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EVALUATION OF TWO TRIGONOMETRIC SUMS

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ABSTRACT. Let α and m be integers with $0 < \alpha < m$. In 1844 Eisenstein used the trigonometric identity

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot \frac{k\pi}{m} = m - 2\alpha$$

to give a proof of the law of quadratic reciprocity. In this paper we give two generalizations of this formula of Eisenstein.

1. Introduction

Let m, n and α be integers with $m \geq 2$, $n \geq 1$, $0 < \alpha < m$. We determine explicitly the values of the trigonometric sums $\sum\limits_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^n \frac{k\pi}{m}$ and $\sum\limits_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^n \frac{k\pi}{m}$ in terms of the values $B_1\left(\frac{\alpha}{m}\right), \dots, B_n\left(\frac{\alpha}{m}\right)$ of Bernoulli polynomials (Theorem, §3). The evaluations are carried out using the infinite partial fraction expansion of $\cot^n x$ (Proposition, §2).

2. Partial fraction expansion of $\cot^n x$

For integers p and v with $0 \leq v \leq p-1$ we define the rational numbers $A(p,v)$ by

$$A(p,0) = 1\,, \qquad p \geq 1\,, \tag{2.1}$$

$$A(p,1) = 0\,, \qquad p \geq 2\,, \tag{2.2}$$

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$$A(p,p-1) = \begin{cases} 0\,, & p\,(\text{even}) \geq 2\,, \\ (-1)^{(p-1)/2}\,, & p\,(\text{odd}) \geq 1\,, \end{cases} \tag{2.3}$$

$$A(p,v) = \left(\frac{p-1-v}{p-1}\right) A(p-1,v) - A(p-2,v-2)\,, \qquad 2 \leq v \leq p-2\,. \tag{2.4}$$

From (2.1)–(2.4) we obtain

$$A(p,v) = 0\,, \qquad 1 \leq v\,(\text{odd}) \leq p-1\,, \tag{2.5}$$

$$A(p,2) = -\frac{p}{3}\,, \qquad p \geq 3\,, \tag{2.6}$$

$$A(p,4) = \frac{p^2}{18} - \frac{7p}{90}\,, \qquad p \geq 5\,, \tag{2.7}$$

$$A(p,6) = -\frac{p^3}{162} + \frac{7p^2}{270} - \frac{62p}{2835}\,, \qquad p \geq 7\,, \tag{2.8}$$

$$A(p,8) = \frac{p^4}{1944} - \frac{7p^3}{1620} + \frac{3509}{340200}p^2 - \frac{127}{18900}p\,, \qquad p \geq 9\,. \tag{2.9}$$

For a fixed even integer $v \geq 2$, $A(p,v)$ $(p \geq v+1)$ is a polynomial in p of degree $v/2$ with no constant term. The coefficient of p^v is $\dfrac{(-1)^v}{3^v v!}$. From (2.1)–(2.9) we obtain the following table of values of $A(p,v)$ $(1 \leq p \leq 10,\ 0 \leq v \leq p-1)$:

$p \setminus v$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	0								
3	1	0	-1							
4	1	0	$-\frac{4}{3}$	0						
5	1	0	$-\frac{5}{3}$	0	1					
6	1	0	-2	0	$\frac{23}{15}$	0				
7	1	0	$-\frac{7}{3}$	0	$\frac{98}{45}$	0	-1			
8	1	0	$-\frac{8}{3}$	0	$\frac{44}{15}$	0	$-\frac{176}{105}$	0		
9	1	0	-3	0	$\frac{19}{5}$	0	$-\frac{818}{315}$	0	1	
10	1	0	$-\frac{10}{3}$	0	$\frac{43}{9}$	0	$-\frac{718}{189}$	0	$\frac{563}{315}$	0

We are now ready to determine the infinite partial fraction expansion of $\cot^p x$ for p a positive integer.

PROPOSITION. *Let p be a positive integer. Set*

$$\varepsilon(p) = \begin{cases} 0, & p \text{ odd}, \\ (-1)^{p/2}, & p \text{ even}. \end{cases}$$

Then, for $x \neq j\pi$ $(j = 0, \pm 1, \pm 2, \dots)$, we have

$$\cot^p x = \varepsilon(p) + \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{p-1} \frac{A(p,v)}{(x+j\pi)^{p-v}}, \tag{2.10}$$

where the prime $(')$ indicates that in the summation the terms for j (≥ 1) and $-j$ are taken together.

P r o o f. We prove (2.10) by induction on p. First

$$\varepsilon(1) + \sum_{j=-\infty}^{\infty}{}' \frac{A(1,0)}{x+j\pi} = \sum_{j=-\infty}^{\infty}{}' \frac{1}{x+j\pi} = \frac{1}{x} + \sum_{j=1}^{\infty} \left\{ \frac{1}{x+j\pi} + \frac{1}{x-j\pi} \right\}$$
$$= \frac{1}{x} + 2x \sum_{j=1}^{\infty} \frac{1}{x^2 - j^2\pi^2} = \cot x$$

by [1; p. 75, formula 4.3.91], proving (2.10) for $p = 1$. Secondly

$$\varepsilon(2) + \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{1} \frac{A(2,v)}{(x+j\pi)^{2-v}} = -1 + \sum_{j=-\infty}^{\infty}{}' \frac{1}{(x+j\pi)^2} = -1 + \operatorname{cosec}^2 x = \cot^2 x$$

by [1; p. 75, formula 4.3.92], proving (2.10) for $p = 2$.

Now we make the inductive hypothesis that (2.10) holds for $p = 1, 2, \dots, k-1$, where $k \geq 3$. As

$$\cot^k x = -\frac{1}{k-1} \frac{\mathrm{d}}{\mathrm{d}x}(\cot^{k-1} x) - \cot^{k-2} x,$$

the inductive hypothesis gives

$$\cot^k x = -\frac{1}{k-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon(k-1) + \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{k-2} \frac{A(k-1,v)}{(x+j\pi)^{k-1-v}}\right)$$
$$- \left(\varepsilon(k-2) + \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{k-3} \frac{A(k-2,v)}{(x+j\pi)^{k-2-v}}\right)$$
$$= \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{k-2} \frac{(k-1-v)}{(k-1)} \frac{A(k-1,v)}{(x+j\pi)^{k-v}} + \varepsilon(k) - \sum_{j=-\infty}^{\infty}{}' \sum_{v=2}^{k-1} \frac{A(k-2,v-2)}{(x+j\pi)^{k-v}}$$
$$= \varepsilon(k) + \sum_{j=-\infty}^{\infty}{}' \sum_{v=2}^{k-2} \left(\frac{(k-1-v)}{(k-1)} A(k-1,v) - A(k-2,v-2)\right) \frac{1}{(x+j\pi)^{k-v}}$$
$$+ \sum_{j=-\infty}^{\infty}{}' \left\{\frac{A(k-1,0)}{(x+j\pi)^k} + \frac{k-2}{k-1}\frac{A(k-1,1)}{(x+j\pi)^{k-1}} - \frac{A(k-2,k-3)}{(x+j\pi)}\right\}$$
$$= \varepsilon(k) + \sum_{j=-\infty}^{\infty}{}' \sum_{v=2}^{k-2} \frac{A(k,v)}{(x+j\pi)^{k-v}}$$
$$+ \sum_{j=-\infty}^{\infty}{}' \left(\frac{A(k,0)}{(x+j\pi)^k} + \frac{A(k,1)}{(x+j\pi)^{k-1}} + \frac{A(k,k-1)}{x+j\pi}\right)$$
$$= \varepsilon(k) + \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{k-1} \frac{A(k,v)}{(x+j\pi)^{k-v}}\,.$$

This completes the inductive step and (2.10) follows by the principle of mathematical induction. □

3. Evaluation of $\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^n \frac{k\pi}{m}$ and $\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^n \frac{k\pi}{m}$

Making use of the Proposition, we prove the following Theorem.

THEOREM. *Let m, n, α be integers with $m \ge 2$, $n \ge 1$, $0 < \alpha < m$. Then*

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^n \frac{k\pi}{m}$$
$$= \begin{cases} 0 & \text{if } n = 2p\,, \\ \displaystyle\sum_{v=0}^{p-1} \frac{(-1)^{p-v}(2m)^{2p-2v-1}}{(2p-2v-1)!} A(2p-1,2v) B_{2p-2v-1}\Big(\frac{\alpha}{m}\Big) & \text{if } n = 2p-1\,, \end{cases}$$

and

$$\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^n \frac{k\pi}{m}$$

$$= \begin{cases} 0 & \text{if } n = 2p-1\,, \\ (-1)^{p+1} + \displaystyle\sum_{v=0}^{p-1} \frac{(-1)^{p-v+1}(2m)^{2p-2v}}{(2p-2v)!} A(2p,2v) \cdot & \\ \qquad \cdot \left[B_{2p-2v}\Big(\frac{\alpha}{m}\Big) - \frac{B_{2p-2v}}{m^{2p-2v}} \right] & \text{if } n = 2p\,. \end{cases}$$

P r o o f. The transformation $k \to m-k$ shows that for any positive integer p we have

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^{2p} \frac{k\pi}{m} = \sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^{2p-1} \frac{k\pi}{m} = 0\,,$$

so that it suffices to evaluate

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^{2p-1} \frac{k\pi}{m} \qquad \text{and} \qquad \sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^{2p} \frac{k\pi}{m}\,.$$

From the Proposition, we obtain

$$\begin{aligned} &\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^{2p-1} \frac{k\pi}{m} \\ &= \sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{2p-2} \frac{A(2p-1,v)}{\left(\frac{k\pi}{m} + j\pi\right)^{2p-v-1}} \\ &= \sum_{v=0}^{2p-2} A(2p-1,v) \Big(\frac{m}{\pi}\Big)^{2p-v-1} \sum_{j=-\infty}^{\infty}{}' \sum_{k=1}^{m-1} \frac{\sin \frac{2k\alpha\pi}{m}}{(k+mj)^{2p-v-1}} \\ &= \sum_{v=0}^{p-1} A(2p-1,2v) \Big(\frac{m}{\pi}\Big)^{2p-2v-1} \sum_{j=-\infty}^{\infty}{}' \sum_{k=1}^{m-1} \frac{\sin \frac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} \end{aligned}$$

by (2.5). Next

$$\begin{aligned}
&\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty}{}' \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} \\
&= \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} + \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k-mj)^{2p-2v-1}} \right\} \\
&= \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} + \sum_{k=1}^{m-1} \frac{\sin \dfrac{2(m-k)\alpha\pi}{m}}{(m-k-mj)^{2p-2v-1}} \right\} \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} + \sum_{j=1}^{\infty} \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{\big(k+m(j-1)\big)^{2p-2v-1}} \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} + \sum_{j=0}^{\infty} \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} ,
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{j=-\infty}^{\infty}{}' \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} &= 2\sum_{j=0}^{\infty} \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} \\
&= 2\sum_{j=0}^{\infty} \sum_{k=1}^{m-1} \frac{\sin \dfrac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v-1}} \\
&= 2\sum_{\ell=1}^{\infty} \frac{\sin \dfrac{2\ell\alpha\pi}{m}}{\ell^{2p-2v-1}} \\
&= \frac{(-1)^{p-v}(2\pi)^{2p-2v-1}}{(2p-2v-1)!} B_{2p-2v-1}\left(\frac{\alpha}{m}\right)
\end{aligned}$$

by [1; p. 805, formula 23.1.17]. Hence

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^{2p-1} \frac{k\pi}{m}$$
$$= \sum_{v=0}^{p-1} \frac{(-1)^{p-v}(2m)^{2p-2v-1}}{(2p-2v-1)!} A(2p-1, 2v) B_{2p-2v-1}\left(\frac{\alpha}{m}\right).$$

Again from the Proposition, we have

$$\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^{2p} \frac{k\pi}{m}$$
$$= \sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \left\{ (-1)^p + \sum_{j=-\infty}^{\infty}{}' \sum_{v=0}^{2p-1} \frac{A(2p,v)}{\left(\frac{k\pi}{m} + j\pi\right)^{2p-v}} \right\}$$
$$= (-1)^p \sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} + \sum_{v=0}^{2p-1} A(2p,v) \left(\frac{m}{\pi}\right)^{2p-v} \sum_{j=-\infty}^{\infty}{}' \sum_{k=1}^{m-1} \frac{\cos \frac{2k\alpha\pi}{m}}{(k+mj)^{2p-v}}$$
$$= (-1)^{p+1} + \sum_{v=0}^{p-1} A(2p, 2v) \left(\frac{m}{\pi}\right)^{2p-2v} \sum_{j=-\infty}^{\infty} \sum_{k=1}^{m-1} \frac{\cos \frac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v}}$$
$$= (-1)^{p+1} + 2\sum_{v=0}^{p-1} A(2p, 2v) \left(\frac{m}{\pi}\right)^{2p-2v} \sum_{j=0}^{\infty} \sum_{k=1}^{m-1} \frac{\cos \frac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v}}$$
$$= (-1)^{p+1} + 2\sum_{v=0}^{p-1} A(2p, 2v) \left(\frac{m}{\pi}\right)^{2p-2v} \left\{ \sum_{j=0}^{\infty} \sum_{k=1}^{m} \frac{\cos \frac{2k\alpha\pi}{m}}{(k+mj)^{2p-2v}} - \sum_{j=0}^{\infty} \frac{1}{(m+mj)^{2p-2v}} \right\}$$
$$= (-1)^{p+1} + 2\sum_{v=0}^{p-1} A(2p, 2v) \left(\frac{m}{\pi}\right)^{2p-2v} \left\{ \sum_{\ell=1}^{\infty} \frac{\cos \frac{2\ell\alpha\pi}{m}}{\ell^{2p-2v}} - \frac{1}{m^{2p-2v}} \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2p-2v}} \right\}$$
$$= (-1)^{p+1} + 2\sum_{v=0}^{p-1} A(2p, 2v) \left(\frac{m}{\pi}\right)^{2p-2v} \left\{ \frac{(-1)^{p-v+1}(2\pi)^{2p-2v}}{2(2p-2v)!} B_{2p-2v}\left(\frac{\alpha}{m}\right) - \frac{(-1)^{p-v+1}(2\pi)^{2p-2v}}{2(2p-2v)!} \frac{B_{2p-2v}}{m^{2p-2v}} \right\}$$

by [1; p. 805, formulae 23.1.18 and 23.1.20]. Hence

$$\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^{2p} \frac{k\pi}{m}$$

$$= (-1)^{p+1} + \sum_{v=0}^{p-1} \frac{(-1)^{p-v+1}(2m)^{2p-2v} A(2p, 2v)}{(2p-2v)!} \left[B_{2p-2v}\left(\frac{\alpha}{m}\right) - \frac{B_{2p-2v}}{m^{2p-2v}} \right].$$

□

From the Theorem and the values of $A(p,v)$ given in the table in §2, we obtain

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot \frac{k\pi}{m} = -2mB_1\left(\frac{\alpha}{m}\right),$$

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^3 \frac{k\pi}{m} = \frac{4m^3}{3} B_3\left(\frac{\alpha}{m}\right) + 2mB_1\left(\frac{\alpha}{m}\right),$$

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^5 \frac{k\pi}{m} = -\frac{4m^5}{15} B_5\left(\frac{\alpha}{m}\right) - \frac{20m^2}{9} B_3\left(\frac{\alpha}{m}\right) - 2mB_1\left(\frac{\alpha}{m}\right),$$

$$\sum_{k=1}^{m-1} \sin \frac{2k\alpha\pi}{m} \cot^7 \frac{k\pi}{m} = \frac{8m^7}{315} B_7\left(\frac{\alpha}{m}\right) + \frac{28m^5}{45} B_5\left(\frac{\alpha}{m}\right) + \frac{392m^3}{135} B_3\left(\frac{\alpha}{m}\right) + 2mB_1\left(\frac{\alpha}{m}\right),$$

and

$$\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^2 \frac{k\pi}{m} = 2m^2 B_2\left(\frac{\alpha}{m}\right) + \frac{2}{3},$$

$$\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^4 \frac{k\pi}{m} = -\frac{2m^4}{3} B_4\left(\frac{\alpha}{m}\right) - \frac{8m^2}{3} B_2\left(\frac{\alpha}{m}\right) - \frac{26}{45},$$

$$\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^6 \frac{k\pi}{m} = \frac{4m^6}{45} B_6\left(\frac{\alpha}{m}\right) + \frac{4m^4}{3} B_4\left(\frac{\alpha}{m}\right) + \frac{46m^2}{15} B_2\left(\frac{\alpha}{m}\right) + \frac{502}{945},$$

$$\sum_{k=1}^{m-1} \cos \frac{2k\alpha\pi}{m} \cot^8 \frac{k\pi}{m} = -\frac{2m^8}{315} B_8\left(\frac{\alpha}{m}\right) - \frac{32m^6}{135} B_6\left(\frac{\alpha}{m}\right) - \frac{88m^4}{45} B_4\left(\frac{\alpha}{m}\right) - \frac{352m^2}{105} B_2\left(\frac{\alpha}{m}\right) - \frac{7102}{14175}.$$

The first of these trigonometric identities was discovered by E i s e n s t e i n [2] in 1844 and used by him to prove the law of quadratic reciprocity.

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