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COMPATIBILITY PROBLEM IN QUASI-ORTHOCOMPLEMENTED POSETS

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ABSTRACT. The conditions when Boolean subalgebras in a quasi-orthocomplemented poset may be embedded into a Boolean σ-algebra are studied.

1. Introduction

One of the actual problems of the mathematical description of quantum mechanics is the problem of simultaneous measurement of several observables. In the classical Kolmogorov model [5], the measurement of non-quantum observables is performed within the framework of Boolean σ-algebra models [9]. For quantum mechanical observables there exists a model of quantum logics [10]. On the other hand, in the quantum logics there are also observables which have the classical character, i.e. their ranges are embedable into a joint Boolean σ-algebra.

The main goal of the present paper is to present conditions showing when the ranges of observables in a quasi-orthocomplemented poset are embeddable into some Boolean σ-algebra. This question is known as the compatibility problem and it has been solved for various classes of quantum logics using various notions of compatibility [1, 4, 6].

We recall that there is a different axiomatic model for measurements of quantum mechanical observables based on fuzzy sets ideas, called an F-quantum space [8], where this problem has been solved, see [2].

We note that our methods are similar to classical ones for quantum logics, however, for the existence of a Boolean sub-σ-algebra we have to use very fine steps.

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2. Quasi-orthocomplemented poset

By a *quasi-orthocomplemented poset* (q.o.p.) we understand a partially ordered set P with a quasi-orthocomplement $\perp\colon P \to P$ such that the following conditions hold:

(i) $(a^\perp)^\perp = a$ for any $a \in P$;
(ii) if $a \leqq b$ then $b^\perp \leqq a^\perp$;
(iii) $a^\perp \neq a$ for any $a \in P$;
(iv) if $\{a_n\}_{n\in\mathrm{N}} \subset P$, $a_i \leqq a_j^\perp$ for $i \neq j$, then

$$\bigvee_{n\in\mathrm{N}} a_n := \sup_{n\in\mathrm{N}} a_n \in P .$$

Example 2.1. Every Boolean σ-algebra is a q.o.p.

Example 2.2. Every quantum logic, i.e. a σ-orthomodular poset (see [7]) is a q.o.p.

Example 2.3. Let (Ω, M) be an F-quantum poset (see [2]), i.e. Ω is a nonvoid set and $M \subset [0,1]^\Omega$ is a system of fuzzy sets such that

(i) if $1(\omega) = 1$ for any $\omega \in \Omega$, then $1 \in M$;
(ii) if $f \in M$, then $f^\perp := (1 - f) \in M$;
(iii) if $1/2(\omega) = 1/2$ for any $\omega \in \Omega$, then $1/2 \notin M$;
(iv) $\bigcup\limits_{n\in\mathrm{N}} f_n \in M$ whenever $f_i \leqq f_j^\perp$ for $i \neq j$ and $\{f_n\}_{n\in\mathrm{N}} \subset M$.

Then M is a q.o.p.

Example 2.4. Let V be an inner product space. Let $L = L(V) = \{A \subset V\colon (A^\perp)^\perp = A\}$, where $A^\perp = \{x \in V\colon (x,y) = 0 \text{ for all } y \in A\}$. Then L is a q.o.p., where the meet denotes the intersection of subspaces and the join is the minimal subspace of L containing given subspaces. We note that if V is a Hilbert space and $L(V) = \{A \subset V\colon (A^\perp)^\perp = A,\ A \text{ is a closed subspace}\}$, then $L(V)$ is a quantum logics.

Example 2.5. Let $X = (0,\infty)$ and the mapping $\perp$, $\perp\colon X \to X$, be a unary operation on X defined via $x \mapsto 1/x$ for any $x \in X$. Let P be a nonempty subset of X such that:

(i) $1 \notin P$;
(ii) if $x \in P$, then $x^\perp := 1/x \in P$;
(iii) if $\{x_n\}_{n\in\mathrm{N}} \subset P$, $x_i \leqq x_j^\perp$ (i.e. $x_i \cdot x_j \leqq 1$), then $\sup\limits_{n\in\mathrm{N}} x_n \in P$.

The operation $\perp$ is a quasi-orthocomplement and P is a q.o.p.

LEMMA 2.6. *Let P be a q.o.p. If $a \vee b \in P$ $(a \wedge b \in P)$, then $a^{\perp} \wedge b^{\perp} \in P$ $(a^{\perp} \vee b^{\perp} \in P)$ and $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$ $((a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp})$.*

Proof. It is simple to verify it in a classical way.

A nonempty set $A \subset P$ is said to be a *Boolean sub-(σ-)algebra of a q.o.p. P* if:

1. There are minimal and maximal elements 0_A and 1_A from A such that $0_A \leqq a \leqq 1_A$ and $a \vee a^{\perp} = 1_A$ for any $a \in A$.
2. With respect to $\vee$, $\wedge$, $\perp$, 0_A, 1_A, A is a Boolean sub-(σ-) algebra (in the sense of Sikorski [9]).

Let $B(\mathbb{R}^1)$ be a Borel σ-algebra of the set of all reals. We say that a mapping $x\colon B(\mathbb{R}^1) \to P$ is an *observable of P* if:

(i) $x(E^c) = x(E)^{\perp}$ for any $E \in B(\mathbb{R}^1)$, where $E^c = \mathbb{R}^1 - E$;

(ii) $x\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigvee_{n\in\mathbb{N}} x(E_n)$ whenever $E_i \cap E_j = \emptyset$ for $i \neq j$ and $\{E_n\}_{n\in\mathbb{N}} \subset B(\mathbb{R}^1)$.

If x is an observable of P, then the *range of x*, that is, the set $R(x) = \{x(E)\colon\ E \in B(\mathbb{R}^1)\}$, is a Boolean subalgebra of P with the minimal and maximal elements $x(\emptyset)$ and $x(\mathbb{R}^1)$, respectively.

Let $a \in P$. We define an observable x_a as a mapping from $B(\mathbb{R}^1)$ into P such that

$$x_a(E) \doteq \begin{cases} a \wedge a^{\perp}, & \text{if } 0,1 \notin E; \\ a^{\perp}, & \text{if } 0 \in E,\ 1 \notin E; \\ a, & \text{if } 0 \notin E,\ 1 \in E; \\ a \vee a^{\perp}, & \text{if } 0,1 \in E; \end{cases}$$

for any $E \in B(\mathbb{R}^1)$. The observable x_a plays the role of the indicator of the event $a \in P$ and the range of x_a is the set $R(x_a) = \{a,\ a^{\perp},\ a \vee a^{\perp},\ a \wedge a^{\perp}\}$.

In accordance with the theory of quantum logics, we say that two elements $a, b \in P$ are

(i) *orthogonal* and write $a \perp b$ if $a \leqq b^{\perp}$;

(ii) *compatible* and write $a \leftrightarrow b$ if $a \wedge b$, $a^{\perp} \wedge b$, $a \wedge b^{\perp} \in P$ and $a = (a \wedge b) \vee (a \wedge b^{\perp})$, $b = (a \wedge b) \vee (a^{\perp} \wedge b)$;

(iii) *strongly compatible* and write $a \overset{s}{\leftrightarrow} b$ if $a \leftrightarrow b \leftrightarrow a^{\perp} \leftrightarrow b^{\perp} \leftrightarrow a$.

It is evident that if $a \leftrightarrow b$, then $a \vee b \in P$.

We note that if $a \leftrightarrow b$, then it is not true, in general, that then $a \overset{s}{\leftrightarrow} b$. Indeed, let (Ω, M) be an F-quantum poset, where M contains two different constant functions f and g with $0 < f < g < 1/2$. Then $f \leftrightarrow g$ and $f \leftrightarrow g^{\perp}$, but $f^{\perp} \not\leftrightarrow g^{\perp}$.

It is easy to verify that $a \overset{s}{\leftrightarrow} b$ if and only if $a \leftrightarrow b^{\perp}$ and $a^{\perp} \leftrightarrow b$. Further, $a \overset{s}{\leftrightarrow} a^{\perp}$, $a \overset{s}{\leftrightarrow} a \wedge a^{\perp} \overset{s}{\leftrightarrow} a^{\perp} \overset{s}{\leftrightarrow} a \vee a^{\perp} \overset{s}{\leftrightarrow} a$, $a \wedge a^{\perp} \overset{s}{\leftrightarrow} a \vee a^{\perp}$ for any $a \in P$.

LEMMA 2.7. *If $a \overset{s}{\leftrightarrow} b$, then $a \vee a^{\perp} = b \vee b^{\perp}$.*

Proof. Calculate

$$\begin{aligned} a \vee a^{\perp} &= \big((a \wedge b) \vee (a \wedge b^{\perp})\big) \vee \big((a^{\perp} \wedge b) \vee (a^{\perp} \wedge b^{\perp})\big) \\ &= \big((a \wedge b) \vee (a^{\perp} \wedge b)\big) \vee \big((a \wedge b^{\perp}) \vee (a^{\perp} \wedge b^{\perp})\big) = b \vee b^{\perp}. \end{aligned}$$

We say that a q.o.p. P has

(i) a *c-f-distributive property* if for any finite subset $\{a, a_1, \dots, a_n\} \subset P$ such that $\bigvee_{i=1}^{n} a_i \in P$ and $a \leftrightarrow a_i$, the equality

$$a \wedge \left(\bigvee_{i=1}^{n} a_i \right) = \bigvee_{i=1}^{n} (a \wedge a_i) \tag{2.1}$$

holds (provided that at least one side of (2.1) exists in P);

(ii) a *c-σ-distributive property* if for any $a \in P$ and any sequence $\{a_n\}_{n\in\mathrm{N}} \subset P$ such that $\bigvee_{n\in\mathrm{N}} a_n \in P$ and $a \leftrightarrow a_n$, the equality

$$a \wedge \left(\bigvee_{n\in\mathrm{N}} a_n \right) = \bigvee_{n\in\mathrm{N}} (a \wedge a_n) \tag{2.2}$$

holds (provided that at least one side of (2.2) exists in P).

Any Boolean σ-algebra, any quantum logic as well as any F-quantum space have the c-σ-distributive property.

PROPOSITION 2.8. *Let a q.o.p. P have the c-f-distributive property. The following statements are equivalent.*

(i) $a \overset{s}{\leftrightarrow} b$.
(ii) *There is an observable x of P such that $x(E) = a$ and $x(F) = b$ for some $E, F \in B(\mathbb{R}^1)$.*
(iii) *There is a Boolean subalgebra of P containing a and b.*

Proof. Lelt (i) hold. Put $x_1 = a \wedge b$, $x_2 = a \wedge b^\perp$, $x_3 = a^\perp \wedge b$, $x_4 = a^\perp \wedge b^\perp$ and define a mapping $x\colon B(\mathbb{R}^1) \to P$ via

$$x(G) = \begin{cases} a \wedge a^\perp, & \text{if } 1,2,3,4 \notin G; \\ \bigvee_i x_i, & \text{if } i \in G, \ i = 1,2,3,4; \end{cases}$$

for any $G \in B(\mathbb{R}^1)$. The straightforward calculation shows that x is an observable of P. If we put $E = \{1,2\}$ and $F = \{1,3\}$, then we get (ii). The statement (ii) evidently gives (iii) and (iii) implies (i).

3. Commensurability

We say that two nonempty subsets A and B of P are (*strongly*) *compatible* and write ($A \overset{s}{\leftrightarrow} B$) $A \leftrightarrow B$ if ($a \overset{s}{\leftrightarrow} b$) $a \leftrightarrow b$ for all $a \in A$ and $b \in B$.

It is clear that if A and B are Boolean subalgebras of P, then $A \overset{s}{\leftrightarrow} B$ if and only if $A \leftrightarrow B$ and moreover $A \cap B \neq \emptyset$ implies $1_A = 1_B$.

We say that a system of nonempty subsets of P, $\{A_t\colon \ t \in T\}$, is (σ-) *commensurable* if there is a Boolelan sub-(σ-)algebra of P containing all A_t.

The main problem of the present section is to give the necessary and sufficient conditions (= compatibility theorem) for a nonempty subset of P to be σ-commensurable.

A nonvoid subset A of P is said to be *f-compatible* ("f" as for finiteness) if for any finite subset $\{a_1, \dots, a_{n+1}\}$ of A we have:

(i) $u := a_1 \wedge \cdots \wedge a_n \wedge a_{n+1} \in P$, $v := a_1 \wedge \cdots \wedge a_n \wedge a_{n+1}^\perp \in P$;
(ii) $u \vee v = a_1 \wedge \cdots \wedge a_n$.

A subset A is *strongly f-compatible* if the set $A \cup A^\perp$ is f-compatible, where $A^\perp = \{a^\perp\colon \ a \in A\}$.

Proposition 3.1.

(i) $a \leftrightarrow b$ ($a \overset{s}{\leftrightarrow} b$) *if and only if* $\{a, b\}$ *is* (*strongly*) *f-compatible.*
(ii) *Every nonempty subset of an* (*strongly*) *f-compatible set is* (*strongly*) *f-compatible.*
(iii) *The* (*strong*) *f-compatibility of* $\{a_1, \dots, a_n\}$ *implies* $\bigwedge_{i=1}^n a_i \in P$ ($\bigvee_{i=1}^n a_i \in P$).

Proof. The first two statements are evident.

If $\{a_1,\dots,a_n\}$ is f-compatible, then from the definition we have easily $\bigwedge_{i=1}^{n} a_i \in P$. Suppose now that $\{a_1,\dots,a_n\}$ is strongly f-compatible. Then $\{a_1^\perp,\dots,a_n^\perp\}$ is f-compatible and therefore $P \ni \bigwedge_{i=1}^{n} a_i^\perp = \left(\bigvee_{i=1}^{n} a_i\right)^\perp$, which implies $\bigvee_{i=1}^{n} a_i \in P$.

PROPOSITION 3.2. *Let P be a q.o.p. with the c-f-distributive property. If $\{a, b_1,\dots,b_n\} \subset P$ is strongly f-compatible, then $a \overset{s}{\leftrightarrow} \bigvee_{i=1}^{n} b_i$ and $a \overset{s}{\leftrightarrow} \bigwedge_{i=1}^{n} b_i$.*

Proof. Denote $J_0 = \{(j_1,\dots,j_n) \in \{0,1\}^n\} - \{(0,0,\dots,0)\}$, $b_i^0 = b_i^\perp$, $b_i^1 = b_i$ for $i = 1,2,\dots,n$. From the strong f-compatibility of $\{a, b_1,\dots,b_n\}$ we have

$$P \ni \bigvee_{J_0} (a \wedge b_1^{j_1} \wedge \dots \wedge b_n^{j_n}) = \bigvee_{i=1}^{n} (a \wedge b_i),$$

$$P \ni \bigvee_{J_0} (a^\perp \wedge b_1^{j_1} \wedge \dots \wedge b_n^{j_n}) = \bigvee_{i=1}^{n} (a^\perp \wedge b_i),$$

therefore

$$\left(a \wedge \left(\bigvee_{i=1}^{n} b_i\right)\right) \vee \left(a \wedge \left(\bigvee_{i=1}^{n} b_i\right)^\perp\right) = \left(\bigvee_{J_0} (a \wedge b_1^{j_1} \wedge \dots \wedge b_n^{j_n})\right) \vee (a \wedge b_1^\perp \wedge \dots \wedge b_n^\perp)$$
$$= \bigvee_{J_n} (a \wedge b_1^{j_1} \wedge \dots \wedge b_n^{j_n}) = \bigvee_{J_{n-1}} (a \wedge b_1^{j_1} \wedge \dots \wedge b_{n-1}^{j_{n-1}}) = \dots = \bigvee_{J_1} (a \wedge b_1^{j_1})$$
$$= (a \wedge b_1) \vee (a \wedge b_1^\perp) = a,$$

where $J_k = \{(j_1,\dots,j_k) \in \{0,1\}^k\}$ for $k = 1,2,\dots,n$;

$$\left(a \wedge \left(\bigvee_{i=1}^{n} b_i\right)\right) \vee \left(a^\perp \left(\bigvee_{i=1}^{n} b_i\right)\right) = \bigvee_{i=1}^{n} (a \wedge b_i \vee a^\perp \wedge b_i) = \bigvee_{i=1}^{n} b_i,$$

which implies $a \leftrightarrow \bigvee_{i=1}^{n} b_i$. Analogously $a^\perp \leftrightarrow \bigvee_{i=1}^{n} b_i$.

It is evident that

$$\left(a \wedge \left(\bigvee_{i=1}^{n} b_i\right)^\perp\right) \vee \left(a^\perp \wedge \left(\bigvee_{i=1}^{n} b_i\right)^\perp\right) = (a \wedge b_1^\perp \wedge \dots \wedge b_n^\perp) \vee (a^\perp \wedge b_1^\perp \wedge \dots \wedge b_n^\perp)$$
$$= b_1^\perp \wedge \dots \wedge b_n^\perp = \left(\bigvee_{i=1}^{n} b_i\right)^\perp$$

therefore, $a \leftrightarrow \left(\bigvee_{i=1}^{n} b_i\right)^{\perp}$, and we have proved that $a \overset{s}{\leftrightarrow} \bigvee_{i=1}^{n} b_i$.

Because the set $\{a, b_1^{\perp}, \dots, b_n^{\perp}\}$ is strongly f-compatible, from the above we have $a \overset{s}{\leftrightarrow} \bigvee_{i=1}^{n} b_i^{\perp}$ and $a \overset{s}{\leftrightarrow} \bigwedge_{i=1}^{n} b_i$, too.

LEMMA 3.3. *Let P be a q.o.p. with the c-f-distributive property and A, B be two different Boolean subalgebras of P. The following statements are equivalent:*

(i) $A \leftrightarrow B$.
(ii) *The set $A \cup B$ is f-compatible.*
(iii) *The set $A \cup B$ is strongly f-compatible.*

Proof. The equivalence of (ii) and (iii) is evident, therefore, $A^{\perp} = \{a^{\perp} : a \in A\} = A$ for any Boolean subalgebra A of P.

Suppose (i). We prove that if $a, c \in A$ and $b, d \in B$, then $c \overset{s}{\leftrightarrow} a \wedge b \overset{s}{\rightarrow} d$. It is clear that $c \wedge (a \wedge b)$, $c^{\perp} \wedge (a \wedge b) \in P$ and $P \ni (c \wedge a^{\perp} \wedge b) \vee (c \wedge a^{\perp} \wedge b^{\perp}) \vee (c \wedge a \wedge b^{\perp}) \vee (c \wedge a^{\perp} \wedge b^{\perp}) = (c \wedge a^{\perp} \wedge b) \vee (c \wedge a^{\perp} \wedge b^{\perp}) \vee \big(((c \wedge a) \vee (c \wedge a^{\perp})) \wedge b^{\perp}\big) = (c \wedge a^{\perp}) \vee (c \wedge b^{\perp}) = c \wedge (a^{\perp} \vee b^{\perp}) = c \wedge (a \wedge b)^{\perp}$. Analogously $c^{\perp} \wedge (a \wedge b)^{\perp} \in P$.

Calculate

$\big(c \wedge (a \wedge b)^{\perp}\big) \vee \big(c^{\perp} \wedge (a \wedge b)^{\perp}\big) = (c \wedge a^{\perp}) \vee (c \wedge b^{\perp}) \vee (c^{\perp} \wedge a^{\perp}) \vee (c^{\perp} \wedge b^{\perp}) = a^{\perp} \vee b^{\perp} = (a \wedge b)^{\perp}$;

$\big(c \wedge (a \wedge b)^{\perp}\big) \vee \big(c \wedge (a \wedge b)\big) = (c \wedge a^{\perp} \wedge b) \vee (c \wedge a^{\perp} \wedge b^{\perp}) \vee (c \wedge b^{\perp} \wedge a) \vee (c \wedge b^{\perp} \wedge a^{\perp}) \vee (c \wedge a \wedge b) = (c \wedge a^{\perp}) \vee (c \wedge a) = c$, that is $c \leftrightarrow (a \wedge b)^{\perp}$.

Further,

$\big(c^{\perp} \wedge (a \wedge b)\big) \vee \big(c \wedge (a \wedge b)\big) = \big((c^{\perp} \wedge a) \vee (c \wedge a)\big) \wedge b = a \wedge b$;

$\big(c^{\perp} \wedge (a \wedge b)\big) \vee \big(c^{\perp} \wedge (a \wedge b)^{\perp}\big) = (c^{\perp} \wedge a \wedge b) \vee (c^{\perp} \wedge a^{\perp} \wedge b) \vee (c^{\perp} \wedge a^{\perp} \wedge b^{\perp}) \vee (c^{\perp} \wedge b^{\perp} \wedge a) = (c^{\perp} \wedge a) \vee (c^{\perp} \wedge a^{\perp}) = c^{\perp}$, therefore $c^{\perp} \leftrightarrow (a \wedge b)$, which gives $c \overset{s}{\leftrightarrow} (a \wedge b)$. Symmetrically $d \overset{s}{\leftrightarrow} (a \wedge b)$.

Let $a_1, a_2, \dots, a_{n+1} \in A \cup B$. Denote $a_i^0 = a_i^{\perp}$, $a_i^1 = a_i$, $i = 1, \dots, n$, $u = a_1^{j_1} \wedge \dots \wedge a_n^{j_n}$, where $(j_1, \dots, j_n) \in \{0,1\}^n$. Only one of the following alternatives holds:

(1) $u \in A$,
(2) $u \in B$,
(3) $u = a \wedge b$,

where $a \in A$ and $b \in B$. In any case $u \overset{s}{\leftrightarrow} a_{n+1}$, which implies $u \wedge a_{n+1}$, $u \wedge a_{n+1}^{\perp} \in P$ and $(u \wedge a_{n+1}) \vee (u \wedge a_{n+1}^{\perp}) = u$, therefore, the set $A \cup B$ is strongly f-compatible.

The converse assertion is evident.

PROPOSITION 3.4. *Let P be a q.o.p. with c-f-distributive property. Then any two compatible Boolean subalgebras of P are commensurable.*

Proof. If A and B are compatible Boolean subalgebras, then $A \cup B$ is the f-compatible set and $1_A = 1_B$. Define $D = \{a \wedge b\colon\ a \in A,\ b \in B\}$. Evidently $D \subset P$ and $A,\ B \subset D$, because $a = a \wedge 1_A = a \wedge 1_B$ and $b = b \wedge 1_B = b \wedge 1_A$.

(i) If $u,\ v \in D$, then $u \overset{s}{\leftrightarrow} v$.

Let $u = a \wedge b$, $v = c \wedge d$, where $a,\ c \in A$ and $b,\ d \in B$. Then $u \wedge v = (a \wedge b) \wedge (c \wedge d) = (a \wedge c) \wedge (b \wedge d) \in D \subset P$. By the proof of Lemma 3.3, we have $c \overset{s}{\leftrightarrow} a \wedge b \overset{s}{\leftrightarrow} d$ and from the c-f-distributive property we get that $(a \wedge b \wedge c^{\perp} \wedge d) \vee (a \wedge b \wedge c^{\perp} \wedge d^{\perp}) \vee (a \wedge b \wedge c \wedge d^{\perp}) = (a \wedge b \wedge c^{\perp}) \vee (a \wedge b \wedge d^{\perp}) = (a \wedge b) \wedge (c^{\perp} \vee d^{\perp}) = u \wedge v^{\perp}$,
therefore $u \wedge v^{\perp} \in P$. Analogously $u^{\perp} \wedge v,\ u^{\perp} \wedge v^{\perp} \in P$.

Calculate
$(u \wedge v^{\perp}) \vee (u^{\perp} \wedge v^{\perp}) = (a \wedge b \wedge c^{\perp}) \vee (a \wedge b \wedge d^{\perp}) \vee (a^{\perp} \wedge b \wedge c^{\perp}) \vee (a^{\perp} \wedge b^{\perp} \wedge c^{\perp}) \vee (a \wedge b^{\perp} \wedge c^{\perp}) \vee (a^{\perp} \wedge b \wedge d^{\perp}) \vee (a^{\perp} \wedge b^{\perp} \wedge d^{\perp}) \vee (a \wedge b^{\perp} \wedge d^{\perp}) = (b \wedge c^{\perp}) \vee (b^{\perp} \wedge c^{\perp}) \vee (b \wedge d^{\perp}) \vee (b^{\perp} \wedge d^{\perp}) = c^{\perp} \vee d^{\perp} = (c \wedge d)^{\perp} = v^{\perp}$.

By the same way we prove that $(u \wedge v^{\perp}) \vee (u \wedge v) = u$, which implies $u \leftrightarrow v^{\perp}$. Symmetrically $u^{\perp} \leftrightarrow v$, therefore, $u \overset{s}{\leftrightarrow} v$.

(ii) $u \wedge u^{\perp} = 0_A$ for any $u \in D$.

Let $u = a \wedge b$, where $a \in A$ and $b \in B$. Then $u \wedge u^{\perp} = (a \wedge b) \wedge (a^{\perp} \vee b^{\perp}) = (a \wedge b \wedge a^{\perp}) \vee (a \wedge b \wedge b^{\perp}) = 0_A \vee 0_B = 0_A$.

(iii) The set D is strongly f-compatible.

Denote $a^0 = a^{\perp}$, $a^1 = a$ for any $a \in P$. Let $u_1, u_2, \dots, u_{n+1} \in D \cup D^{\perp}$. Then there is a set $J \subset \{(j_1, \dots, j_n, j_{n+1}, \dots, j_{2n}) \in \{0,1\}^{2n}\}$ such that $u_1 \wedge \dots \wedge u_n = \bigvee\limits_J w_j$, where $w_j = a_1^{j_1} \wedge \dots \wedge a_n^{j_n} \wedge b_1^{j_{n+1}} \wedge \dots \wedge b_n^{j_{2n}}$, $a_i^{j_i} \in A$ and $b_i^{j_{n+i}} \in B$ for $i = 1, 2, \dots, n$. Evidently $w_j \in D$ and $\bigvee\limits_J w_j \in P$, because $w_j \perp w_m$ for $j \neq m$.

Without loss of generality we can assume that $u_{n+1} = a_{n+1} \wedge b_{n+1}$. Due to the above we have $w_j \overset{s}{\leftrightarrow} u_{n+1}$, therefore the elements $w_j \wedge u_{n+1}$ and $w_j \wedge u_{n+1}^{\perp}$ exist in P, moreover, $(w_j \wedge u_{n+1}) \vee (w_j \wedge u_{n+1}^{\perp}) = w_j$. Using the c-f-distributive property we get that

$$P \ni \bigvee_J (w_j \wedge u_{n+1}) = \Big(\bigvee_J w_j\Big) \wedge u_{n+1} = u_1 \wedge \dots \wedge u_n \wedge u_{n+1} =: u\,,$$

$$P \ni \bigvee_J (w_j \wedge u_{n+1}^{\perp}) = \Big(\bigvee_J w_j\Big) \wedge u_{n+1}^{\perp} = u_1 \wedge \dots \wedge u_n \wedge u_{n+1}^{\perp} =: v\,.$$

Then $u \vee v = \bigvee_J (w_j \wedge u_{n+1}) \vee \bigvee_J (w_j \wedge u_{n+1}^\perp) = \bigvee_J (w_j \wedge u_{n+1} \vee w_j \wedge u_{n+1}^\perp) = \bigvee_J w_j = u_1 \wedge \cdots \wedge u_n$, which implies that the set D is strongly f-compatible.

Finally, denote by $U = \left\{ u = \bigvee_{i=1}^{n} u_i : \ u_i \in D, \ n \geq 1 \right\}$.

We claim to show that U is a Boolean subalgebra of P containing the Boolean algebras A and B.

(1) If $u, v \in U$, then $u \wedge v \in U$.

Let $u \in U$, $u = \bigvee_{i=1}^{n} u_i$ and $v \in D$. The set $\{v, u_1, \ldots, u_n\}$ is strongly f-compatible, then by Proposition 3.2, $v \overset{s}{\leftrightarrow} u$, which implies $u \wedge v \in P$, moreover, $u \wedge v = \left(\bigvee_{i=1}^{n} u_i \right) \wedge v = \bigvee_{i=1}^{n} (u_i \wedge v) \in U$.

Suppose now that $u, v \in U$, $u = \bigvee_{i=1}^{n} u_i$, $v = \bigvee_{j=1}^{m} v_j$. The set $\{v_j, u_1, \ldots, u_n\}$ is strongly f-compatible and, therefore, $v_j \overset{s}{\leftrightarrow} u$ for any $j = 1, 2, \ldots, m$. Then $U \ni \bigvee_{j=1}^{m} \bigvee_{i=1}^{n} (u_i \wedge v_j) = \bigvee_{j=1}^{m} (u \wedge v_j) = u \wedge v$.

(2) $u^\perp \in U$ for any $u \in U$.

This result follows from the strong f-compatibility of the set D (see (iii)).

(3) If $u, v \in U$, then $u \vee v \in U$.

This result follows from (1) and (2).

(4) $u \wedge u^\perp = 0_A$ and $0_A \leq u \leq 1_A$ for any $u \in U$.

If $u = \bigvee_{i=1}^{n} u_i$, then from the strong compatibility of u_k and u for any $k = 1, 2, \ldots, n$, we have $u \wedge u^\perp = \bigvee_{k=1}^{n} \left(u_k \wedge \bigwedge_{i=1}^{n} u_i^\perp \right) = 0_A$ and $0_A = u \wedge u^\perp \leq u \leq u \vee u^\perp = 1_A$ for any $u \in U$.

(5) $u \overset{s}{\leftrightarrow} v$ for every $u, v \in U$.

In view of the above, $u \wedge v$, $u^\perp \wedge v^\perp \in P$. Let $u = \bigvee_{i=1}^{n} u_i$ and $v = \bigvee_{j=1}^{m} v_j$. Then $v_j \overset{s}{\leftrightarrow} u$ for any $j = 1, \ldots, m$ and the strong f-compatibility of the set $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ implies that $P \ni \bigvee_{J_0} \left(u_1^\perp \wedge \cdots \wedge u_n^\perp \wedge v_j^{j_1} \wedge \cdots \wedge v_m^{j_m} \right) = \bigvee_{j=1}^{m} (u^\perp \wedge v_j) = u^\perp \wedge \left(\bigvee_{j=1}^{m} v_j \right) = u^\perp \wedge v$, where the set J_0 is the same as in the proof of the Proposition 3.2. Symmetrically $u \wedge v^\perp \in P$.

Calculate

$$(u^{\perp} \wedge v) \vee (u^{\perp} \wedge v) = \Big(\bigvee_{J_0} u^{\perp} \wedge v_1^{j_1} \wedge \cdots \wedge v_m^{j_m} \Big) \vee (u^{\perp} \wedge v_1^{\perp} \wedge \cdots \wedge v_m^{\perp}) = \cdots = u^{\perp}$$

(see proof of the Proposition 3.2),

$$(u^{\perp} \wedge v) \vee (u \wedge v) = \Big(\bigvee_{j=1}^{m} (u^{\perp} \wedge v_j) \Big) \vee \Big(\bigvee_{j=1}^{m} (u \wedge v_j) \Big)$$
$$= \bigvee_{j=1}^{m} \big((u^{\perp} \wedge v_j) \vee (u \wedge v_j) \big) = \bigvee_{j=1}^{m} v_j = v \,,$$

which gives $u^{\perp} \leftrightarrow v$. Symmetrically $u \leftrightarrow v^{\perp}$, therefore $u \overset{s}{\leftrightarrow} v$.

(6) The distributivity in U follows from the c-f-distributive property and from (5).

From (1) – (6) is evident that U is a Boolean subalgebra of P.

PROPOSITION 3.5. *Let $A_1, \dots, A_n$ be Boolean subalgebras of a q.o.p. P with the c-f-distributive property. The algebras $A_1, \dots, A_n$ are commensurable if and only if the set $\bigcup_{i=1}^{n} A_i$ is f-compatible.*

Proof. If $A_1, \dots, A_n$ are commensurable, then there is a Boolean subalgebra B such that $\bigcup_{i=1} A_i \subset B$ and every Boolean algebra is f-compatible.

The sufficiency follows from the observation that the Boolean subalgebra containing $A_1, \dots, A_n$ consists of the elements of the form $\bigvee_{i=1}^{m} a_{1i} \wedge a_{2i} \wedge \cdots \wedge a_{ni}$, where $a_{ki} \in A_k$ for $k = 1, \dots, n$ and $m \geqq 1$. To prove that, we use the same arguments as in the proof of Proposition 3.4.

The statement of Proposition 3.5 is incorrect if we assume only the mutual compatibility of $A_1, \dots, A_n$. Indeed, let $X = \{1, 2, \dots, 8\}$ and S be a system of all subsets of X with even number of elements. The system S is a q.o.p. Put $A = \big\{\{1,2,3,4\}, \{5,6,7,8\}, X, \emptyset\big\}$, $B = \big\{\{1,2,5,6\}, \{3,4,7,8\}, X, \emptyset\big\}$, $C = \big\{\{1,3,6,8\}, \{2,4,5,7\}, X, \emptyset\big\}$. Then A, B, C are pairwisely compatible Boolean subalgebras of S, but $\{1,2,3,4\} \cap \{1,2,5,6\} \cap \{1,3,6,8\} = \{1\}$, so A, B, C are no commensurable.

THEOREM 3.6. *A system $\{A_t \colon t \in T\}$ of Boolean subalgebras of a q.o.p. with the c-f-distributive property is commensurable if and only if the set $\bigcup_{t \in T} A_t$ is f-compatible.*

P r o o f. Let T_0 be any finite nonempty subset of T. In view of Proposition 3.5, there is a Boolean subalgebra $A(T_0)$ containing all A_t for $t \in T_0$. Write $A = \bigcup_{T_0 \subset T} \{A(T_0) \colon\ T_0 \text{ is a finite subset of } T\}$. Simple verification shows that A is a Boolean subalgebra of P including all A_t, $t \in T$.

PROPOSITION 3.7. *Let P be a q.o.p.with c-σ-distributive property. Then any Boolean subalgebra of P is contained in a maximal one and a maximal Boolean subalgebra of P is necessarily a Boolean sub-σ-algebra.*

The proof of the proposition depends on the following results.

LEMMA 3.8. *Let P be a q.o.p.with the c-σ-distributive property, let A be a Boolean subalgebra of P, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise orthogonal elements of A and let b be any element of A. Put $a = \bigvee_{n\in\mathbb{N}} a_n$. Then*

(1) $a_i \leftrightarrow a$, $a_i \leftrightarrow a^\perp$ *for any* $i \in \mathbb{N}$;
(2) $a \wedge a^\perp = 0_A$;
(3) $a_i \overset{s}{\leftrightarrow} a$ *for any* $i \in \mathbb{N}$;
(4) $a \wedge b$, $a^\perp \wedge b$, $a \wedge b^\perp$, $a^\perp \wedge b^\perp \in P$;
(5) $a_i \leftrightarrow a \wedge b$, $a_i \leftrightarrow (a \wedge b)^\perp$, $a_i \leftrightarrow a \wedge b^\perp$, $a_i \leftrightarrow a^\perp \wedge b$, $a_i \leftrightarrow a^\perp \wedge b^\perp$ *for any* $i \in \mathbb{N}$;
(6) $a \leftrightarrow a^\perp \wedge b$, $a \leftrightarrow a^\perp \wedge b^\perp$, $a^\perp \leftrightarrow a \wedge b$, $a^\perp \leftrightarrow a \wedge b^\perp$, $b \leftrightarrow a \wedge b^\perp$, $b^\perp \leftrightarrow a \wedge b$;
(7) $a \overset{s}{\leftrightarrow} a_i \wedge b$, $a \overset{s}{\leftrightarrow} a_i \wedge b^\perp$ *for any* $i \in \mathbb{N}$;
(8) $(a \wedge b)^\perp \leftrightarrow (a_i^\perp \wedge b)^\perp$ *for any* $i \in \mathbb{N}$;
(9) $a \leftrightarrow b$, $a \leftrightarrow b^\perp$;
(10) $b \overset{s}{\leftrightarrow} a^\perp \wedge b$, $b \overset{s}{\leftrightarrow} a^\perp \wedge b^\perp$;
(11) $a_i^\perp \wedge b^\perp \overset{s}{\leftrightarrow} a^\perp \wedge b$ *for any* $i \in \mathbb{N}$;
(12) $a \overset{s}{\leftrightarrow} b$.

P r o o f o f P r o p o s i t i o n 3.7. The first statement follows easily from Zorn's Lemma.

In order to prove the second, suppose that A is a maximal Boolean subalgebra of P. Let $\{a_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of elements from A. Without loss of generality we may assume that $a_i \leqq a_j^\perp$ for $i \neq j$. Put $a = \bigvee_{n\in\mathbb{N}} a_n$. If b is any element of A, then by Lemma 3.8, $a \overset{s}{\leftrightarrow} b$. It is clear that $b \overset{s}{\leftrightarrow} a \vee a^\perp = 1_A$, $b \overset{s}{\leftrightarrow} a \wedge a^\perp = 0_A$, which implies that $A \leftrightarrow A_a$, where $A_a = \{a, a^\perp, a \vee a^\perp, a \wedge a^\perp\}$. Referring to Proposition 3.4, there is a Boolean subalgebra B containing Boolean subalgebras A and A_a, which gives $A = B$. Then $A_a \subset A$ and, therefore, the element a is from A, which implies that A is a Boolean sub-σ-algebra of P.

From the Proposition 3.7 is evident that the commensurability and σ-commensurability are equivalent notions.

THEOREM 3.9. *Let A be a nonempty set of a q.o.p. P with the c-σ-distributive property. The following statements are equivalent.*

(i) *A is strongly f-compatible.*
(ii) *A is σ-commensurable.*

Proof. For any $a \in A$, define a Boolean subalgebra A_a via $A_a = \{a, a^\perp, a \vee a^\perp, a \wedge a^\perp\}$. It is clear that the set $\bigcup\limits_{a\in A} A_a$ is f-compatible. Referring to Theorem 3.6 and Proposition 3.7, the proof is finished.

4. Calculus for compatible observables and a joint observable

In the present section we apply the compatibility theorem for Boolean subalgebras of a q.o.p. P to build up the so-called functional calculus for observables of P and for the existence of a joint observable. We note that for compatible observables of a quantum logic, the functional calculus has been build up by Varadarajan [10] and for F-observables of an F-quantum space by Dvurečenskij and Riečan [3].

Throughout this section we shall assume that P is a q.o.p. with the c-σ-distributive property.

It is well known that if x is an observable of P and if f is a Borel measurable real-valued function, then a mapping $y = x \circ f^{-1}$ defined via

$$y(E) = x\big(f^{-1}(E)\big), \qquad E \in B(\mathbb{R}^1),$$

is an observable of P.

A Boolean sub-σ-algebra A of P is said to be *separable* if A contains a generator of itself with countably many elements.

LEMMA 4.1. *A Boolean sub-σ-algebra A of P is separable if and only if there is an observable x of P such that $A = R(x) = \{x(E) : E \in B(\mathbb{R}^1)\}$. Moreover, there is a measurable space (Ω, S), a σ-homomorphism h from S onto A and an S-measurable mapping $g : \Omega \to \mathbb{R}^1$ such that*

$$x(E) = h\big(g^{-1}(E)\big), \qquad E \in B(\mathbb{R}^1). \tag{4.1}$$

Proof. The sufficiency is evident. Conversely, if A be separable, due to the Loomis-Sikorski theorem (see, for example [9]), there is a σ-algebra S of subsets of some set Ω and a σ-homomorphism h from S onto A. According to Varadarajan [10], there is a measurable mapping $g\colon \Omega \to \mathbb{R}^1$ such that (4.1) holds.

We recall that an observable x and an observable y are *compatible* if $x(E) \leftrightarrow y(F)$ for any $E \in B(\mathbb{R}^1)$ and $F \in B(\mathbb{R}^1)$. Analogously we say that $\{x_t\colon t \in T\}$ is a *system of f-compatible observables* if $\bigcup_{t\in T} R(x_t)$ is an f-compatible set in P.

THEOREM 4.2. *Let P be a q.o.p. with the c-σ-distributive property and let $\{x_t\colon t \in T\}$ be a family of observables of P. If the observables x_t, $t \in T$, are f-compatible, then there is a measurable space (Ω, S), real-valued S-measurable functions g_t on Ω, and a σ-homomorphism h of S into P such that*

$$x_t(E) = h\big(g_t^{-1}(E)\big) \tag{4.2}$$

for all $t \in T$ and $E \in B(\mathbb{R}^1)$. Suppose further that either P is separable in the sense that every Boolean sub-σ-algebra of P is separable, or that T is countable. Then there is an observable x and real-valued Borel functions f_t of a real variable such that for all $t \in T$,

$$x_t = x \circ f_t^{-1}\,. \tag{4.3}$$

Proof. Let $\{x_t\colon t \in T\}$ be a family of f-compatible observables. According to the compatibility theorem (Theorem 3.6), there is a Boolean sub-σ-algebra A of P such that $R(x_t) \subset A$ for all $t \in T$. The Loomis-Sikorski theorem entails that there is a measurable space (Ω, S) and a σ-homomorphism h from S onto A. Let S_t be a sub-σ-algebra of S such that $h_t := h/S_t$ is a σ-homomorphism of S_t onto the range $R(x_t)$ of x_t for any $t \in T$. Due to Lemma 4.1, we see that there is an S_t-measurable $g_t\colon \Omega \to \mathbb{R}^1$ such that $x_t(E) = h_t\big(g_t^{-1}(E)\big) = h\big(g_t^{-1}(E)\big)$ for any $E \in B(\mathbb{R}^1)$. This proves the equation (4.2). Theorem 6.9 of [10] entails that there are an observable x and Borel measurable real-valued functions f_t such that (4.3) holds.

The characterization of simultaneous observability given in Theorem 4.2 enables us to construct a calculus of functions of several observables which are f-compatible.

Let $x_1, x_2, \dots, x_n$ be f-compatible observables. Then we may define the sum of observables via

$$x_1 + x_2 + \cdots + x_n = x \circ (f_1 + f_2 + \cdots + f_n)^{-1}\,, \qquad \text{where} \quad x_i = x \circ f_i^{-1}\,.$$

Finally we apply Theorem 4.2 to the problem of existence of a joint observable of f-compatible observables.

A collection $\{x_i\colon\ i = 1,\dots,n\}$ of observables of P is said to *have a joint observable* if there is a σ-homomorphism $w\colon B(\mathbb{R}^n) \to P$ such that

$$w\bigl(p_i^{-1}(E)\bigr) = x_i(E) \qquad \text{for any} \quad E \in B(\mathbb{R}^1)\,, \quad i = 1,2,\dots,n\,,$$

where p_i is the projection of $\mathbb{R}^n$ on $\mathbb{R}^1$. We call w a *joint observable.*

We note that the joint observable in a quantum logic, which is not a lattice, need not exist even in the case when $\{x_i\colon\ i = 1,\dots,n\}$ are mutually compatible (see [6, Example 6]).

THEOREM 4.3. *Let P be a q.o.p. with the c-σ-distributive property. A system $\{x_i\colon\ i = 1,2,\dots,n\}$ of observables of P has a joint observable if and only if $x_1, x_2,\dots,x_n$ are f-compatible.*

Proof. If $x_1,\dots,x_n$ are f-compatible observables, by Theorem 4.2 there is an observable x and real-valued Borel functions f_t such that $x_i = x \circ f_i^{-1}$, $i = 1,\dots,n$.

Define a function $f\colon \mathbb{R}^1 \to \mathbb{R}^n$ via

$$f(t) = \bigl(f_1(t),\dots,f_n(t)\bigr)\,, \qquad t \in \mathbb{R}^1\,.$$

The function f is $B(\mathbb{R}^1)$-measurable, i.e. $f^{-1}(H) \in B(\mathbb{R}^1)$ for any $H \in B(\mathbb{R}^n)$. Now we define a mapping $w\colon B(\mathbb{R}^n) \to P$ such that

$$w(H) = x\bigl(f^{-1}(H)\bigr) \qquad \text{for} \quad H \in B(\mathbb{R}^n)\,.$$

It is evident that the mapping w is a σ-homomorphism.
Therefore, $f^{-1}\bigl(p_i^{-1}(E)\bigr) = \bigl\{t \in \mathbb{R}^1\colon\ f(t) \in p_i^{-1}(E)\bigr\} = \bigl\{t \in \mathbb{R}^1\colon\ f_i(t) \in E\bigr\} = f_i^{-1}(E)$ for any $E \in B(\mathbb{R}^1)$, we have $w\bigl(p_i^{-1}(E)\bigr) = x\bigl(f^{-1}\bigl(p_i^{-1}(E)\bigr)\bigr) = x\bigl(f_i^{-1}(E)\bigr) = x_i(E)$, which implies that w is a joint observable of $x_1,\dots,x_n$. It is simple to verify that the joint observable is unique.

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