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THE COMPLEXITY OF UNIFORM DISTRIBUTION

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ABSTRACT. We investigate the notion of testable sequence which was proposed in [3], and we show that the set of uniformly distributed sequences is $\mathbf{\Pi}^0_3$-complete, hence not refutable.

0. Introduction

In [3], Winkler discussed various properties of pseudorandom sequences and how to "test" them. The purpose of this paper is to characterize this testability in the more familiar terms of the Borel hierarchy, and to answer two questions left open by [3] for the case of finite measure spaces.

Let X be a finite set, and μ be a probability measure on X. To simplify the notation and to exclude trivialities, we assume that all singletons of X are measurable and have positive measure. Let ${}^nX$ and ${}^\omega X$ be furnished with the product measure.

A *test* is a measurable function $t\colon \bigcup\limits_{n\in\omega} {}^nX \to [0,1]$, where $\omega = \{0,1,2,\dots\}$. For $\mathbf{x} = (x_0, x_1, \dots) \in {}^\omega X$, we write $t_n(\mathbf{x})$ for $t(x_0,\dots,x_{n-1})$. (See 0.1 for notation.)

t is called a *discrete test* if all values of t are in $\{0,1\}$.

The *acceptance set of a test* t is defined by

$$X_{t,1} := \bigl\{\mathbf{x} \in {}^\omega X : \lim_{n\to\infty} t_n(\mathbf{x}) = 1\bigr\}.$$

A set is *testable* if it is the acceptance set of some test, and a set is *refutable* if its complement is testable.

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A test t, or equivalently, its acceptance set $X_{t,1}$, is called *invariant*, if $X_{t,1}$ has the property $\mathbf{x} \in X_{t,1}$ *if and only if* $x^- \in X_{t,1}$, where

$\mathbf{x}^-$ (the *shift* of $\mathbf{x}$) is defined by $\mathbf{x}^-(n) = \mathbf{x}(n+1)$ for all n.

t is *generous* if $X_{t,1}$ has full measure.

The above definitions are from [3], where also an extensive list of pointers to the relevant literature can be found.

One of the questions left open in [3] is whether there are any nontrivial generous invariant tests on a finite set X. We show that there are many such tests; in fact, any null set can be refuted by some invariant generous test. This observation is due to Saharon Shelah.

The motivation of [3] comes from the set

$$\mathbf{U} := \left\{ \mathbf{x} \in {}^{\omega}2 : \lim_{n\to\infty} \frac{|\{i < n : x_i = 1\}|}{n} = \frac{1}{2} \right\}$$

of uniformly distributed sequences. It is clear that $\mathbf{U}$ is testable, but it was left open if $\mathbf{U}$ could be discretely testable, or refutable.

The main result proved here is that $\mathbf{U}$ is $\mathbf{\Pi}^0_3$-complete, that is, $\mathbf{U}$ is as complicated as any testable set can be. In particular, there is a (recursively defined) continuous function $\mathbf{F}\colon {}^{\omega}2 \times {}^{\omega}2 \to {}^{\omega}2$ such that for every $F_{\sigma\delta}$ set $\mathbf{X}$ there is $\mathbf{a} \in {}^{\omega}2$ such that

$$\forall \mathbf{x} \in {}^{\omega}2 : \ \mathbf{x} \in \mathbf{X} \iff \mathbf{F}(\mathbf{x}, \mathbf{a}) \in \mathbf{U},$$

i.e., every $F_{\sigma\delta}$ set is a continuous preimage of $\mathbf{U}$. Consequently, $\mathbf{U}$ is not refutable ($= \mathbf{\Sigma}^0_3$). (This was to be expected; see, e.g., the relevant remark in [2; p. 330].)

0.1. NOTATION. We use standard (set-theoretical) notation.

$\omega = \{0, 1, 2, \dots\}$ is the set of natural numbers, and we identify each natural number with the set of its predecessors, $n = \{0, \dots, n-1\}$.

${}^{B}A$ is the set of functions from B to A. (By "function" we mean the graph of a function.) Occasionally we write functions $f \in {}^{n}X$ or ${}^{\omega}X$ as sequences $(f_0, \dots, f_{n-1})$ or $(f_0, f_1, \dots)$. We let ${}^{<\omega}X = \bigcup_{n\in\omega} {}^{n}X$.

If $f \in {}^{A}B$, $A' \subseteq A$, then we write $f \restriction A'$ for the restriction of f to A' (so $f \restriction A' \subseteq f$).

The cardinality of a set x is denoted by $|x|$. In particular, for $f \in {}^{n}X$ we have $|f| = n$.

If $n \in \omega$, $t \in {}^nX$, $x \in X$, then $t^\frown x$ denotes the extension of t by x (i.e., $t^\frown x$ is a function with domain $n+1 = n \cup \{n\}$, $t^\frown x$ extends t and $(t^\frown x)(n) = x$).

On any topological space, we let $\mathbf{\Sigma}^0_1$ be the family of open sets and $\mathbf{\Pi}^0_1$ the family of closed sets. $\mathbf{\Sigma}^0_2$ are the F_σ sets, $\mathbf{\Pi}^0_2$ are the G_δ sets, $\mathbf{\Sigma}^0_3$ the $G_{\delta\sigma}$ sets, etc. $\mathbf{\Delta}^0_1 := \mathbf{\Sigma}^0_1 \cap \mathbf{\Pi}^0_1$ are the clopen sets, etc.

When a set $R \subseteq {}^{<\omega}2$ defines in a natural way a subset of ${}^\omega 2$, then we denote this subset of ${}^\omega 2$ by $\mathbf{R}$. Similarly, a function from ${}^\omega 2$ to ${}^\omega 2$ which is induced by a function $F\colon {}^{<\omega}2 \to {}^{<\omega}2$ will be called $\mathbf{F}$. See 2.1 and 2.2.

1. The complexity of testable sets

It is clear that any discretely testable set must be a $\mathbf{\Sigma}^0_2$ set, i.e., a countable union of closed sets. Here we show that also the converse is true.

The following two lemmas were pointed out by Shelah.

1.1. Fact. *If $A \subseteq {}^\omega X$ is a $\mathbf{\Sigma}^0_2$ set, then there is a discrete test t such that $A = X_{t,1}$.*

P r o o f. Let $A = \bigcup_n A_n$, where each A_n is closed, and $A_0 \subseteq A_1 \subseteq \ldots$. For $s \in {}^{<\omega}2$ let

$$n(s) = \min\{k : \exists \mathbf{x} \in A_k \ s \subseteq \mathbf{x}\}$$

(and $n(s) = \infty$ if no such $\mathbf{x} \in A$ exists). Then we have

(1) For each $\mathbf{x} \in {}^\omega 2$ the sequence $\big(n(\mathbf{x} \restriction k) : k \in \omega\big)$ is nondecreasing.
(2) If $\mathbf{x} \in A_i$, then this sequence is eventually constant with value $\leq i$.
(3) If $n(\mathbf{x} \restriction k) = i < \infty$ for almost all k, then $\mathbf{x} \in A_i$.

(1) and (2) are clear. To show (3), let $\mathbf{x} \restriction k \subseteq \mathbf{x}_k$ for almost all k, where $\mathbf{x}_k \in A_i$. As A_i is closed, also $\mathbf{x} \in A_i$.

Now define $t(s) = 1$ if and only if $|s| > 0$ and $n(s) = n\big(s \restriction (|s|-1)\big) < \infty$. By (1)–(3), $\mathbf{x} \in A$ if and only if $\big(t(\mathbf{x} \restriction n) : n \in \omega\big)$ is eventually equal to 1, i.e., if $\mathbf{x} \in X_{t,1}$.

As a consequence of 1.1 we get the following solution to Winkler's problem:

1.2. Corollary. *For any measure zero set A there is a generous discrete invariant test t such that t refutes a superset of A, i.e., $A \cap X_{t,1} = 0$.*

P r o o f. Let A be a measure zero set. Let A' be the smallest invariant set containing A. Then A', as a countable union of measure zero sets, is still a measure zero set. Note that the measure on ${}^\omega X$ is regular, i.e., we can approximate any measurable set with closed sets from below. So there is a closed set B of positive measure disjoint from A', and let B' be the smallest invariant set containing B. So B' still disjoint from A', B' is a $\mathbf{\Sigma}^0_2$ set, and by the well-known 0-1-law, B' must have measure 1. So by 1.1, there is a discrete test t with $X_{t,1} = B'$, and in particular t refutes a superset of A.

We can similarly characterize testable sets:

1.3. Fact. *A is testable if and only if A is a $\mathbf{\Pi}^0_3$ set.*

Proof. If $A = X_{t,1}$ for some test t, then

$$A = \left\{ \mathbf{x} : \ \forall k \, \exists n \, \forall m \geq n : \ t(\mathbf{x} \restriction m) \geq 1 - \frac{1}{k} \right\},$$

so A is clearly $\mathbf{\Pi}^0_3$.

To show the converse, it is enough to show that the family

$$\{X_{t,0} : \ t \ \text{a test}\}$$

is closed under countable intersections (where $X_{t,0} := \left\{ \mathbf{x} \in {}^\omega X : \ \lim_{n\to\infty} t_n(\mathbf{x}) = 0 \right\}$ $= X_{1-t,1}$). So let $t_0, \ldots, t_n, \ldots$ be tests. Then it is easy to see that

$$\bigcap_n X_{t_n,0} = X_{t,0} \, ,$$

where t is defined by $t(s) = \sum_{n\in\omega} \dfrac{t_n(s)}{2^{n+1}}$.

We can summarize the relations described above in the following diagram, where arrows denote (proper) inclusions:

$$\begin{tikzcd}
\text{testable} = \mathbf{\Pi}^0_3 & & \mathbf{\Sigma}^0_3 = \text{refutable} \\
& \mathbf{\Delta}^0_3 = \text{testable and refutable} \arrow[ul] \arrow[ur] & \\
\text{discretely refutable} = \mathbf{\Pi}^0_2 \arrow[ur] & & \mathbf{\Sigma}^0_2 = \text{discretely testable} \arrow[ul] \\
& \mathbf{\Delta}^0_2 = \begin{array}{l}\text{discretely testable}\\ \text{and discretely refutable}\end{array} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

2. U is a complete $\mathbf{\Sigma}^0_3$ set

To simplify the notation (but really without loss of generality) we will in this section consider only the case where our finite discrete measure space X has only two elements of the same probability, so $X = 2 = \{0,1\}$.

Recall that $\mathbf{U}$ is the set of uniformly distributed sequences in ${}^\omega 2$. Clearly $\mathbf{U}$ is $\mathbf{\Pi}^0_3 = F_{\sigma\delta}$, hence testable. Is $\mathbf{U}$ refutable ($\mathbf{\Sigma}^0_3$, $G_{\delta\sigma}$) or maybe even discretely testable ($\mathbf{\Sigma}^0_2$, F_σ)?

We answer this question negatively by showing that $\mathbf{U}$ is in fact a complete $\mathbf{\Pi}^0_3$ set. In particular, every $\mathbf{\Pi}^0_3$ set is a continuous preimage of $\mathbf{U}$, whereas it is well known (and easy to prove, see 2.3.(2)) that there are $\mathbf{\Pi}^0_3$ sets which are not $\mathbf{\Sigma}^0_3$, hence not a continuous preimage of any refutable set.

2.1. Notation. The variables $\mathbf{x}$ and $\mathbf{t}$ will range over ${}^\omega 2$, x and t will range over ${}^{<\omega}2$. If $F\colon {}^{<\omega}2 \to {}^{<\omega}2$ is weakly increasing, i.e., satisfies

$$x_1 \subseteq x_2 \implies F(x_1) \subseteq F(x_2),$$

and for all $\mathbf{x} \in {}^\omega 2$ the domain of $\bigcup_{n\in\omega} F(\mathbf{x} \restriction n)$ is all of ω, then F naturally induces a continuous function from ${}^\omega 2$ to ${}^\omega 2$, which we denote by $\mathbf{F}$:

$$\mathbf{F}(\mathbf{x}) = \bigcup_{n\in\omega} F(\mathbf{x} \restriction n).$$

Conversely, every continuous function $\mathbf{F}\colon {}^\omega 2 \to {}^\omega 2$ is induced by such a function $F\colon {}^{<\omega}2 \to {}^{<\omega}2$.

2.2. Main Theorem.

(1) *For every recursive relation $R \subseteq \omega \times \omega \times \omega \times {}^{<\omega}2$ there is a recursive function F such that for all $\mathbf{x} \in {}^\omega 2$:*

$$\forall k\, \exists n\, \forall m\, (k,n,m,\mathbf{x} \restriction m) \in R \iff \mathbf{F}(\mathbf{x}) \in \mathbf{U}.$$

(*We will write* $\mathbf{R}$ *for the set* $\{\mathbf{x} : \forall k\, \exists n\, \forall m\, (k,n,m,\mathbf{x} \restriction m) \in R\}$.)

(2) *Moreover, the construction of F is uniform in R, that is: If R is a (recursive) relation, $R \subseteq \omega \times \omega \times \omega \times {}^{<\omega}2 \times {}^{<\omega}2$, then there is a (recursive) function $F\colon {}^{<\omega}2 \times {}^{<\omega}2 \to {}^{<\omega}2$ such that for all $\mathbf{a} \in {}^\omega 2$, for all $\mathbf{x} \in {}^\omega 2$:*

$$\forall k\, \exists n\, \forall m\, (k,n,m,\mathbf{x} \restriction m, \mathbf{a} \restriction m) \in R \iff \mathbf{F}(\mathbf{x},\mathbf{a}) \in \mathbf{U}.$$

2.3. CONCLUSION.

(1) *Every $F_{\sigma\delta}$ set is a continuous preimage of* $\mathbf{U}$.

(2) $\mathbf{U}$ *is not a* $\mathbf{\Sigma}^0_3$ *set.*

Proof.

(1) There is a (recursive) relation R such that the set

$$\mathbf{R} := \{(\mathbf{x}, \mathbf{a}) : \ \forall k\ \exists n\ \forall m\ (k, n, m, \mathbf{x} \restriction m, \mathbf{a} \restriction m) \in R\}$$

is a universal $\mathbf{\Pi}^0_3$-set, i.e., every $\mathbf{\Pi}^0_3$ set is of the form

$$\mathbf{R_a} := \{\mathbf{x} \in {}^{\omega}2 : \ \forall k\ \exists n\ \forall m\ (k, n, m, \mathbf{x} \restriction m, \mathbf{a} \restriction m) \in R\}$$

for some $\mathbf{a} \in {}^{\omega}2$. (Why? Any $\mathbf{\Pi}^0_3$-set is of the form $\bigcap_k \bigcup_m \bigcap_n \{\mathbf{x} : \ f(k,n,m) \subseteq \mathbf{x}\}$ for some function $f\colon \omega\times\omega\times\omega \to {}^{<\omega}2$. Let $H\colon \omega\times\omega\times\omega\times{}^{<\omega}2 \to \omega$ be a recursive bijection, and let $\mathbf{a}(\ell) = 1$ if and only if ℓ is of the form $H(k,n,m,f(k,n,m))$ for some k, n, m.)

Now find F as in the theorem, let $\mathbf{F_a}(\mathbf{x}) := \mathbf{F}(\mathbf{x},\mathbf{a})$, then $\mathbf{R_a} = \mathbf{F}_{\mathbf{a}}^{-1}(\mathbf{U})$.

(2) If $\mathbf{U}$ were $\mathbf{\Sigma}^0_3$, then also $\{\mathbf{x} : \ \mathbf{F}(\mathbf{x},\mathbf{x}) \in \mathbf{U}\}$ would be $\mathbf{\Sigma}^0_3$, so its complement would be $\mathbf{\Pi}^0_3$, so for some $\mathbf{a}$ would have

$$\{\mathbf{x} : \ \mathbf{F}(\mathbf{x},\mathbf{x}) \notin \mathbf{U}\} = \{\mathbf{x} : \ \mathbf{F}(\mathbf{x},\mathbf{a}) \in \mathbf{U}\},$$

which is a contradiction.

2.4. Remark. The word "recursive" is not really necessary in the above theorem. It just emphasizes the fact that all constructions in the proof below (e.g., the computation of t' from t in fact 2.8) are effective.

Proof of the main theorem. (We will prove only (1). (2) is similar – we just have to add a parameter a or $\mathbf{a}$ everywhere.)

So fix a recursive relation $R \subseteq \omega \times \omega \times \omega \times {}^{<\omega}2$.

2.5. First Step. From $\mathbf{x} \in {}^{\omega}2$ we can uniformly and recursively find a family $(A_k(\mathbf{x}) : \ k \in \omega)$ of r.e. sets such that $\mathbf{x} \in \mathbf{R}$ if and only if $\forall k\ A_k(\mathbf{x})$ is finite.

We accomplish this as follows:
We define a recursive function $f\colon \omega \times \omega \times {}^{<\omega}2 \to \omega$ by

$$f(k,M,x) = \begin{cases} \text{the least } n < M \text{ such that } \forall m \le M\ (k,n,m,x \restriction m) \in R; \\ M, \text{ if either } |x| < M \text{ or such an } n \text{ as above does not exist.} \end{cases}$$

Note that for given k and M, $f(k,M,x)$ depends only on $x \restriction M$, so we will abuse notation and write $f(k,M,\mathbf{x})$ for $f(k,M,\mathbf{x}\restriction M)$ whenever $\mathbf{x} \in {}^{\omega}2$.

To explain the motivation for this definition, let us fix $\mathbf{x}$ and k. If there is an n such that

$$\forall m\ (k,n,m,\mathbf{x}\restriction m) \in R,$$

then for all large enough values of M, $f(k,M,\mathbf{x})$ will be the minimal such n. Otherwise we will have $f(k,M,\mathbf{x}) = M$ for infinitely many M.

For $x \in {}^{<\omega}2$ let

$$A_k(x) = \{f(k,M,x) :\ M < |x|\},$$

and for $\mathbf{x} \in {}^{\omega}2$ let

$$A_k(\mathbf{x}) = \{f(k,M,\mathbf{x}) :\ M < \omega\} = \bigcup_{n\in\omega} A_k(\mathbf{x}\restriction n).$$

By the above argument, we have for all $\mathbf{x} \in {}^{\omega}2$:

$$\mathbf{x} \in \mathbf{R} \iff \forall k\ A_k(\mathbf{x}) \text{ is finite.}$$

2.6. Second Step. From $\mathbf{t} \in {}^{\omega}2$ we can uniformly and recursively find a family $(B_k(\mathbf{t}) :\ k \in \omega)$ of r.e. sets such that $\mathbf{t} \in \mathbf{U}$ if and only if $\forall k\ B_k(\mathbf{t})$ is finite.

For $t \in {}^{n}2$ let

$$D(t) = \left| \frac{|\{i < n :\ \mathbf{t}(i) = 1\}|}{n} - \frac{1}{2} \right| \cdot 2.$$

For $\mathbf{t} \in {}^{\omega}2$ we let

$$\mathbf{D}(\mathbf{t}) = \limsup_{k\to\infty} D(\mathbf{t}\restriction k).$$

For $\mathbf{t} \in {}^{\omega}2$ let

$$B_k(\mathbf{t}) = \left\{ i < |\mathbf{t}| :\ D(\mathbf{t}\restriction i) > \frac{1}{k+1} \right\}$$

(this definition makes sense also for $\mathbf{t} \in {}^{<\omega}2$).

Clearly, if $B_k(\mathbf{t})$ is infinite, then $\mathbf{D}(\mathbf{t}) \geq \frac{1}{k+1}$, and if $B_k(\mathbf{t})$ is finite, then $\mathbf{D}(\mathbf{t}) \leq \frac{1}{k+1}$. So $\mathbf{t} \in \mathbf{U}$ if and only if for all k the set $B_k(\mathbf{t})$ is finite.

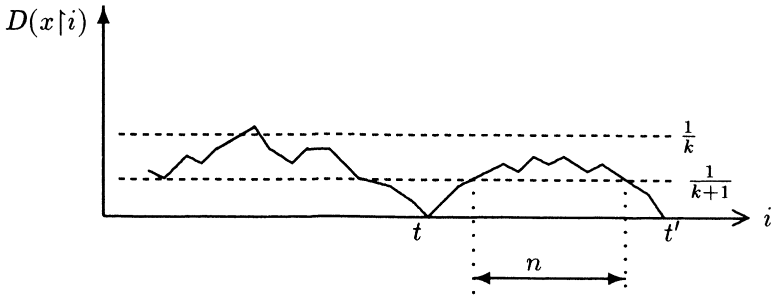
Before we continue, we mention the following two facts:

2.7. Fact. *If $D(t) = 0$, and $|t| \geq k$, $|t| > 0$, then, letting $t' := t^\frown 0^\frown 1$ we get: $D(t') = 0$ and $B_k(t') = B_k(t)$.*

2.8. Fact. *If $D(t)=0$, $|t|\geq k$, $|t|>0$, $n\in\omega$, then there is t' such that*

- $t\subset t'$,
- $D(t')=0$,
- $|B_k(t')|=|B_k(t)|+n$,
- *For all* $k'<k$, $B_{k'}(t')=B_{k'}(t)$.

We will only hint at the (easy) proof of 2.8 with a picture. For some sequence $\mathbf{x}$ extending t and t' as in 2.8 we plot the function $i\mapsto D(\mathbf{x}\restriction i)$:



2.9. Third Step. Now we have prepared all the necessary notation for translating our general set $\mathbf{R}$ into the set $\mathbf{U}$. We will find a strictly increasing recursive function $F\colon {}^{<\omega}2\times{}^{<\omega}2\to{}^{<\omega}2$ such that for the induced function $\mathbf{F}\colon {}^{\omega}2\times{}^{\omega}2\to{}^{\omega}2$ the following holds:

- If for all k the set $A_k(\mathbf{x})$ is finite, then also all the sets $B_k\big(\mathbf{F}(\mathbf{x})\big)$ will be finite.
- If k^* is the minimal k such that $A_k(\mathbf{x})$ is infinite, then for all $k\geq k^*$ the set $B_k\big(\mathbf{F}(\mathbf{x})\big)$ will be infinite.

Let $(k_i : i\in\omega)$ be a recursive enumeration of ω in which each number occurs infinitely often. Also assume that for all i we have $k_i\leq i$.

We will define F by induction. We let $F(\emptyset)$ be the 2-element sequence $(0,1)$. F will satisfy

$$\forall x\in{}^{<\omega}2:\ D\big(F(x)\big)=0\qquad\text{and}\qquad |F(x)|>|x|\,.$$

Let $i:=|y|-1$, $x:=y\restriction i$. We assume that $F(x)$ is already defined, and we have to define $F(y)$. We distinguish two cases:

Case 1: $|A_{k_i}(y)|\leq\big|B_{k_i}\big(F(x)\big)\big|$. In this case we let $F(y)=F(x)^\frown 0^\frown 1$.

So in this case we have:

$$|A_{k_i}(y)|\leq\big|B_{k_i}\big(F(y)\big)\big|\,,\tag{1}$$

$$B_\ell\big(F(x)\big)=B_\ell\big(F(y)\big)\qquad\text{for}\quad \ell\leq|F(x)|\,.\tag{2}$$

(Note that $k_i \leq i = |x| < |F(x)|$, so equation (2) holds in particular for all $\ell \leq i$, and even more so for all $\ell \leq k_i$.)

Case 2: Otherwise. By 2.8, we can find $F(y) \supset F(x)$ such that

$$D(F(y)) = 0,$$

$$|A_{k_i}(y)| \leq |B_{k_i}(F(y))|, \tag{3}$$

$$B_\ell(F(y)) = B_\ell(F(x)) \qquad \text{for} \quad \ell < k_i. \tag{4}$$

2.10. Last Step. We have to check that F satisfies all the requirements.

Clearly F is recursive, and strictly increasing, and it defines a function $\mathbf{F}\colon {}^\omega 2 \to {}^\omega 2$. We have to show $\mathbf{x} \in \mathbf{R} \iff \mathbf{F}(\mathbf{x}) \in \mathbf{U}$.

First consider $\mathbf{x} \notin \mathbf{R}$. So for some k the set $A_k(\mathbf{x})$ is infinite. Since $A_k(\mathbf{x})$ is the increasing union of the sets $A(\mathbf{x} \restriction i)$, this means that $\lim_{i\to\infty} |A_k(\mathbf{x} \restriction i)| = \infty$.

For infinitely many i (namely, for all numbers i for which $k = k_i$) we have $|A_k(\mathbf{x} \restriction i{+}1)| \leq |B_k(F(\mathbf{x} \restriction i{+}1))|$. Hence the set $B_k(\mathbf{F}(\mathbf{x}))$ will be infinite, so $\mathbf{F}(\mathbf{x}) \notin \mathbf{U}$.

Now let $\mathbf{x} \in \mathbf{R}$. So for each $\ell \in \omega$ the set $A_\ell(\mathbf{x})$ is finite. This means that for each natural number ℓ we can find a number j_ℓ such that

$$A_\ell(\mathbf{x} \restriction j_\ell) = A_\ell(\mathbf{x} \restriction j_\ell{+}1) = \cdots = A_\ell(\mathbf{x}).$$

Moreover, (by increasing j_ℓ, if necessary) we may assume that $j_\ell > \max(\ell, j_{\ell-1})$ and that

$$k_{j_\ell - 1} = \ell.$$

Applying (1) and (3) to $x := \mathbf{x} \restriction j_\ell - 1$, $y := \mathbf{x} \restriction j_\ell$ we get

$$|A_\ell(\mathbf{x} \restriction j_\ell)| \leq |B_\ell(F(\mathbf{x} \restriction j_\ell))|$$

and also (since the right side increases with increasing i while the left side stays constant)

$$\forall i \geq j_\ell: \ |A_\ell(\mathbf{x} \restriction i{+}1)| \leq |B_\ell(F(\mathbf{x} \restriction i))|. \tag{5}$$

Now fix k. To show that $B_k(\mathbf{x})$ is finite, it will be enough to show that the sequence $\big(B_k(\mathbf{x} \restriction i) : \ j_k \leq i < \infty\big)$ is constant. So let us fix $i \geq j_k$.

If $k < k_i$, then it is clear from (2) and (4) that $B_k(\mathbf{x} \restriction i) = B_k(\mathbf{x} \restriction i{+}1)$, so we may assume that $k \geq k_i$. Let $\ell := k_i$. So $k \geq \ell$, hence $j_k \geq j_\ell$ and therefore $i \geq j_\ell$. Hence, by (5), we have $|A_\ell(\mathbf{x} \restriction i{+}1)| \leq |B_\ell(F(\mathbf{x} \restriction i))|$.

This means that $F(\mathbf{x} \restriction i{+}1)$ was defined in Case 1: $F(\mathbf{x} \restriction i{+}1) = F(\mathbf{x} \restriction i)^\frown 0^\frown 1$. Since $k \leq j_k \leq i$, the remark after equation (2) tells us that $B_k(F(\mathbf{x} \restriction i)) = B_k(F(\mathbf{x} \restriction i{+}1))$, and we are done.

This finishes the proof of 2.2.

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