}_{k,l}|=\big(1-\max(x_m,x_k)\big)\big(1-\max(y_n,y_l)\big)\,.$$

In a similar way as in Paragraph 1 (4°), we define the counting function

$$\tilde{A}\big((f,g),\omega_N\big)=\sum_{n=1}^{N}f(x_n)g(y_n)-\frac{1}{N}\sum_{n=1}^{N}f(x_n)\sum_{n=1}^{N}g(y_n)$$

on the cartesian product $\mathbf{X}\times\mathbf{X}$ of $\mathbf{X}=\{f\colon[0,1]\to\mathbb{R}\,;\ f(0)=0,\ f$ is continuous$\}$, and we compute the $L^2$ discrepancy

$$\int_{\mathbf{X}}\int_{\mathbf{X}}\left(\frac{\tilde{A}\big((f,g),\omega_N\big)}{N}-0\right)^2\mathrm{d}f\,\mathrm{d}g$$
$$=\frac{1}{N^2}\sum_{m,n}^{N}\frac{\min(x_m,x_n)}{2}\frac{\min(y_m,y_n)}{2}+\frac{1}{N^4}\sum_{m,n,k,l=1}^{N}\frac{\min(x_m,x_n)}{2}\frac{\min(y_k,y_l)}{2}$$
$$-\frac{2}{N^3}\sum_{m,k,l=1}^{N}\frac{\min(x_m,x_k)}{2}\frac{\min(y_m,y_l)}{2}\,.$$

Then we apply Theorem 1 with Remark 2 and we obtain that this $L^2$ discrepancy is a discrepancy of the class $\boldsymbol{\Omega}$. Finally, it can be verified

$$\int_0^1\int_0^1\left(\frac{A^*([0,x)\times[0,y),\,\omega_N)}{N}-0\right)^2 \mathrm{d}x\,\mathrm{d}y = 4\int_{\mathbf{X}}\int_{\mathbf{X}}\left(\frac{\tilde{A}((f,g),\omega_N)}{N}-0\right)^2 \mathrm{d}f\,\mathrm{d}g\,,$$

which leads to $2^\circ$. $\square$

## REFERENCES


[1] JACOBS, K.: *Measure and Integral*, Academic Press, New York-San Francisco-London, 1978.

[2] KUIPERS, L.—NIEDERREITER, H.: *Uniform Distribution of Sequences*, John Wiley & Sons, New York, 1974.

[3] KUIPERS, L.: *Remark on the Weyl-Schoenberg criterion in the theory of asymptotic distribution of real numbers*, Nieuw Arch. Wisk. (3) **16** (1968), 197–202.

[4] LEVEQUE, W. J.: *An inequality connected with Weyl's criterion for uniform distribution.* In: Theory of Numbers. Proc. Sympos. Pure Math., Calif. Inst. Tech., Pasadena, Calif. 1963. Vol. VIII, Amer. Math. Soc., Providence, R. I., 1965, pp. 22–30.

[5] DE MATHAN, B.: *Approximations diophantiennes dans un corps local*, Bull. Soc. Math. France, Mémoire **21** (1970).

[6] SCHOENBERG, I. J.: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199.

[7] STRAUCH, O.: *On the $L^2$ discrepancy of distances of points from a finite sequence*, Math. Slovaca **40** (1990), 245–259.

[8] STRAUCH, O.: *A new moment problem of distribution functions in the unit interval*, Math. Slovaca **44** (1994), 171–211.

[9] STRAUCH, O.: *Duffin-Schaeffer conjecture and some new types of real sequences*, Acta Math. Univ. Comenian. **XL–XLI** (1982), 233–265.

[10] WINKLER, R.: *Some remarks on pseudorandom sequences*, Math. Slovaca **43** (1993), 493–512.

[11] ZINTERHOF, P.: *Über einige Abschätzungen bei der Approximation von Funktionen mit Gleichverteilungsmethoden*, Sitzungsber. Österreich. Akad. Wiss. Math.-Natur. Kl II **185** (1976), 121–132.

[12] GOLDSTERN, M.: *The complexity of uniform distribution*, Math. Slovaca **44** (1994), 491–500.

[13] DOBROVOĹSKIĬ, N. M.: *An effective proof of Roth's theorem on quadratic deviation* (Russian), Uspekhi Mat. Nauk **39** (1984), 155–156.

[14] WEYL, H.: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352.

[15] VAN DER CORPUT, J. G.—PISOT, C.: *Sur la discrépance modulo un*, Indag. Math. **1** (1939), 143–153, 184–195, 260–269.

[16] FAST, H.: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[17] SCHOENBERG, I. J.: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.

[18] NIEDERREITER, H.: *Application of diophantine approximations to numerical integration.* In: Diophantine Approximation and Its Applications (C. F. Osgood, ed.), Academic Press, New York, 1973, pp. 129–199.

[19] KOKSMA, J. F.: *Een algemeene stelling uit de theorie der gelijmatige verdeeling modulo* 1, Mathematica B (Zutphen) **11** (1942/43), 7–11.

[20] GRABNER, P. J.—TICHY, R. F.: *Remarks on statistical independence of sequences*, Math. Slovaca **44** (1994), 91–94.

[21] GEL'FAND, I. M.—JAGLOM, A. M.: *Integration in functional spaces and its application in quantum physics* (Russian), Uspekhi Mat. Nauk **11** (1956), 77–114.

[22] SHOHAT, J. A.—TAMARKIN, J. D.: *The Problem of Moments. Mathematical Surveys*, Amer. Math. Soc., Providence, Rhode Island, 1943.

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