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ON MINIMAL IDEALS IN SEMIGROUPS WITH RESPECT TO THEIR SUBSETS, II

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ABSTRACT. In this paper, the concepts of a partial left group and of a completely simple semigroup with respect to the set B of a semigroup S are defined. The aim of this paper is to study the following structures:

i) the partial left group with respect to the set of the semigroup S,
ii) the minimal left ideals with respect to the set B of the semigroups S in the completely semigroup with respect to its subset,
iii) the completely simple semigroups with respect to its subset.

In [13], the concept of a minimal left (right, two-sided) ideal with respect to a subset B ($\emptyset \neq B \subseteq S$) of a semigroup S was introduced.

In this paper, the following concepts are defined:

i) a partial left group (see Definition 1),
ii) a completely simple semigroup with respect to a subset B of the semigroup S (see Definitions 2, 3).

Under certain conditions on the subset B of a semigroup S, the following structures are investigated:

a) the structure of minimal left ideals with respect to B (see Definition 01, Theorems 1, 2),
b) the structure of partial left groups (see Theorem 4),
c) the structure of completely simple semigroups with respect to their subsets (see Theorem 7).

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The results of this paper are generalizations of results of [3], [9].

We note that some of our results can also be obtained using some well-known theorems on ideals of semigroups (however not the main results, see for example Theorems 1, 2 and 7). We shall proceed in such a way that these known results will be obtained as corollaries of our assertions.

In forthcoming papers, we shall study properties of minimal left (right, two-sided) ideals and of quasiideals with respect to a subset B of a semigroup S, under weaker conditions (in some sense) for B as compared to the conditions considered for B in this paper.

First, we introduce notations and definitions of concepts (assertions on these concepts), which will be used through the paper. Notations and definitions of concepts (resp. assertions on these concepts), which will be used but not introduced in this paper, will be employed in the current sense (see e.g. [1], [6]).

Let S be a semigroup, and let $\emptyset \neq B \subseteq S$.

$\mathscr{L}$ ($\mathscr{J}$, $\mathscr{R}$) be the *Green $\mathscr{L}$-equivalence* (*$\mathscr{J}$-equivalence*, *$\mathscr{R}$-equivalence*) on S (see [1]);

$L(B)$ (L_B) will denote the set $\bigcup\{L(b) \mid b \in B\}$ ($\bigcup\{L_b \mid b \in B\}$).

The sets $R(B)$, $J(B)$ (R_B, J_B) are defined similarly.

$NL(B)$ ($N(B)$, $NR(B)$) be the set of all $x \in S$ such that, for each $b \in B$, $L_b \nleq L_x$ ($J_b \nleq L_x$, $R_b \nleq R_x$).

$D_\ell(B)$ ($D_r(B)$) be the set of all elements $b \in B$ such that $bB = B$ ($Bb = B$), and let

$E(B)$ denote the set of all idenpotents e of S such that $e \in B$.

Note that:

a) $X \subset Y$ will mean that X is a proper subset of the set Y (to distinguish it from $X \subseteq Y$ which means either $X \subset Y$ or $X = Y$).
b) If $A \subseteq S$, then $\bar{A}$ will denote the set $S \setminus A$.

DEFINITION 01. (see [13]) Let S be a semigroup, and let $\emptyset \neq B \subseteq S$. A left ideal L of the semigroup S is called a *minimal left ideal with respect to B* if $L \cap B \neq \emptyset$, and there is no left ideal N' of the semigroup S such that $N' \cap B \neq \emptyset$, and $N' \subset N$.

A *minimal right* (*two-sided*) *ideal with respect to* B will be defined similarly.

In the following, the definitions of new concepts (will be mostly omitted) and the theorems about them will be given only for left ideals of S. Theorems on left ideals of S will also be used (without mentioning) in case of analogous theorems (concepts) concerning right ideals of S.

R e m a r k . If we put $B = S$ ($B = S \setminus \{0\}$) in Definition 01, then for each nonempty subset L of the semigroup S (the semigroup S with 0) the following holds:

L is a minimal left (0-minimal left) ideal in S if and only if L is a minimal left ideal with respect to the subset B of the semigroup S (of the semigroup S with 0).

Let S be a semigroup with kernel K (i.e. K is the intersection of all two-sided ideals in S, and $K \neq \emptyset$). Put $B = S \setminus K$. In [13], it is shown how to get theorems on simple left ideals of S with kernel K (a left ideal L of S is called a simple ideal of S with the kernel K if $K \subset L$, and there is no left ideal L' in S, such that $K \subset L' \subset L$ (see [10])) using theorems on minimal left ideals with respect to the subset B of the semigroup S.

Examples can show that:

a) There exists s semigroup S (see e.g. Example 5) not containing any minimal left ideal and containing infinitely many pairwise different subsets such that with respect to each of them the set of minimal left ideals in S is nonempty.

b) There exists a semigroup S with kernel K not containing any simple left ideal and containing infinitely many pairwise different subsets such that with respect to each of them the set of minimal left ideals of S is nonempty (see [13]).

We shall say that a semigroup S satisfies *condition* m_{LB} (m_{RB}), if the set of all minimal left (right) ideals with respect to the subset B of S is nonempty.

THEOREM 01. (see [13]) *Let a semigroup S satisfy condition m_{LB}. Then we have:*

(a) *For each subset L of S, L is a minimal left ideal with respect to the subset B of the semigroup S if and only if there exists an element $b \in B$ such that $L = L(b)$ and L_b is a minimal element in $\overline{NL(B)}/\mathscr{L}$.*

(b) *For each $b \in B$, $L(b)$ is a minimal left ideal with respect to B if and only if $L(b) \cap \overline{NL(B)} = L_b$.*

A semigroup S is called a *partial group* if $D_r(S) \neq \emptyset$ and $D_r(S) = D_\ell(S)$ (see [6: p. 339]).

R e m a r k . We shall use the following assertions (without mentioning them). Let S be a semigroup, and let $\emptyset \neq B \subseteq S$. Then we have:

(a) If $NL(B) \neq \emptyset$, then $NL(B)$ is a left ideal in S (see [12]).

(b) For each left ideal of a semigroup L there holds:

$$L \cap \overline{NL(B)} \neq \emptyset \iff L \cap B \neq \emptyset .$$

(c) If $c \in \overline{NL(B)}$, then $L_c \subseteq \overline{NL(B)}$.

1.

LEMMA 1. *Let H be a filter of a semigroup S, and $Sx = S$ for all $x \in H$. Let $E(H) \neq 0$, and let $e \in E(H)$. Then there hold:*

(a) *e is a right unit element of S;*
(b) *the equation $xa = b$ has exactly one solution in S for each $a \in H$ and $b \in S$.*

Proof.

(a) is trivial.

(b) By the assumption for each $a \in H$ and $b \in S$ the equation $xa = b$ has a solution in S. Then for each $a \in H$ there exists an element $\bar{a} \in S$ such that $\bar{a}a = e$. Suppose that there exist elements $a \in H$, $b \in S$ and elements $x_1, x_2 \in S$ such that $x_1 a = b$, and $x_2 a = b$. Then $(a\bar{a})(a\bar{a}) = a(\bar{a}a)\bar{a} = (ae)\bar{a} = a\bar{a}$. Hence $e^* = a\bar{a} \in E(H)$. Then $(x_1 a)\bar{a} = (x_2 a)\bar{a}$, i.e. $x_1 e^* = x_2 e^*$. As a consequence we obtain $x_1 = x_2$.

LEMMA 2. *Let a semigroup S satisfy the assumption of Lemma 1. Then we have:*

(a) *eS is a subsemigroup of S and e is a unit element of eS;*
(b) *$D_r(eS) = eS \cap H$ is a group;*
(c) *$D_r(eS) = D_\ell(eS)$.*

Proof.

(a) is trivial.

(b) I. Let $a \in D_r(eS)$. If $a \in S \setminus H$, then $e \in eS = eSa \subseteq S \setminus H$, which is a contradiction. Hence $D_r(eS) \subseteq eS \cap H$.

II. Let $a \in eS \cap H$. Then $eS = eSa$. It follows that $eS \cap H \subseteq D_r(eS)$, and there exists an element $a' \in eS$ such that $a'a = e$. Suppose that $a' \in S \setminus H$. Then $e = a'a \in S \setminus H$, which is a contradiction. It follows, we obtain (b).

(c) Let $a \in D_r(eS)$. By (b), there exists an element $a' \in D_r(eS)$ such that $aa' = e$. By (a), for all $b \in eS$ it holds $b = eb = (aa')b = a(a'b) \in aeS$. Hence $eS \subseteq aeS$. By our assumption, $aeS \subseteq eS$. It follows that $a \in D_\ell(eS)$.

Let $a \in D_\ell(eS)$. If $a \in S \setminus H$, then $e \in eS = aeS \subseteq S \setminus H$, which is a contradiction. Hence $eS = eSa$, i.e. $a \in D_r(eS)$.

COROLLARY 1. *Let a semigroup S satisfy the following assumptions:*

(a) *$D_r(S) \neq \emptyset$,*
(b) *$D_r(S)$ contains the left unit element e of the semigroup S,*
(c) *For each $a \in D_r(S)$ there exists an element $a^{-1} \in D_r(S)$ such that $a^{-1}a = e$.*

Then S is a partial group.

Proof. Clearly, $D_r(S)$ is a group.

We will show that $D_r(S)$ is a filter of S. Let $a, b \in S$, and suppose that $ab \in D_r(S)$. If $b \notin D_r(S)$, then $Sab \subseteq Sb \subseteq S \setminus D_r(S)$, which is a contradiction. Hence $b \in D_r(S)$. Then there exists an element $b^{-1} \in D_r(S)$ such that $bb^{-1} = e$. Then $(ab)b^{-1} \in D_r(S)$. Hence $S = S(abb^{-1}) = Sae = Sa$. It follows that $D_r(S)$ is a filter of S. By Lemma 2, we have that S is a partial group.

THEOREM 1. *Let the semigroup S satisfy condition m_{LB}. Let $L(c)$, $c \in B$, be a minimal left ideal with respect to B. Let L_c be a filter of the semigroup $L(c)$, and let $E(L_c) \neq \emptyset$. Put $G_e = eL(c)$ for each $e \in E(L_c)$. Then we have:*

- (a) *G_e is a subsemigroup of the semigroup $L(c)$, and e is a unit element of G_e;*
- (b) *$D_r(G_e) = G_e \cap L_c$ is a group;*
- (c) *G_e is a partial group.*

The proof follows from Lemma 2.

COROLLARY 2. *Let a semigroup S satisfy the assumptions of Theorem 1. Then $L_c \subseteq \bigcup\{G_e \mid e \in E(L_c)\}$.*

Proof. Let a be an element of L_c. By the assumption and Lemma 2, there holds $L(c)a = L(c)$. Hence there exists an element $e \in L(c)$ such that $ea = a$. Suppose $e \notin L_c$. Then $e \in L(c) \setminus L_c$. By the assumption, we obtain that $ea \in L(c) \setminus L_c$. This is in contradiction with $a \in L_c$. From the above, we have that $e^2a = ea$. By Lemma 1, it follows that $e^2 = e$ and $e \in E(L_c)$. It means that $a \in eL(c)$, and $e \in E(L_c)$.

Remark. An example can show that there exists a semigroup S and its nonempty subset B such that:

- a) There exists an element $c \in B$ such that $R(c)$ is a minimal right ideal with respect to B, and R_c is a filter of semigroup $R(c)$;
- b) $E(R_c) \neq \emptyset$ and there does not hold $R(c) = \bigcup\{R(c)e \mid e \in E(R_c)\}$, i.e. $R(c)$ is a set-union of partial groups.

Example 1. Let $S_1 = \{a, b, c, d, e, f, g, h\}$, and let a binary operation on S_1 be given by the following table:

	a	b	c	d	e	f	g	h
a	a	b	c	d	c	d	c	d
b	a	b	c	d	c	d	c	d
c	a	b	c	d	c	d	c	d
d	a	b	c	d	c	d	c	d
e	a	b	c	d	e	f	g	h
f	a	b	c	d	e	f	g	h
g	a	b	c	d	g	h	e	f
h	a	b	c	d	g	h	e	f

Then S is a semigroup. Put $B = \{e, f, g, h, \}$. Then the following hold:

a) $R(h)$ is a minimal right ideal with respect to B, and R_h is a filter in $R(h)$;

b) $E(R_h) \neq \emptyset$;

c) $\bigcup\{R(h)f \mid f \in E(R_h)\} \neq R(h)$, i.e. $R(h)$ is not a set-union of partial groups;

d) $L(h)$ is a minimal left ideal with respect to B, and L_h is a filter in $L(h)$;

e) $E(L_h) \neq \emptyset$;

f) $\bigcup\{fL(h) \mid f \in E(L_h)\} = L(h)$, i.e. $L(h)$ is a set-union of pairwise disjoint partial groups, none of which is a group.

Example 2. Let $S = \{a, b, c, d\}$, and let a binary operation on S be given by the following table:

	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	c	d
d	a	b	c	d

Then S is a semigroup. Put $B = \{c, d\}$. Then one has:

a) $R(d)$ is a minimal right ideal with respect to B, and R_d is a filter of the semigroup $R(d)$;

b) $E(R_d) = \{c, d\} \neq \emptyset$;

c) $R(d) = \bigcup\{R(d)f \mid f \in E(R_d)\}$, i.e. $R(d)$ is a set-union of partial groups, and there holds: $c, d \in E(R_d)$, $R(d)c \neq R(d)d$, and $R(d)c \cap R(d)d \neq \emptyset$, i.e. the groups are not pairwise disjoint.

LEMMA 3. *Let S be a semigroup, and let $E(S) \neq \emptyset$. Let $e, f \in E(S)$. If $Se = S = Sf$, then the semigroups eS and fS are isomorphic.*

P r o o f. Put $\varphi\colon eS \to fS$, where $\varphi(x) = fx$. For $x, y \in eS$ we have $\varphi(xy) = f(xy) = (fx)(fy) = \varphi(x)\varphi(y)$. If $y \in fS$, then $\varphi(ey) = f(ey) = fy = y$. If $\varphi(x) = \varphi(y)$ for $x, y \in eS$, then $fx = fy$, and so $x = ex = efx = efy = ey = y$.

THEOREM 2. *Let a semigroup S satisfy the assumptions of Theorem 1. Then for each $e, f \in E(L_c)$ the partial groups $eL(c)$ and $fL(c)$ are isomorphic.*

The proof follows from Lemma 3.

COROLLARY 3. (see [8], [9], [11]). *Let L be a minimal left ideal of the semigroup S, and let $E(L) \neq \emptyset$. Then we have:*

(a) *eL is a subgroup of the semigroup L for each $e \in E(L)$;*

(b) *$L = \bigcup\{eL \mid e \in E(L)\}$;*

(c) *the groups eL and fL are isomorphic for each $e, f \in E(L)$.*

P r o o f. Put $B = S$. Then $\overline{NL(B)} = S$. Let $c \in L$; by the assumption and by Theorem 01, we have $L = L(c) = L_c$. Clearly, L_c is a filter of the semigroup $L(c)$. By the assumption, we obtain that the semigroup S satisfies condition m_{LB}, and $E(L_c) \neq \emptyset$. By Theorem 1 and Theorem 2, we have then Corollary 3.

LEMMA 4. *Let a semigroup S satisfy condition m_{LB}. Let $\overline{NL(B)}$ be a filter in S. Let $L(c)$, $c \in B$, be a minimal left ideal with respect to B. Then L_c is a filter of the semigroup $L(c)$.*

P r o o f. Let a and b be elements of $L(c)$. Then by the assumption and by Theorem 01, we have $ab \in L_c$ if and only if $a, b \in L_c$. Lemma 4 is proved.

THEOREM 3. *Let a semigroup S satisfy condition m_{LB}. Let $\overline{NL(B)}$ be a filter of S. Let $L(c)$, $c \in B$, be a minimal left ideal with respect to B such that $E(L_c) \neq \emptyset$. Then $R = eS$ is a minimal right ideal with respect to the subset $\overline{NL(B)}$ of the semigroup S for each $e \in E(L_c)$.*

P r o o f. By the assumption and by Lemma 4, we have $R \cap L_c \neq \emptyset$, and $R \cap \overline{NL(B)} \neq \emptyset$. Suppose that there exists a right ideal R' of the semigroup S such that $R' \subset R$ and $R' \cap \overline{NL(B)} \neq \emptyset$. Let $b \in R' \cap \overline{NL(B)}$. According to the assumption and Theorem 01, we get $[R'L(c)] \cap L_c \neq \emptyset$. From $R'L(c) \subseteq R'$, we have $R' \cap L_c \neq \emptyset$. Let a be an element of $R' \cap L_c$. Hence $a \in R' \subset eS$. Therefore there exists an element $u \in S$ such that $a = eu$. Hence $ea = e(eu) = e^2u = eu = a$. Since $a \in L_c$ and L_c is a filter of the semigroup $L(c)$, by the assumption and Corollary 1, there is an element $\bar{e} \in E(L_c)$ such that $a \in \bar{e}L(c)$. Therefore $\bar{e}a = a$. By Lemma 1, we have that $e = \bar{e}$. Therefore $a \in eL(c) \cap L_c$. By Theorem 1, we obtain that $a \in D_r(eL(c))$. Using Theorem 1 we have that to the element a there is an element $\bar{a} \in D_r(eL(c))$ such that $a\bar{a} = e$. Hence $R = eS = (a\bar{a})S \subseteq aS \subseteq R'$. This is in contradiction with $R' \subset R$. Theorem 3 is proved.

COROLLARY 4. (see [10]) *Let L be a minimal left ideal of a simple semigroup without zero. Let $E(L) \neq \emptyset$. Then $R = eS$ is a minimal right ideal of S for each $e \in E(L)$.*

Proof. Put $B = S$. Then $\overline{NL(B)} = S$. Hence by Theorem 3, we obtain Corollary 4.

An example will show the existence of a semigroup S_1 and its nonempty subset B such that there is $c \in B$ such that $L(c)$ is a minimal left ideal with respect to B, $E(L_c) \neq \emptyset$, and there exists $e \in E(L_c)$ such that $R = eS$ is not a minimal right ideal with respect to the subset $\overline{NL(B)}$ of the semigroup S_1.

Example 3. Let $S_1 = \{0, \alpha, \beta, u, v, e\}$, and let a binary operation on S_1 be given by the following multiplication table:

	α	β	u	v	e
α	α	0	0	v	e
β	0	β	u	0	0
u	u	0	0	β	u
v	0	v	e	0	0
e	e	0	0	v	e

Then S_1 is a semigroup. Put $B = \{\alpha, \beta, v\}$. Then we have:

a) the set of all minimal left ideals with respect to the set B is the set $\{L(\alpha), L(\beta)\}$;
b) $E(L_\alpha) \neq \emptyset$, $\alpha \in E(L)$, and αS_1 is not a minimal right ideal with respect to B.

DEFINITION 1. A semigroup S will be called a *partial left group* if:

(i) $D_r(S)$ is a filter in S;

(ii) for each $a, b \in D_r(S)$, $R(a) \neq R(b)$ it follows that $R(a) \cap R(b) = \emptyset$.

Example 4. Let S be a semigroup of the Example 2. Then $D_\ell(S) = \{c, d\}$, and $D_\ell(S)$ is a filter in S. The set of all minimal left ideals with respect to the subset $D_r(S)$ of S is $\{L(c), L(d)\}$, where $L(c) = \{a, b, c\}$, $L(d) = \{a, b, d\}$. Then $L(c) \neq L(d)$, and $L(c) \cap L(d) \neq \emptyset$. Note that $R(c) = \{a, b, c, d\}$ is a minimal right ideal with respect to the subset $D_\ell(S)$ of the semigroup S. Hence not each minimal right ideal with respect to $D_\ell(S)$ is a partial right group.

Example 5. Let $S = \{a, b, c, d, e, f, g, h\}$ be the semigroup from Example 1. Then it is easy to prove that S is a partial right group.

CONVENTION. In the next, if S is a partial left group, then instead of $D_\ell(S)$ ($D_r(S)$) write F (H), i.e. $F = D_\ell(S)$ ($H = D_r(S)$).

LEMMA 5. *Let S be a partial left group. Then:*

(a) *H is a left simple semigroup;*

(b) *$R(a)$ is a minimal right ideal with respect to the subset H of the semigroup S for each $a \in H$;*

(c) *for each $a \in H$ there hold:*

(c$_1$) $R_a = R(a) \cap H$,

(c$_2$) *R_a is a minimal ideal of the semigroup H,*

(c$_3$) $E(R_a) \neq \emptyset$.

Proof.

(a) According to the assumption, the equation $xa = b$ has for each pair $a, b \in H$ one solution in S. If $x \in \bar{H} = S \setminus H$, then $b = xa \in \bar{H}$. It is in contradiction with $b \in H$. It follows that (a) holds.

(b) Let $a \in H$. Suppose that there exists a right ideal R of the semigroup S that $R \subset R(a)$ and $R \cap H \neq \emptyset$. Let $b \in R \cap H$. Then $R(b) \subset R(a)$. Hence $R(a) \neq R(b)$ and $R(a) \cap R(b) \neq \emptyset$, which is a contradiction. It follows that (b) holds.

(c) (c$_1$) By the assumption, $\overline{NR(H)} = H$. Hence by (b) and by Theorem 01 (more precisely, by the theorem dual to Theorem 01), we get the assertion (c$_1$).

(c$_2$) Let $a \in H$. By (b) and by Theorem 01, R_a is a minimal element in $H/\mathscr{R}$. It follows by (a) and by Theorem 01 ($B = S$) that R_a is a minimal right ideal in H for each $a \in H$, i.e. assertion (c$_2$) holds.

(c$_3$) Let $a \in H$. Then there exists an element $e \in S$ such that $ea = a$. Suppose that $e \notin H$. Then by the assumption $ea = S \setminus H$, which is a contradiction. Then $e \in R_a$. It means, there exists an element $\bar{a} \in R_a$ such that $e = a\bar{a}$. Then $e = a\bar{a} = e(a\bar{a}) = e^2$.

Let T be a subsemigroup of a semigroup S. By $R^T(a)$ (R_a^T), we shall denote the right ideal (the $\mathscr{R}$-class) in T generated by the element a (containing the element a) in T ($T/\mathscr{R}$).

Lemma 6. *Let S be a partial left group. Let $T = \bigcup\{R(a) \mid a \in H\}$. Then we have:*

(a) $R(a) = R^T(a)$, $R(a) = aS = aT$ *for each* $a \in H$;
(b) $T = \bigcup\{R^T(a) \mid a \in H\}$;
(c) *for each* $a \in H$ *there exists an element* $e \in E(H)$, *such that* $R(a) = eS = eT$;
(d) $Tb = T$ *for each* $b \in H$;
(e) $D_r(T) = H$;
(f) T *is a partial left group.*

P r o o f.

(a) Let $a \in H$. Then $aT \cap H \neq \emptyset$. By the assumption, aT is a right ideal of the semigroup S. By Lemma 5, we have that $R(a) = aT = R^T(a)$. Therefore $aS \subseteq aT$, and $Ta \subseteq Sa$. The assertion (a) follows.

(b) Assertion (b) can be proved using assertion (a).

(c) The proof of (c) follows from Lemma 5.

(d) Let b be an element of H. Then by (a) and (b),

$$Tb = \Big(\bigcup\{aS \mid a \in H\}\Big)b = \bigcup\{(aS)b \mid a \in H\} = \bigcup\{aS \mid a \in H\} = T.$$

(e) By (d), we obtain that $H \subseteq D_r(T)$. Suppose that there exists an element $c \in D_r(T)$ such that $c \notin H$ and $Tc = T$. Then $T = Tc \subseteq S \setminus H$, i.e. $T \cap H = \emptyset$. It is in contradiction with $H \subset T$.

(f) Assertion (f) can be proved using assertions (a) and (e).

Lemma 7. *Let S be a partial left group. Put $T = \bigcup\{R(a) \mid a \in H\}$. Then there hold:*

(a) $T = \bigcup\{eT \mid e \in E(H)\}$, *and for each* $e, f \in E(H)$, $e \neq f$, *one has* $eT \cap fT = \emptyset$;
(b) eT *is a partial group for each* $e \in E(H)$;
(c) *partial groups* eT, fT *are isomorphic for each* $e, f \in E(H)$.

P r o o f.

(a) By (c) of Lemma 6, we have that $T = \bigcup\{eT \mid e \in E(H)\}$. Let $e, f \in E(H)$, and let $e \neq f$. Suppose $eS = fS$. Hence $e, f \in R_f$. For any

$e \in R_f$ we have $eR(f) = R(f)$. By Lemma 1, we get that e is a unit element of $R(f)$ for each $e \in E(R_f)$. Then $e = ef = f$. It is in contradiction with $e \neq f$. Hence $eS \neq fS$. Now by the assumption and Lemma 6, we have that $eT \cap fT = \emptyset$.

(b) Let $c \in H$. By Lemma 6, $L(c) = T$, $L_c = H$, and L_c is a filter in $L(c)$. By Lemma 6, we have $E(L_c) \neq \emptyset$. By Theorem 1 and Theorem 2, we get assertions (b) and (c).

Note that an example can show that the following assertion does not hold: If S is a partial right group, then $S = \bigcup\{R(a) \mid a \in H\}$ (see Example 1).

THEOREM 4. *For a semigroup S, the following assertions are equivalent:*

(i) *S is a partial left group, and $S = \bigcup\{R(a) \mid a \in H\}$.*

(ii) *$H = D_r(S)$ is a filter, $E(H) \neq \emptyset$, $S = \bigcup\{eS \mid e \in E(H)\}$, and for each $e, f \in E(H)$, $e \neq f$, we have $eS \cap fS = \emptyset$.*

(iii) *S is isomorphic with the direct product $G \times E$ of the partial group G and the semigroup of left zeros E* (see [5]).

Proof.

(i) $\Longrightarrow$ (ii): From (i) follows (ii) by Lemma 6 and Lemma 7.

(ii) $\Longrightarrow$ (iii): Let e, f be elements of $E(H)$. Then by Lemma 1, we have $ef = e$. It follows that $E(H)$ is a subsemigroup of left zeros of the semigroup H. Let g be an element of $E(H)$. Put $G = R(g)$ and $E = E(H)$. Put $\varphi\colon G \times E \to S$, where $\varphi(a, f) = fa$ and $G \times E$ is a direct product of semigroups G, E. Then by Lemma 1, for each two elements $(a, e), (b, f) \in G \times E$ the following holds:

$$\varphi(a, e) \cdot \varphi(b, e) = (ea)(fb) = e(ab) = \varphi(ab, e) = \varphi\big[(a, e), (b, f)\big] .$$

Suppose that there exist two elements $(a, e), (b, f) \in G \times E$ such that $\varphi(a, e) = \varphi(b, f)$, i.e. $ea = fb$. Further suppose that $e \neq f$. Then by Lemma 7, we have $R(e) \cap R(f) = \emptyset$, which is in contradiction with $ea = fb$. Hence $e = f$. Then $ea = eb$. By the assumption, we have $a, b \in G$ and g is a unit element in G, hence $a = ga = (ge)a = g(ea) = g(eb) = (ge)b = gb = g$.

Let a be an element of S. Since by the assumption, $S = \bigcup\{R(a) \mid a \in H\}$, by Lemma 7, there exists $f \in E$ such that $a \in R(f)$. Therefore $ga \in G$ and $(ga, f) \in G \times E$. Since f is a unit element of the semigroup $R(f)$, we have $\varphi(ag, f) = f(ga) = fa = a$.

(iii) $\Longrightarrow$ (i): Clearly, this implication holds.

LEMMA 8. *Let the semigroup S be a left group. Then S is a partial left group and $S = \bigcup\{R(a) \mid a \in H\}$.*

P r o o f. By the assumption, $S = H = L(c) = L_c$ for each $c \in S$. Hence $E(L_c) \neq \emptyset$. By Corollary 4, $R = eS$ is a minimal right ideal of the semigroup S. Then the set-union M of all minimal right ideals in S is a two-sided ideal in S. By the assumption, $M = S$. From the above, we obtain that

1) H is a filter in S,
2) for each $a, b \in H$, $R(a) \neq R(b)$ implies $R(a) \cap R(b) = \emptyset$,
3) $S = \bigcup\{R(a) \mid a \in H\}$.

Hence S is a partial left group, and $S = \bigcup\{R(a) \mid a \in H\}$.

COROLLARY 5. *For a semigroup S, the following conditions are equivalent:*

(i) *S is a left group.*
(ii) *$D_r(S) = S$ and $E(S) \neq \emptyset$.*
(iii) *S is isomorphic with the direct product $G \times E$ of the group G and the semigroup of left zeros E* (see [5]).

P r o o f. By Lemma 8, it is enough to show that the set G defined in the proof of Theorem is a group.

DEFINITION 2. Let S be s semigroup, and let $\emptyset \neq B \subseteq S$. The semigroup S will be called *simple with respect to B* if:

(i) for each two-sided ideal N of S, $N \cap B \neq \emptyset$ implies that $N = S$;
(ii) $\overline{N(B)}$ is a filter in S;
(iii) $S = \bigcup\{L(b) \mid b \in B\} = \bigcup\{R(b) \mid b \in B\}$.

DEFINITION 3. A semigroup S will be called *completely simple with respect to its subset B* if:

(i) S is a simple with respect to B;
(ii) S contains at least one minimal left ideal with respect to B and at least one minimal right ideal with respect to B.

R e m a r k. It is clear that the following assertion holds:

Each simple (completely simple) semigroup is also simple (completely simple) with respect to the subset $B = S$.

LEMMA 9. *Let S be a simple semigroup with respect to its subset B. Let S contain at least one minimal left ideal with respect to B. Then the following hold:*

(a) *L_c is a minimal left ideal of the semigroup $\overline{N(B)}$ for each $c \in \overline{N(B)}$;*

(b) $\overline{N(B)} = \bigcup\{L_c \mid c \in \overline{N(B)}\}$;

(c) $\overline{N(B)}$ *is a simple semigroup.*

P r o o f.

(a) By the assumption and Theorem 01, there exists an element $b \in B$ such that L_b is a minimal left ideal of the semigroup $\overline{N(B)}$. Let M be the set-union of all minimal left ideals in $\overline{N(B)}$. Put $P = \big(S \setminus \overline{N(B)}\big) \cup M$. Then P is a two-sided ideal in S, and $P \cap B \neq \emptyset$. According to the assumption, we have $P = S$. It follows that $\overline{N(B)}$ is equal to the set-union of all minimal left ideals of the semigroup $\overline{N(B)}$.

(b) Using Theorem 01 ($B = S$) we obtain assertion (b).

(c) Assertion (c) follows from assertions (a) and (b).

THEOREM 5. *Let a semigroup S satisfy conditions of Lemma 9. Then we have:*

(a) *$L(c)$ is a minimal left ideal with respect to B for each $c \in B$.*

(b) *S is a set-union of the set of all left minimal ideals with respect to B.*

P r o o f.

(a) By the assumption, Lemma 10, and Theorem 01, we see that $L(c)$ is a minimal ideal with respect to the subset $\overline{N(B)}$ of the semigroup S. Using $B \subseteq \overline{N(B)}$, we obtain the assertion (a).

(b) By the assumption and by (a), we obtain (b).

LEMMA 10. *Let S be a completely simple semigroup with respect to a subset of it. Then we have:*

(a) *for each $a \in H$ there exists an element $e \in E(H)$ such that $e \in R_a \cap L_a$;*

(b) *$eL(e) = R(e)L(e) = R(e) \cap L(e) = eSe = R(e)e$ for each $e \in E(H)$.*

P r o o f.

(a) Let $a \in H$. By Theorem 01, Lemma 1, and Lemma 9, there exists an element $a \in L(a)$ such that $ea = a$ and $e \in R_a \cap L_a$. Hence there exists an element $\bar{a} \in R(a)$ such that $a\bar{a} = e$. Then $e = a\bar{a} = (ea)\bar{a} = e(a\bar{a}) = e^2$, i.e. $e \in E(H)$.

(b) Let $e \in E(H)$. Clearly, $eL(e) \subseteq R(e)L(e) \subseteq R(e) \cap L(e)$.

Let $x \in R(e) \cap L(e)$. Then there exist elements $u, v \in S$ such that $x = ue = ev$. Then $x = ue = (ue)e = (ev)e = eve \in eSe = eL(e) = R(e)e$, which proves (b).

COROLLARY 6. (see [13]) *Let S be a completely simple semigroup with respect to a subset B. Let $u, v \in B$, and put $G_{u,v} = R(u)L(v)$. Then:*

(a) *$G_{u,v}$ is a partial group.*
(b) *$L(v) = Se$, $R(u) = eS$ and $R(u) \cap L(v) = eSe$, where $e \in E(R_v \cap L_u)$.*
(c) *$G_{u,v} = R(u) \cap L(v)$.*

Proof. By Theorem 5, Theorem 01, and Lemma 9, we have $[R(u) \cap L(v)] \cap \overline{N(B)} = [R(u) \cap \overline{NL(B)}] \cap [L(u) \cap \overline{NL(B)}] = R_u \cap L_v \neq \emptyset$. Let $a \in R_u \cap L_v$. Then by Lemma 10, there exists an element $e \in E(H)$ such that $e \in R_a \cap L_b$. Assertions (a), (b), (c) follow.

THEOREM 6. *Let a semigroup S satisfy the assumptions of Lemma 9. Then S is completely simple with respect to the subset B if and only if $E(\overline{N(B)}) \neq \emptyset$.*

Proof.

I. Let S be completely simple with respect to the subset B. Then by Theorem 01 (more precisely, by the theorem dual to Theorem 01), there exist elements $b, c \in B$ such that $L(b)$ is a minimal left ideal with respect to B, and $R(c)$ is a minimal right ideal with respect to B. Then by Lemma 9, $E(\overline{N(B)}) \neq \emptyset$.

II. Let the assumptions of Lemma 9 be satisfied, and let $e \in E(\overline{N(B)})$. Then by Lemma 9, there exists an element $d \in B$ such that $e \in L_d$. Now, by Theorem 3, we see that $R = eS$ is a minimal right ideal with respect to B. It follows that S is a completely simple semigroup with respect to B.

COROLLARY 7. (see [9]) *Let S be a simple semigroup and contain at least one minimal ideal. Then S is completely simple if and only if $E(S) \neq \emptyset$.*

Proof. Put $B = S$. Then, according to the assumption, the semigroup S is simple with respect to B and contains at least one minimal ideal with respect to B. Clearly, $\overline{N(B)} = S$. Then using Theorem 3 and Corollary 4 we obtain Corollary 7.

THEOREM 7. *Let S be a completely simple semigroup with respect to its subset B. Then S is a set-union of mutually isomorphic partial groups.*

Proof. Put $G_{u,v} = R(u) \cap L(v)$ for each two elements $u, v \in B$. Let x be an element of S. Then by the assumption, there exist elements $a, b \in B$ such that $u \in R(a)$ and $u \in L(b)$. Hence $u \in R(a) \cap L(b)$. It follows that $S \subseteq \bigcup\{G_{u,v} \mid u, v \in B\} \subseteq S$. Let u, v be elements of B. By Corollary 6, $G_{u,v}$ is a partial group. By $e_{u,v}$, we shall denote a unit element of $G_{u,v}$. Let a, b, c, d be elements of B. Then by Corollary 6, $G_{a,b} = e_{a,b}L(b)$, $G_{c,d} = R(c)e_{c,d}$, and $G_{c,b} = e_{c,b}L(b) = R(c)e_{c,b}$. Using Theorem 2 we get the assertion of Theorem 7.

COROLLARY 8. (see [9]) *Let S be a completely simple semigroup. Then S is a union of mutually disjoint isomorphic groups.*

Proof. Put $B = S$. By the assumption, the semigroup S is completely simple with respect to B. Theorem 7 and Theorem 01 imply the assertion.

Remark. An example will show that

a) There exists a semigroup containing infinitely many subsemigroups, each of them being a partial left group with respect to its subset, and none of them being a left group.

b) There exists a semigroup containing infinitely many subsemigroups, each of them being completely simple with respect to its subset, and none of them being a completely simple semigroup.

Example 6. Let S_1 be the set of all real numbers such that $0 < x < 1$. Let the binary operation on S_1 be defined as follows: $x \cdot y = \min\{x, y\}$ for each $x, y \in S_1$. Then S_1 is a semigroup. Let $S_2 = \{c, d, e, f, g, h\}$ be a subsemigroup of the semigroup S_1 of Example 1. Let $S_3 = S_1 \times S_2$ be the direct product of semigroups S_1, S_2. Then it is easy to show that $H_x = L(x) \times L(e)$ is a partial left group with respect to the subset $M_x = \{(x, e)\}$ for each $x \in S_1$. Further, it is easy to prove that none of its subsemigroups is a left group with respect to its subset.

Example 7. Let S_1 be the semigroup of Example 4. Let S_2 be the semigroup of Example 2. Let $S_3 = \{a_1, a_2, a_3, a_4\}$, and let a binary operation on S_3 be defined by the following table:

	a_1	a_2	a_3	a_4
a_1	a_1	a_2	a_3	a_3
a_2	a_2	a_2	a_4	a_4
a_3	a_1	a_2	a_3	a_3
a_4	a_2	a_2	a_4	a_4

Then S_3 is a semigroup. Let $S_4 = S_1 \times S_2 \times S_3$ be the direct product of semigroups S_1, S_2, S_3. Then each subsemigroup $H_x = L(x) \times S_2 \times S_3$ of the semigroup S_3 is a completely simple semigroup with respect to its subset $B_x = \{x\} \times \{c, d\} \times \{a_1, a_2\}$ for each $x \in S_1$. Further, it is easy to prove that none of its subsemigroups is a completely simple semigroup.

Finally, let us remark that if we weaken condition (ii) in Definition 2, then we speak about a so called *o-simple* (*completely o-simple*) *semigroup with respect to* *a subset B of a semigroup S.* The structure of completely o-simple semigroups with respect to their subsets will be a subject of our forthcoming paper.

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