Mathematica Slovaca

Jaromír Kuben

Time-optimal control of two-dimensional systems and regular synthesis

Mathematica Slovaca, Vol. 40 (1990), No. 3, 303--320

Persistent URL: `http://dml.cz/dmlcz/136512`

Math. Slovaca 40, 1990, No. 3, 303—320

TIME-OPTIMAL CONTROL OF TWO-DIMENSIONAL SYSTEMS AND REGULAR SYNTHESIS

JAROMÍR KUBEN

ABSTRACT. A time-optimal control of a system $u' = v - F(u)$, $v' = -g(u) + w(t)$, $|w| \le K$, $F, g \in C^1(R)$ is studied. Pontryagin's maximum principle is used to prove that optimal controls are piecewise constant. An optimal feedback control is studied and a construction of a locus of switching is described. Then a regular synthesis in Boltyanskii's sense is defined and its existence is proved in special cases. In the last part the obtained results are compared with those of Boltyanskii and Lee and Markus.

1. Introduction

In [13—17] Lee and Markus studied the time-optimal control of the second order differential equation

$$x'' + f(x, x') = w, \qquad |w| \leqslant 1. \tag{1}$$

Their results are applicable to the controlled generalized Liénard equation

$$x'' + f(x)x' + g(x) = w, \qquad |w| \leqslant 1. \tag{2}$$

Some other results concerning the control of the equation (2) and of the Van der Pol equation are in [1, 8, 9, 10, 18].

Boltyanskii [2—4] introduced the concept of regular synthesis and proved its existence for the equation

$$x'' = h(x, x', w), \qquad |w| \leqslant 1, \tag{3}$$

having some "oscillating" properties. The concept of regular synthesis was considerably developed by Brunovský [5—7].

The aim of this paper is to prove the existence of the regular synthesis for the first order two-dimensional systems (4) described below onto which the equa-

AMS Subject Classification (1980): Primary 49B10.
Key words: Maximal trajectories, Pontryagin's maximum principle, Regular synthesis.

tion (2) can be transformed. This approach allows us to assume less smoothness of f in (2).

In the first part of the paper besides some basic facts about optimal trajectories which are an immediate consequence of Pontryagin's maximum principle, especially qualitative properties of solutions of the system (4) and its maximal trajectories are studied. Boltyanskii's approach (the use of polar coordinates) is not possible in the case of the system (4) and more complicated topological methods had to be used instead to obtain the need results concerning the properties of the locus of switching. Then the synthesis of (4) is constructed and its regularity verified.

In the last part the obtained results are compared with those ones of Boltyanskii and Lee and Markus. A class of special equations of the type (2) is presented ($f(x) \equiv 0$) which is not covered by the papers of the mentioned authors. Some deeper results on second order linear differential equations with a periodic coefficient had to be used there. Then it is observed that no nonlinear equation (1) which is "oscillating" in Boltyanskii's sense exists. Finally, an assumption of Lee and Markus, which does not seem to be quite correct, is discussed and it is stated that no nonlinear equation of the special type (25) fulfilling this assumption exists.

2. Formulation of the control problem

Consider a control system

$$\begin{aligned} u' &= v - F(u) \\ v' &= -g(u) + w(t), \end{aligned} \tag{4}$$

where $(u, v) \in R^2$ are state variables, w is a control and $F, g \in C^1(R)$.

Let $K > 0$ b fixed. We shall consider two kinds of admissible controls w:

i) $w \in L^{\infty}_{loc}(R)$, $|w| \leq K$, where $L^{\infty}_{loc}(R)$ is a set of all locally essentially bounded Lebesgue's measurable functions on R,

ii) $w \in M$, where M is a set of all piecewise continuous bang-bang functions, i.e. $w = \pm K$.

The response (u, v) will be a couple of absolutely continuous functions fulfilling (4) almost everywhere (a.e.). Denote (S_+), (S_-), (S_0) the system (4), where $w \equiv +K$, $w \equiv -K$, $w \equiv 0$, respectively.

The aim of the control is to steer an initial state (u_0, v_0) into a target state $O = (0, 0)$ in a minimal possible time. Let $D = L^{\infty}_{loc}(R)$ or $D = M$. Denote $W(D)$ a set of controllability of (4) for $w \in D$, i.e. $(u_0, v_0) \in W(D)$ iff there exists $w \in D$ steering (u_0, v_0) to O, and $W_{opt}(D)$ a set of the states that can be steered optimally.

Theorem 1. *Let* $F(0) = g(0)$. *Then* $W(L'_{loc}(R))$ *is a region in* R^2.

Proof. Consider the linearization of (4). Let

$$A = \begin{pmatrix} -F(0) & 1 \\ -g'(0) & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As $rg(B, AB) = 2$ the assertion is a consequence of [13; p. 399; Th. 1]. Consider now the equation (2), where $f, g \in C^0(R)$. This equation is equivalent to the system

$$x' = y$$
$$y' = -f(x)y - g(x) + w.$$

It is easy to verify that the tranformation

$$u = x, \qquad v = y + F(x),$$

where $F'(x) = f(x)$, transforms (2) into (4) with $F \in C^1(R)$, $g \in C^0(R)$ so that (4) is more general than (2) (if $F \in C^0(R)$, then the back transformation is impossible).

In the sequel the following results concerning the qualitative behaviour of (S_0) will be needed. For their proofs see [11] or [12].

Lemma 1. *Let* $g(u)(u - a) > 0$ *for* $u \neq a$, $a \in R$. *Denote* $U(I)$, $U(II)$, $U(III)$, $U(IV)$ *the regions bounded in* R^2 *with a line* $u = a$ *and a graph of a function* $F(u)$. *Let* Γ *be a trajectory corresponding to a nonstationary solution of* (S_0). *Then*:

i) *Each connected arc of* Γ *lying entire in some above defined region is a graph of a function of the variable* u; *this function is decreasing in* $U(I)$ *and* $U(III)$ *and increasing in* $U(II)$ *and* $U(IV)$,

ii) *If* Γ *intersects the bounds of these regions, then it goes from* $U(I)$ *to* $U(IV)$, *from* $U(IV)$ *to* $U(III)$, *from* $U(III)$ *to* $U(II)$ *and from* $U(II)$ *to* $U(I)$ *with increasing* t.

Let ω_+ (ω_-) be a right-hand (left-hand) end of the maximal interval of the existence of a considered solution. Let $0 \in (\omega_-, \omega_+)$.

Lemma 2. *Let* $(u(t), v(t))$ *be a noncontinuable solution of* (S_0) *and*

$$g(u) > 0 \qquad \text{for } u > a,\ a \in R. \tag{5}$$

Let $u(0) \geqslant a$, $v(0) > F(u(0))$. *Then*
either there exists $t_1 > 0$ *such that* $v(t_1) = F(u(t_1))$,
$v(t) > F(u(t))$, $t \in \langle 0, t_1)$,
or $v(t) > F(u(t))$, $t \in \langle 0, \omega_+)$ *and* $\lim_{t \to \omega_+} u(t) = +\infty$.

If moreover

$$\liminf_{u \to +\infty} g(u) > 0 \qquad \text{and} \qquad \liminf_{u \to +\infty} F(u) > -\infty, \tag{6}$$

the first possibility holds.

Lemma 3. *Let* $(u(t), v(t))$ *be as in Lemma* 2 *and* (5) *holds. Let* $u(0) > a$, $v(0) > F(u(0))$. *Then*
either $\omega_- = -\infty$ *and* $(u(t), v(t)) \to (a, F(a))$ *for* $t \to -\infty$, $v(t) > F(u(t))$, $t \in (-\infty, 0)$,
or there exists $t_2 < 0$ *such that* $u(t_2) = a$, $v(t_2) > F(a)$ *and* $v(t) > F(u(t))$, $t \in \langle t_2, 0)$.
The first possibility can occur only if $(a, F(a))$ *is a singular point of* (S_0).

Similar assertions are available in $U(II) - U(IV)$.

3. Properties of optimal controls

Denote

$$H(\eta_1, \eta_2, u, v, w) = \eta_1(v - F(u)) + \eta_2(-g(u) + w)$$

and

$$M(\eta_1, \eta_2, u, v) = \max_{|w| \leq K} H(\eta_1, \eta_2, u, v, w).$$

The system

$$\begin{aligned} \eta_1' &= F'(u)\,\eta_1 + g'(u)\,\eta_2 \\ \eta_2' &= -\eta_1 \end{aligned} \tag{7}$$

is called adjoint to (4).

Definition 1. *We shall say that a control* $w \in L'_{\mathrm{loc}}(R)$ *and its response* $(u(t), v(t))$, $t \in \langle 0, t_1\rangle$, *are maximal (satisfy Pontryagin's maximum principle) if*

i) *there exists a nontrivial solution* (η_1, η_2) *of* (7) *such that*

$$H[\eta_1(t), \eta_2(t), u(t), v(t), w(t)] = M[\eta_1(t), \eta_2(t), u(t), v(t)]$$

a.e. on $\langle 0, t_1\rangle$,

ii) *the function* $M(t) = M[\eta_1(t), \eta_2(t), u(t), v(t)]$ *is constant on* $\langle 0, t_1\rangle$ *and* $0 \leq M(0)$.

It is well known that each optimal control and its response must be maximal — see, e.g., [13; p. 344].

Theorem 2. *Let* $w(t)$, $t \in \langle 0, t_1\rangle$ *be a maximal control. Then*

$$w(t) = K\,.\,\mathrm{sign}\,\eta_2(t) \qquad \textit{a.e. on } \langle 0, t_1\rangle\,. \tag{8}$$

Further, $\eta_2(t)$ *has only a finite number of roots on* $\langle 0, t_1\rangle$ *and these ones are single, i.e. we can suppose that* $w \in M$.

Proof. Evidently, $(\eta_1, \eta_2) \in C^1$. If $\eta_2(t)$ had infinitely many roots on $\langle 0, t_1\rangle$, at an accumulation point $\bar{t}$ it would be $\eta_2(\bar{t}) = \eta_2'(\bar{t}) = 0$. But (7) then implies $\eta_1(\bar{t}) = 0$ and we have a contradiction. If $\eta_2(t) = 0$, then $\eta_1(t) \neq 0$ and from (7) we obtain $\eta_2'(t) \neq 0$ so that the roots are single. Finally, (8) is a consequence of the fact that $\eta_2(t)\,w(t)$ must be maximal.

With respect to the preceding assertion we shall suppose without repeating it in the sequel that $w \in M$ for a maximal w.

Lemma 4. *If a control $w(t)$ and its response $(u(t), v(t))$, $t \in \langle 0, t_1 \rangle$, satisfy the first condition of the maximum principle, then $M(t)$ is constant on $\langle 0, t_1 \rangle$.*

Proof. It is easy to see that theorem 2 remains available so that $M(t)$ is continuous on $\langle 0, t_1 \rangle$. As $\frac{\mathrm{d}M}{\mathrm{d}t} \equiv 0$ on each opern subinterval noncontaining zeros of η_2, $M(t)$ must be constant.

The next lemma will serve as the main tool in our further considerations.

Lemma 5. *Let the assumptions of lemma 4 be fulfilled and $(\eta_1(t), \eta_2(t))$ is the corresponding solution of (7). Let $0 \leqslant \xi_1, \xi_2 \leqslant t_1$, $\xi_1 \neq \xi_2$. Then the next implications hold:*

i) *If $\eta_2(\xi_1) = \eta_2(\xi_2) = 0$ and $v(\xi_1) = F(u(\xi_1))$, then $v(\xi_2) = F(u(\xi_2))$.*

ii) *If $\eta_2(\xi_1) = \eta_2(\xi_2) = 0$ and $v(\xi_1) \neq F(u(\xi_1))$, then $v(\xi_2) \neq F(u(\xi_2))$, but there exists ξ_3 lying between ξ_1 and ξ_2 such that $v(\xi_3) = F(u(\xi_3))$.*

iii) *If $v(\xi_1) - F(u(\xi_1)) = v(\xi_2) - F(u(\xi_2)) = 0$, $v(t) - F(u(t)) \neq 0$ for t between ξ_1, ξ_2 and $\eta_2(\xi_1) = 0$, then $\eta_2(\xi_2) = 0$.*

iv) *If $v(\xi_1) - F(u(\xi_1)) = v(\xi_2) - F(u(\xi_2)) = 0$, $v(t) - F(u(t) \neq 0$ for t between ξ_1, ξ_2 and $\eta_2(\xi_1) \neq 0$, then $\eta_2(\xi_2) \neq 0$, but there exists ξ_3 lying between ξ_1 and ξ_2 such that $\eta_2(\xi_3) = 0$.* The proof is similar to that of [13; p. 463; Th. 1]. For the details see [11], lemma 4.6.

The preceding lemma shows that if $v(t) - F(u(t))$ has isolated roots, then these either coincide with the roots of $\eta_2(t)$ or they separate and are separated by those of $\eta_2(t)$.

Lemma 6. *If $w(t)$ and $(u(t), v(t))$, $t \in \langle 0, t_1 \rangle$, are maximal and $\eta_2(\xi) = 0$, $v(\xi) \neq F(u(\xi))$, $\xi \in (0, t_1)$, then*

$$\operatorname{sign}(v(\xi) - F(u(\xi))) = -\operatorname{sign} \eta_2'(\xi),$$

i.e. $w(t)$ changes from K to $-K$ (from $-K$ to K) above (below) the graph of F.

Proof. As $M(\xi) \leqslant 0$ and ξ is a single root of η_2, we obtain $\eta_1(\xi)(v(\xi) - f(u(\xi))) > 0$. From the second equation of (7) we have $\eta_1(\xi) \neq 0$ and $\operatorname{sign} \eta_2'(\xi) = -\operatorname{sign} \eta_1(\xi)$.

4. Locus of switching

Definition 2. *Denote V a set of all points in $W'(L'_{\mathrm{loc}}(R))$ in which some maximal response which terminates in O has not the derivative. Let $O \in V$. Then V is called the locus of switching.*

We shall describe the construction of V. Denote

$$(u(t, u_0, v_0), v(t, u_0, v_0)) \tag{9}$$

a solution of (4) such that $(u(0, u_0, v_0), v(0, u_0, v_0)) = (u_0, v_0)$. Let $(u_0, v_0) \in R^2$ and (9) is a corresponding solution of (S_-). Consider a solution (η_1, η_2) of (7) corresponding to (9) such that $\eta_1(0) \neq 0$, $\eta_2(0) = 0$. Denote $T^-(u_0, v_0)$ the largest negative root of $\eta_2(t)$ (if it exists). Denote Ω^- a set of points in R^2 for which T^- is defined. We have

$$T^-: \Omega^- \to (-\infty, 0). \tag{10}$$

Let

$$\Lambda^-(u_0, v_0) = (u(T^-(u_0, v_0), u_0, v_0), v(T^-(u_0, v_0), u_0, v_0)), \tag{11}$$

i.e. $\Lambda^-: \Omega^- \to R^2$.
T^+, Ω^+ and Λ^+ are defined in a similar way using the system (S_+). Put $\Omega_1^- = \Omega^-$, $\Lambda_1^- = \Lambda^-$.
If $k \geqslant 2$ is odd and $G_k^- = \Lambda_{k-1}^-(\Omega_{k-1}^-) \cap \Omega_1^- \neq \emptyset$, put

$$\Omega_k^- = (\Lambda_{k-1}^-)^{-1}(G_k^-) \text{ and } \Lambda_k^- = \Lambda_1^- \circ \Lambda_{k-1}^-. \tag{12a}$$

If $k \geqslant 2$ is even and $G_k^- = \Lambda_{k-1}^-(\Omega_{k-1}^-) \cap \Omega_1^+ \neq \emptyset$, put

$$\Omega_k^- = (\Lambda_{k-1}^-)^{-1}(G_k^-) \text{ and } \Lambda_k^- = \Lambda_1^+ \circ \Lambda_{k-1}^-. \tag{12b}$$

If $G_k^- = 0$, we define Λ_k^- as an empty map $\emptyset \to R^2$. So we have

$$\Lambda_k^-: \Omega_k^- \to R^2, \qquad k \in N.$$

Analogously we introduce Λ_k^+.

Let $F(0) = g(0) = 0$. Consider a solution $(u(t, 0, 0), v(t, 0, 0))$ of (S_+). There is $v(t) < F(u(t))$, $u(t) > 0$ for small $t < 0$. If these inequalities hold for all negative $t > \omega_-$, put $J = (\omega_-, 0\rangle$. If there exists the largest $t_1 < 0$ such that $v(t_1) = F(u(t_1))$, put $J = \langle t_1, 0\rangle$. Denote

$$V_+^1 = \{(u(t), v(t)): t \in J\}. \tag{13}$$

Analogously, using a trajectory of (S_-), we define V_-^1.
Now let

$$\begin{aligned} V_\mp^{k+1} &= \Lambda_k^-(V_\pm^1 \cap \Omega_k^-), \qquad k \in N \ (+ \text{ for } k \text{ even and } - \text{ for } k \text{ odd}) \\ V_\pm^{k+1} &= \Lambda_k^+(V_\mp^1 \cap \Omega_k^+), \qquad k \in N \ (- \text{ for } k \text{ even and } + \text{ for } k \text{ odd}). \end{aligned} \tag{14}$$

Theorem 3. *Let V be the locus of the switching of* (4) *and* $F(0) = g(0) = 0$. *Then*

$$V = \bigcup_{k=1}^{\infty} (V_+^k \cup V_-^k).$$

Proof. Let $p \in V$, $p \neq O$, and $\tau_1 < \ldots < \tau_k$ be roots of a maximal control $w(t)$, $t \in \langle 0, t_1\rangle$, steering p to O. Then $\tau_1 = 0$, $\tau_k \leqslant t_1$. If $\tau_k < t_1$, then $p_k = (u(\tau_k), v(\tau_k)) \in V_+^1$, $p_{k-1} = (u(\tau_{k-1}), v(\tau_{k-1})) = \Lambda^+(p_k) \in V_+^2$ etc. We have

$p = p_1 \in \in V_+^k \cup V_-^k$. If $\tau_k = t_1$, there is $p_k = O$, $p_{k-1} \in V_+^1$ and analogously $p \in V_+^{k-1} \cup V_-^{k-1}$, so that $V \subset \bigcup\limits_{k=1}^{\infty} (V_+^k \cup V_-^k)$.

The converse inclusion will be proved by induction. Let $p \in V_+^1$, $p \neq O$. Put $w \equiv K$, $(u(0), v(0)) = p$, $\eta_1(0) = -1$, $\eta_2(0) = 0$. Let t_1 be the time when O will be reached. According to lemma 5, i), or ii), and (13) we obtain that $\eta_2(t) \neq 0$ on $(0, t_1)$, i.e. $w(t)$ is maximal. We have $V_+^1 \subset V$ and similarly $V_-^1 \subset V$.

Let $V_\pm^k \subset V$, $k = 1, \ldots, n$, $p \in V_-^{n+1}$ and $p = \Lambda^-(q)$, $q \in V_+^n$. There exists a maximal control $w(t)$, $t \in \langle 0, t_1 \rangle$ steering q to O. For the corresponding $(\bar{\eta}_1, \bar{\eta}_2)$ there is $\bar{\eta}_2(0) = 0$. Let $\hat{w}(t) \equiv -K$, $t \in \langle T^-(q), 0 \rangle$ and $(\hat{\eta}_1, \hat{\eta}_2)$ be a solution of (7) used in the definition of T^-; we can suppose that $\hat{\eta}_1(0) = -1$, $\hat{\eta}_2(0) = 0$. Define

$$w(t) = \begin{cases} \hat{w}(t + T\ (q)) & \text{for } t \in \langle 0, -T\ (q)) \\ \bar{w}(t + T\ (q)) & \text{for } t \in \langle -T\ (q), t_1 - T^-(q) \rangle \end{cases}$$

and

$$\eta_i(t) = \begin{cases} -\bar{\eta}_1(0)\, \hat{\eta}_i(t + T\ (q)) & \text{for } t \in \langle 0, -T\ (q) \rangle \\ \bar{\eta}_i(t + T\ (q)) & \text{for } t \in \langle -T^-(q), t_1 - T^-(q) \rangle \end{cases}, \qquad i = 1, 2.$$

Evidently (η_1, η_2), $t \in \langle 0, t_1 - T\ (q) \rangle$, is a solution of (7) which corresponds to the control $w(t)$ and its response (u, v); moreover $w(t)$ steers p to O. As according to lemma 4, the function $M(t)$ is constant on $\langle 0, t_1 - T\ (q) \rangle$ and $M(t) \geq 0$ on $\langle -T\ (q), t_1 - T\ (q) \rangle$, the control $w(t)$ is maximal. Hence $V^{n+1} \subset V$ and similarly $V_+^{n+1} \subset V$.

5. Properties of the locus of switching

To prove some other properties of the locus of switching we must suppose a higher smoothness of the right hand sides of (4).

Lemma 7. *Let $F, g \in C^2(R)$ in (4). Then the sets Ω^+ are open, the functions T^+ are differentiable and the maps Λ^+ are global diffeomorhisms.*

Proof. The assertion will be proved for Ω , T and Λ .

Let $(\hat{u}_0, \hat{v}_0) \in \Omega$ and denote $T_0 = T\ (\hat{u}_0, \hat{v}_0)$. Consider the solution $(u(t, \hat{u}_0, \hat{v}_0), v(t, \hat{u}_0, \hat{v}_0))$ of $(S\ )$ on $\langle \tau, 0 \rangle$, $\tau < T_0$. According to the theorem on the continuous dependence and the differentiability of solutions of differential equations with respect to initial conditions, we can find a neighbourhood O_1 of $(\hat{u}_0, \hat{v}_0)$ such that for $(u_0, v_0) \in O_1$ the solution (9) of $(S\ )$ is defined on $\langle \tau, 0 \rangle$. Moreover continuous partial derivatives $\dfrac{\partial u}{\partial u_0}, \dfrac{\partial u}{\partial v_0}, \dfrac{\partial v}{\partial u_0}, \dfrac{\partial v}{\partial v_0}$ exist in (t, u_0, v_0),

$t \in \langle \tau, 0 \rangle$, $(u_0, v_0) \in O_1$. Denote

$$(\eta_1(t, u_0, v_0), \eta_2(t, u_0, v_0)) \tag{15}$$

the solution of the system (7) corresponding to the solution (9) of (S_-) satisfying the initial conditions

$$\eta_1(0) = -1, \ \eta_2(0) = 0. \tag{16}$$

As $F, g \in C^2(R)$, the theorem on the differentiability of solutions of differential equations with respect to parameters ensures the existence of continuous partial derivatives $\frac{\partial \eta_i}{\partial u_0}, \frac{\partial \eta_i}{\partial v_0}$, $i = 1, 2$ in (t, u_0, v_0), $t \in \langle \tau, 0 \rangle$, $(u_0, v_0) \in O_1$.

Consider now the equation

$$\eta_2(t, u_0, v_0) = 0. \tag{17}$$

We know that $\eta_2(T_0, \hat{u}_0, \hat{v}_0) = 0$ and (7) and (17) imply

$$\frac{\eta_2(T_0, \hat{u}_0, \hat{v}_0)}{\partial t} \neq 0.$$

Due to the implicit function theorem a neighbourhood $O_2 \subset O_1$ of $(\hat{u}_0, \hat{v}_0)$ and $\delta > 0$, $0 > T_0 + \delta > T_0 - \delta > \tau$ exist such that for each $(u_0, v_0) \in O_2$ there exists the unique solution $T(u_0, v_0)$ of (17), $T(u_0, v_0) \in (T_0 - \delta, T_0 + \delta)$, i.e. $\eta_2(T(u_0, v_0), u_0, v_0) = 0$. Moreover, T is differentiable. We shall show that for (u_0, v_0) near $(\hat{u}_0, \hat{v}_0)$ the solution $T(u_0, v_0)$ is the largest negative root of (17). As

$$\frac{\partial \eta_2(0, \hat{u}_0, \hat{v}_0)}{\partial t} = 1,$$

there exist a neighbourhood $O_3 \subset O_2$ of $(\hat{u}_0, \hat{v}_0)$ and an $\varepsilon < 0$ sufficiently small such that for $(u_0, v_0) \in O_3$ and $t \in \langle \varepsilon, 0 \rangle$

$$\frac{\partial \eta_2(t, u_0, v_0)}{\partial t} > \frac{1}{2},$$

so that $\eta_2(t, u_0, v_0) < \frac{1}{2} t$. On the interval $\langle T_0 + \delta, \varepsilon \rangle$ there is $\eta_2(t, \hat{u}_0, \hat{v}_0) < 0$. As η_2 depends continuously on (u_0, v_0), we can find a neighbourhood $O_4 \subset O_3$ such that $\eta_2(t, u_0, v_0) < 0$ for $(u_0, v_0) \in O_4$, $t \in \langle T_0 + \delta, \varepsilon \rangle$. Then (17) has the unique solution on $(T_0 - \delta, 0)$ for $(u_0, v_0) \in O_4$, i.e. $O_4 \subset \Omega^-$ and $T^- = T$ on O_4.

Further Λ^- as a composition of differentiable mappings is itself differentiable. Repeating the construction of Λ^-, but considering the least positive root $\hat{T}^-$ of η_2, we obtain the map $\hat{\Lambda}^-$ which has the same properties as Λ^-, namely it is differentiable. Evidently, $\hat{\Lambda}^- \circ \Lambda^-$ is an identical map and we receive for their Jacobians

$$J(\hat{\Lambda}^-)J(\Lambda^-) = 1,$$

i.e. $J(\Lambda^-) \neq 0$. As Λ^- is injective due to the uniquenes of solutions of (S_-) with respect to initial conditions, we obtain that Λ^- is the global diffeomorphism.

Consequence. *All nonempty $\Omega_k^\pm$ are open and $\Lambda_k^\pm$ are global diffeomorphisms.*

Theorem 4. *Let $F, g \in C^2(R)$, $F(0) = g(0) = 0$. Then each nonempty set $V_\pm^{k+1}$, $k \in N_0$, has an at most countable number of components and these are regular Jordan's arcs. The locus of the switching V of (4) is a union of at most denumerably many regular Jordan's arcs.*

Proof. As $V_\pm^1$ are graphs of functions, the assertion is available for $k = 0$. Let $k \geqslant 1$. Then $V_+^1 \cap \Omega_k^-$ is a union of an at most countable system of its components and these are pairwise disjoint regular Jordan's arc U_i, $i = 1, 2, \ldots$, which represent graphs of functions defined on the intervals I_i. Denote $E_i = \{(u, v) \in R^2 \colon u \in I_i\}$. For each $p \in U_i$ there exists a circle K_p such that $p \in K_p \subset \Omega_k^-$. Put $\tilde{D}_i = \bigcup_{p \in U_i} (K_p \cap E_i)$; $\tilde{D}_i$ are open and connected. If I_i is open, define $D_i = \tilde{D}_i$. If I_i contains a boundary point corresponding to $p \in U_i$ (it is possible for $p = O$ or for $p = \Lambda^+(0)$), define $D_i = \tilde{D}_i \cup K_p$. Then D_i are pairwise disjoint regions, $U_i \subset D_i \subset \Omega_k^-$. According to (14)

$$V_\pm^{k+1} = \Lambda_k^-\left(\bigcup_i U_i\right) = \bigcup_i \Lambda_k^-(U_i)$$

and $\Lambda_k^-(U_i)$ are connected Jordan's arcs. As Λ_k^- is the diffeomorphism, they are regular and $\Lambda_k^-(D_i)$ are open. Further

$$V_\pm^{k+1} \cap \Lambda_k^-(D_j) = \Lambda_k^- \bigcup_i U_i) \cap \Lambda_k^-(D_j) = \Lambda_k^-\left(\left(\bigcup_i U_i\right) \cap D_j\right) = \Lambda_k^-(U_j),$$

i.e. $\Lambda_k^-(U_j)$, $j = 1, 2, \ldots$, are open in $V_\pm^{k+1}$, so that they form components of $V_\pm^{k+1}$. The remainder of the assertion is the consequence of theorem 3.

6. Regular synthesis

Consider the controlled system (4) with $w \in M$.

Definition 3. *Let an open set $G \subset R^2$, piecewise smooth sets $P^0 \subset P^1 \subset P^2 = G$ and a function $w \colon G \to \langle -K, K \rangle$ (feedback control) be given. The sets P^i, $i = 0, 1, 2$ and the function w are said to define a regular synthesis of (4) in G if the following conditions are fulfilled:*

i) *$O \in P^0$ and P^0 has no cluster points in G. Each component of $P^i - P^{i-1}$, $i = 1, 2$, (called cell) is an i-dimensional smooth manifold in G. Points of P^0 are* *called null-dimensional cells. The function w is C^1 on each cell σ and can be extended into a C^1-function on some neighbourhood of σ.*

ii) *The set $\mathscr{S}$ of all cells is a union of disjoint sets $\mathscr{S}_1$ — the cells of type I — and $\mathscr{S}_2$ — the cells of type II. All two-dimensional cells are from $\mathscr{S}_1$, all null-dimensional cells are from $\mathscr{S}_2$.*

iii) *There exist Π: $\mathscr{S}_1 \to \mathscr{S}$ and Σ: $\mathscr{S}_2 \to \mathscr{S}_1$ with the following properties:*

If $\sigma \in \mathscr{S}_1$ is i-dimensional, then a unique trajectory of

$$\begin{aligned} u' &= v - F(u) \\ v' &= -g(u) + w(u, v) \end{aligned} \tag{18}$$

goes through each point of σ and this trajectory intersects after a finite time transversally with nonzero velocity the $(i-1)$-dimensional cell $\Pi(\sigma)$.

If $\sigma \in \mathscr{S}_2 - \{O\}$ is i-dimensional, then a unique trajectory of (18) *starts in each point of σ and this trajectory goes through the $(i+1)$-dimensional cell $\Sigma(\sigma)$; moreover, w is C^1 on $\sigma \cup \Sigma(\sigma)$.*

iv) *Every trajectory starting in an arbitrary point of G and continued according to* iii) *reaches O going only through a finite number of cells and satisfies the maxium principle. These trajectories are called distinguished.*

v) *The time in which the distinguished trajectory reaches O is a continuous function of an initial state.*

Theorem 5. *If there exists a regular synthesis of* (4) *in G, then all distinguished trajectories are time-optimal in G.*

For the proof see [2; p. 266; Th. 3.19].

To prove the existence of a regular synthesis for (4) we must first give some lemmas concerning the properties of maximal trajectories and of a locus of switching. We shall introduce the next assumptions:

F, $g \in C^2(R)$, $F(0) = g(0) = 0$, $ug(u) > 0$ for $u \neq 0$, the equations $g(u) = K$, $g(u) = -K$ have the unique roots u^+, u^-, respectively, and g is increasing in u^+ and u^-. (19)

$F, g \in C^2(R)$, $F(0) = g(0) = 0$ and for each $p \in V$ nonlying on the graph of F the tangent to V in this point is not parallel to the axis v. (20)

Denote $A_0 = B_0 = O$ and $B_i = \Lambda^-(A_{i-1})$ for $A_{i-1} \in \Omega^-$ and $A_i = \Lambda^+(B_{i-1})$ for $B_{i-1} \in \Omega^+$, $i \in N$. Let $A_i = (c_i^+, d_i^+)$, $B_i = (c_i^-, d_i^-)$. The points A_{i-1} and A_i (B_{i-1} and B_i) will be called ends of V_+^i (V_-^i) if they exist. Further $\mathring{V}_\pm^i$ means $V_\pm^i$ without its ends.

If the proofs for the cases "plus" and "minus" are the same, we shall consider only one possibility in the sequel.

Lemma 8. *Let F, $g \in C^1$, $F(0) = g(0) = 0$.* Then

i) *All the ends A_i, B_i, $i \in N_0$ (if they exist) lie on the graph of F.*

ii) *Each $\mathring{V}_+^i$ ($\mathring{V}_-^i$) lies below (above) the graph of F.*

iii) $\mathring{V}_{\pm}^{i} \cap \mathring{V}_{\pm}^{i+1} = \emptyset$, $i \in N$.

Proof. The parts i) and ii) are an immediate consequence of lemma 5. As to iii), it is sufficient to prove that $\mathring{V}_{\pm}^{1} \cap \mathring{V}_{\pm}^{2} = 0$. The general case then implies from (14) and the fact that $\Lambda^{\pm}$ are injective.

Let $p \in \mathring{V}_{-}^{1} \cap \mathring{V}_{-}^{2}$, $p = \Lambda^{-}(q)$, $q \in \mathring{V}$. A solution (u, v) of (S_{-}) passing through p in $t = 0$ achieves q in $t = -T^{-}(q)$. There exists t_0, $0 < t_0 < -T^{-}(q)$ such that $(u(t_0), v(t_0)) = O$. Thus $v(t) - F(u(t)) < 0$ for $t > t_0$ near t_0 due to lemma 1, ii). According to lemma 5, iv) this inequality holds for $t \in (t_0, -T^{-}(q)\rangle$ so that $u(t)$ is decreasing and q lies to the left of the axis v, which is impossible.

Lemma 9. *Let* (19) *hold. Then the tangents in the ends of* $V_{\pm}^{k}$, $k \in n$, *are parallel to the axis* v.

Proof. Let $p \in \Omega^{-}$ lying on the graph of F not be a singular point of (S_{-}). Evidently Λ^{-} maps each trajectory of (S_{-}) into itself. According to lemma 5, i) $\Lambda^{-}(p)$ then lies on the graph of F. Tangents of such a trajectory in p and $\Lambda^{-}(p)$ are parallel to the axis v. It implies that a tangent mapping induced by Λ^{-} transforms vectors parallel to v into vectors with the same property. As with respect to (19) no end can be a singular point, the proof is finished.

Lemma 10. *Let* (20) *hold. Then each component of a set* $V_{\pm}^{i} \neq 0$, $i \in N$, *is a graph of a function of the variable* u.

Proof. Theorem 4 gives that each component U is a regular Jordan's arc. Let $\varphi = (\varphi_1, \varphi_2)\colon I \to R^2$ be its parametrization. If there exist $p_k = \varphi(t_k) \in U$, $t_k \in I$, $k = 1, 2$, such that $\varphi_1(t_1) = \varphi_1(t_2)$, $\varphi_2(t_1) \neq \varphi_2(t_2)$, then due to Rolle's theorem $\varphi_1'(t_3) = 0$ for some $t_3 \in (t_1, t_2)$. But then $\varphi_2'(t_3) \neq 0$ and a tangent in $\varphi(t_3)$ is parallel to v, which is a contradiction as $\varphi(t_3)$ does not lie on the graph of F.

Suppose that some $V_{\pm}^{k}$ is connected and (20) holds. Then it represents a graph of a function $h_k^{\pm}$. If the ends of $V_{\pm}^{k}$ exist, then $h_k^{+}\colon \langle c_{k-1}^{+}, c_k^{+}\rangle \to R$ and $h_k^{-}\colon \langle c_k^{-}, c_{k-1}^{-}\rangle \to R$.

Lemma 11. *Let the next assumptions be satisfied*:

i) (19) *holds.*

ii) *there exist the ends* A_{k-1}, B_{k-1} *for some* $k \in N$, $k \geqslant 2$.

iii) $V_{\pm}^{i}$ *are connected,* $i = 1, \dots, k$.

iv) *The set* $\bigcup_{i=1}^{k} (V_{+}^{i} \cup V_{-}^{i})$ *is a graph of a function.*

Let $u_0 \in \operatorname{int}\operatorname{dom} h_j^{-}$ $(u_0 \in \operatorname{int}\operatorname{dom} h_j^{+})$, $2 \leqslant j \leqslant k$. *Denote* m_1, m_2 *functions defined in a neighbourhood of* u_0 *the graphs of which are trajectories of* (S_{+}), (S_{-}) *through* $(u_0, h_j^{-}(u_0))$ $((u_0, h_j^{+}(u_0)))$. *Then*

$$m_1'(u_0) > m_2'(u_0) > h_j^{-\prime}(u_0) \qquad (m_2'(u_0) > m_1'(u_0) > h_j^{+\prime}(u_0))\,.$$

Proof. As B_{k-1} exists, we have $\operatorname{dom} h_i^{-} = \langle c_i^{-}, c_{i-1}^{-}\rangle$, $i = 1, \dots, k-1$. If B_k exists, there is $\operatorname{dom} h_k^{-} = \langle c_k^{-}, c_{k-1}^{-}\rangle$. Otherwise there is $\operatorname{dom} h_k^{-} = (\alpha, c_{k-1}^{-}\rangle$, $\alpha < c_{k-1}^{-}$. The inequality $m_1'(u_0) > m_2'(u_0)$ is evident because of the relation

$h_j^-(u_0) > F(u_0)$. According to lemma 1 $m_2'(u_0) > 0$. Let $i = 2$. As $h_2^{-\prime}(c_1^-) = -\infty$, the assertion holds in a left neighbourhood of c_1^-. If it does not hold on the whole int dom h_2^-, there exists u_0 such that $m_2'(u_0) = h_2^{-\prime}(u_0)$. Let $\Lambda^-(p, q) = (u_0, h_2^-(u_0)) \in V_-^2$, $(p, q) \in V_+^1$, and L is an arc of a trajectory of (S_-) joining these points. The diffeomorphism Λ^- maps each trajectory of (S_-) into itself. As L and V_+^1 are not tangent in (p, q), the same must be true in $(u_0, h_2^-(u_0))$, which is a contradiction. Similarly we proceed by induction for $j = 3, \ldots, k$.

Lemma 12. *Let the assumptions* i)—iv) *of lemma* 11 *be fulfilled and L is an arc of a trajectory of* (S) *joining* $(p, q) \in \mathring{V}_+^i$ *and* $\Lambda^-(p, q) \in \mathring{V}_-^{i+1}$, $1 \leq i \leq k - 1$. *Then*

i) *L intersects* V_+^i *and* V_-^{i+1} *at a unique point and with nonzero angle.*

ii) *L intersects the graph of F at a unique point* $(u_0, F(u_0))$, $c_{i-1}^+ < u_0 < c_i^+$.

Proof. Lemma 5 implies that $\Lambda_-(p, q)$ cannot be an end of V_-^{i+1} and that L intersects the graph of F at the unique point. According to lemma 11, L can intersect $\mathring{V}_+^i$ and $\mathring{V}_-^{i+1}$ in at most one point. As L goes to the right in the region $U(IV)$ with increasing u, it cannot go through A_{i-1}. A more detailed analysis (comparison of trajectories of (S_-) and (S_+) going through A_i) shows that L cannot go through A_i — for the details see [11; p. 47].

Lemma 13. *Let* (19) *and* (20) *hold and* A_k, B_k *exist for some* $k \in N$. *Then all sets* $V_\pm^i$, $i = 1, \ldots, k$, *are nonvoid and connected and a set* $\bigcup_{i=1}^{k} (V_+^i \cup V_-^i)$ *represents a graph of a function defined on* $\langle c_k^-, c_k^+ \rangle$.

Proof. The existence of A_k, B_k implies the existence of all A_i, B_i, $i = 1, \ldots, k - 1$. First we shall prove that $V_\pm^i$, $i = 1, \ldots, k$, are nonvoid, connected and represent graphs of functions. We shall proceed by induction.

The assertion is evident for V_-^1. Let $i > 1$. Denote L an arc of a trajectory of (S_-) joining A_{i-1} and $B_i = \Lambda^-(A_{i-1})$. Let $C \in \mathring{V}_+^{i-1}$. Consider a trajectory (u, v) of (S_-), $(u(0), v(0)) = C$. Denote K the part of this trajectory for $t \in (\omega_-, 0\rangle$. Then K intersects the graph of F at a point $C_1 = (u(t_1), v(t_1))$, $t_1 < 0$, $c_{i-2}^+ < u(t_1) < c_{i-1}^+$. Further K cannot intersect V_-^1 as it is an arc of a trajectory of (S_-), i.e. K cannot converge to the singular point $(u^-, F(u^-))$, so that according to lemma 3, there exists $t_2 < t_1$ such that $u(t_2) = u^-$, $v(t_2) > F(u(t_2))$. As K cannot intersect L, the same lemma ensures the existence of $t_3 < t_2$ in which $v(t_3) = F(u(t_3))$. Using lemma 5, iv) we obtain that $C \in \Omega^-$, i.e. $\mathring{V}_+^{i-1} \subset \Omega^-$. Therefore V_-^i is connected and due to lemma 10 it represents a graph of a function. The rest of the proof is now evident.

Lemma 14. *Let* (19) *hold,* A_1, B_1 *exist and*

$$\liminf_{u \to +\infty} g(u) > K, \qquad \limsup_{u \to -\infty} g(u) < -K, \tag{21}$$

$$\liminf_{u \to -\infty} F(u) > -\infty, \qquad \liminf_{u \to +\infty} F(u) < +\infty. \tag{22}$$

Then there exist the ends A_i, B_i for each $i \in N$.
If moreover F is nondecreasing, the assumption of the existence of A_1, B_1 can be omitted.

Proof. The assertion in an immediate consequence of lemmas 2 and 3 and the definition of the ends.

Now we are able to construct the regular synthesis of (4). In the remainder of this paragraph we shall suppose that (19) and (20) holds.

Definition 4. *The locus of switching V will be said to have a property (G) if V is connected, represents a graph of a function and*
either A_k, B_k exist for each $k \in N$
or there exist $k_1, k_2 \in N$ such that $V_-^i = \emptyset$ for $i > k_1$, $V_+^i = \emptyset$ for $i > k_2$ and B_i exist for $i < k_1$, A_i exist for $i < k_2$.
In the second case evidently $|k_1 - k_2| \leqslant 1$.

Further we shall introduce the next notation:
Let $C \in R^2$. Then $M^+(C)$ is an arc of a trajectory (u, v) of (S_+), $(u(0), v(0)) = C$, $t \in (T^+(C), 0)$ if $C \in \Omega^+$, and $t \in (\omega_-, 0)$ if $C \notin \Omega^+$. $M^-(C)$ is defined similarly.

If A_k (B_k), $k \in N_0$, exists, denote $L_-^{k+1} = M^-(A_k)$ $(L_+^{k+1} = M^+(B_k))$. Suppose that V has a property (G). If A_{k+1} (B_{k+1}), $k \geqslant 1$, exists, denote H_+^k (H_-^k) a region bounded by V_-^k, V_+^{k+1}, L_+^k and L_+^{k+1} $(V_+^k$, V_-^{k+1}, L^k and $L_-^{k+1})$. If for some $k \geqslant 1$ the end A_{k+1} (B_{k+1}) does not exist, but $V_-^k \neq \emptyset$ $(V_+^k \neq \emptyset)$, put $H_+^k = \bigcup_{C \in V_-^k} M^+(C)$ $\left(H_-^k = \bigcup_{C \in V_+^k} M^-(C)\right)$.

Due to the continuous dependence of solutions of $(S_\pm)$ on initial values and a special form of trajectories — see lemma 1 — $H_\pm^k$ are also regions.

Denote
P^0 the set of all ends A_k, B_k, $k \in N_0$, that exist,
$\tilde{P}^1$ the set of all points of $\mathring{V}_\pm^k$ and $L_\pm^k$, $k \in N$, that exist,
$\tilde{P}^2$ the set of all points of $H_\pm^k$, $k \in N$, that exist.
Put $P^1 = \tilde{P}^1 \cup P^0$, $P^2 = \tilde{P}^2 \cup P^1$, $G = P^2$; evidently G is a region. Define a function w: $G \to \langle -K, K \rangle$ in the following way:

$$w(u, v) = \begin{cases} K & \text{for } (u, v) \in H_+^k \cup V_+^{k+1}, \quad k \geqslant 1 \\ -K & \text{for } (u, v) \in H_-^k \cup V_-^{k+1}, \quad k \geqslant 1 \\ K & \text{for } (u, v) \in L_+^k \cup \{A_k\}, \quad k \geqslant 1 \\ -K & \text{for } (u, v) \in L_-^k \cup \{B_k\}, \quad k \geqslant 1 \\ 0 & \text{for } (u, v) = (0, 0). \end{cases} \tag{23}$$

Theorem 6. *Let* (19) *and* (20) *hold and a locus of switching V of* (4) *has a* *property* (G). *Then the sets* P^i, $i = 0, 1, 2$, *and the function* w *given by* (23) *define a regular synthesis of* (4) *in* G.

Proof. Choose A_k, B_k, $k \geqslant 0$, as nul-dimensional cells of type II, $L_{\pm}^k$, $k \geqslant 1$, as one-dimensional cells of type I, $V_{\pm}^k$, $k \geqslant 2$, as one-dimensional cells of type II and $H_{\pm}^k$, $k \geqslant 1$, as two-dimensional cells of type I. Further put

$$\Pi(H_{\pm}^k) = V_{\mp}^k,\ k \geqslant 1,\ \Pi(L_{+}^k) = B_{k-1},\ \Pi(L_{-}^k) = A_{k-1},\ k \geqslant 1,\ \Pi(L_{\pm}^1) = 0 \text{ and}$$

$$\Sigma(V_{\pm}^k) = H_{\pm}^{k-1},\ k \geqslant 2,\ \Sigma(A_k) = L_{+}^k,\ \Sigma(B_k) = L_{-}^k,\ k \geqslant 1\,.$$

Lemma 11 and the construction of cells imply that the condition iii) of the definition of the regular synthesis is fulfilled. Distinguished trajectories consist in general of an arc OC, $C \in V_{+}^1 \cup \{A_1\}$, which is a part of V_{+}^1, further of arcs $M^-(C)$, $M^+(\Lambda_1^-(C))$, $M^-(\Lambda_2^-(C))$, ..., $M^{\pm}(\Lambda_k^-(C))$, which join points $\Lambda_1^-(C)$, ..., $\Lambda_{k+1}^-(C)$, and of an arc $\Lambda_{k+1}^-(C)\,D$, $D \in M^{\mp}(\Lambda_{k+1}^-(C))$, which is a part of $M^{\mp}(\Lambda_{k+1}^-(C))$; here D is an initial state. The definition of $\Lambda^{\pm}$ and theorem 3 and its proof imply that all distinguished trajectories are maximal. Likewise all the other conditions of definition 3 are evidently fulfilled except v). Thus to finish the proof it suffices to show that the time of transfer from an arbitrary state $D \in G$ to O along distinguished trajectories is a continuous function of D.

Let $W_{\pm} = \{(u, v) \in G: w(u, v) = \pm K\}$. Evidently W_+ (W_-) is a set of all points in G which lie on V on the right (left) of O and under (above) V. Suppose that $D \in W_- \setminus V_-^1$. Consider a trajectory of (S_-) going through D in $t = 0$ which in $t_1 > 0$ intersects for the first time the locus of switching V at a point $C = (u_1, v_1) \in V$, $u_1 > 0$. We shall show that C and t_1 depend continuously on D. We shall use a system $(\hat{S}_-)$

$$\begin{aligned} u' &= F(u) - v \\ v' &= g(u) + K, \end{aligned}$$

which has the same trajectories as (S_-), but oppositely oriented. Let $(\bar{u}, \bar{v})$ be a solution of $(\hat{S}_-)$, $(\bar{u}(0), \bar{v}(0)) = C$. Then $(\bar{u}(t_1), \bar{v}(t_1)) = D$. Due to the continuous dependence on initial values there exists a neighbourhood O_1 of C such that for $(u_2, v_2) \in O_1$ a solution $(u(t), v(t))$ of $(\hat{S}_-)$, $(u(0), v(0)) = (u_2, v_2)$, exists at least on $\langle 0, t_1 + \delta)$, where $\delta > 0$ is sufficiently small. Let h be a function the graph of which is V; then $h(u_1) = v_1$. Choose $\varepsilon > 0$ such that $(u, h(u)) \in O_1$ for $u \in (u_1 - \varepsilon, u_1 + \varepsilon)$. Define a mapping γ: $(u_1 - \varepsilon, u_1 + \varepsilon) \times (-\delta, \delta) \to R^2$ in the following way:

$$\gamma(u_2, \tau) = (u(t_1 + \tau, u_2, h(u_2)), v(t_1 + \tau, u_2, h(u_2)))\,.$$

Then γ is continuous and lemma 12 implies that it is an injection. Moreover, $\gamma(u_1, 0) = D$. Hence γ is a homeomorphism and $O_2 = \gamma((u_1 - \varepsilon, u_1 + \varepsilon) \times (-\delta, \delta)) \cap W_-$ is a neighbourhood of D in W_-. That is why τ and u_2 depend continuously on $\hat{D} \in O_2$. As $t_1 + \tau(\hat{D})$ is the time of transfer from $\hat{D} \in O_2$ to $\hat{C} \in V$ and $\hat{C} = (u_2(\hat{D}),\ h(u_2(\hat{D})))$, a function $\vartheta^-(\hat{D}) = t_1 + \tau(\hat{D})$ and a map $\mu^-(\hat{D}) = \hat{C}$, $\hat{D} \in O_2$, are continuous. Similarly it can be shown that ϑ^- and μ^- are continuous on W_- even for $D \in \mathring{V}_-^1 \cup \{B_1\}$. The same is true for ϑ^+ and μ^+ defined analogously on W_+.

Let $D \in G$ and, e.g., $D \in H_-^k$, $k \in N$. Then a distinguished trajectory transfers a state D to $\mu^-(D) = C_1 \in \mathring{V}_+^k$ in a time $\vartheta^-(D)$, a state C_1 to $\mu^+(C_1) = C_2 \in \mathring{V}_-^{k-1}$ in a time $\vartheta^+(C_1)$ etc. up to a state C_{k-1} $(C_0 = D)$ to $\mu^{\pm}(C_{k-1}) = C_k \in V_{\mp}^1$ in a time $\vartheta^{\pm}(C_{k-1})$ and a state C_k to O in a time $\vartheta^{\mp}(C_k)$. As $\mu^{\pm}$ are continuous and $C_1, \ldots, C_k$ depend continuously on D, the whole time of transfer $\vartheta^-(D) + \vartheta^+(C_1) + \ldots + \vartheta^{\pm}(C_{k-1}) + \vartheta^{\mp}(C_k)$ is a continuous function of D. It is easy to verify that the same is true even for $D \in V_-^k$ or L_-^k, $k \in N$.

Thus we have according to theorem 5 the next

Consequence. *The distinguished trajectories described in the proof of theorem* 6 *are time-optimal* (*with respect to controls from* M).

Theorem 7. *Let* (19), (20), (21) *and* (22) *hold and F be nondecreasing. Then there exists a regular synthesis of* (4) *in* R^2 *and the distinguished trajectories are time-optimal with respect to controls from* $L^{\infty}_{\mathrm{loc}}(R)$.

Proof. Lemmas 13 and 14 imply that V has a property (G) and we can use theorem 6. Further it can be proved that if F is nondecreasing, then $W(L^{\infty}_{\mathrm{loc}}(R)) = W_{\mathrm{opt}}(L^{\infty}_{\mathrm{loc}}(R)) = R^2$ — see [11; p. 25; Th. 3.4.] or [12; Th. 9]. Pontryagin's maximum principle and theorem 2 imply that time-optimal trajectories are maximal and the corresponding controls are from M. Theorem 3 and its proof show that such trajectories are just exactly distinguished trajectories from theorem 6, that is why $G = R^2$.

Comment. Boltyanskii in [2] studied a regular synthesis of an equation

$$x'' = h(x, x', w), \qquad |w| \le 1 . \tag{24}$$

He defined a concept of an oscillating system and proved that such a system then satisfies (20). An oscillating system must fulfil:

i) $h \in C^2(R)$, $\dfrac{\partial h(x, y, w)}{\partial w} > 0$, $h(0, 0, 1) > 0 > h(0, 0, -1)$

for arbitrary x, y and $|w| \leqslant 1$,

ii) $\dfrac{\partial h}{\partial x} < -\dfrac{1}{4}\left(\dfrac{\partial h}{\partial y}\right)^2$,

iii) $\left(\dfrac{\partial^2 h}{\partial x \partial y}\right)^2 \leqslant \dfrac{\partial^2 h}{\partial x^2}\dfrac{\partial^2 h}{\partial y^2}$,

iv) $w\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right)\leqslant 0$,

where ii)—(iv) hold for arbitrary x, y and $w = \pm 1$.

Equation (1) is a special case of (24). It is not difficult to prove that each function $f(x, y)$ from (1) fulfilling i)—iv) is linear, i.e. $f(x, y) = ax + by$, $b^2 - 4a < 0$ — see [11; p. 55; Com. 6.19]. Thus Boltyanskii's results do not cover nonlinear equations of type (1).

Further we shall show that the equation (2) which can be considered as a special case of (4) cannot be nonoscillating in Boltyanskii's sense — see [2], p. 278 — if the assumptions of lemma 14 are fulfilled, i.e. especially if $\int_0^x f(t)\,\mathrm{d}t$ is bounded. In this case a maximal trajectory can have arbitrarily many switchings while each maximal trajectory of a nonoscillating system can have at most one switching — see [2; p. 282; L. 3.20].

We shall give an example of a class of equations to which theorem 7 can be applied and such that the assumption (20) is replaced by a less "noneffective" assumption.

Consider the equation

$$u'' + g(u) = w(t), \qquad |w| \leqslant K$$

which is equivalent to

$$\begin{aligned} u' &= v \\ v' &= -g(u) + w(t), \end{aligned} \tag{25}$$

i.e. to the system (4) with $F \equiv 0$. Suppose that

$$\begin{gathered} g \textit{ fulfils } (19) \textit{ and moreover the functions } g(u), \\ g_-(u) = g(u^- + u) - K,\ g_-(u) = g(u^- + u) + K \textit{ are odd.} \end{gathered} \tag{26}$$

Evidently, every function g fulfilling (26) can be obtained in the following way: Let $\hat{g} \in C^2\langle 0, u^-\rangle$, $u^- > 0$, $\hat{g}(0) = 0$, $\hat{g}(u^-) = K$, $\hat{g}''(0) = \hat{g}''(u^+) = 0$, $0 < \hat{g}(u) < K$ for $u \in (0, u^-)$ be arbitrary. We put $g(u) = \hat{g}(u)$ on $\langle 0, u^+\rangle$ and enlarge $g(u)$ on R in a unique way using the fact that g, g_+ and g_- must be odd.

Lemma 15. *Let* (26) *hold and let for an arbitrary nonconstant solution* $u(t)$ *of*

$$u'' + g(u) = K \tag{27}$$

all the solutions of

$$\eta'' + g'[u(t)]\,\eta = 0 \tag{28}$$

be bounded. Then the locus of switching V *of* (25) *has the property* (G) *and* (20) *holds.*

Proof. We shall show that Λ^-, Λ^- are rotations by an angle π with centres $(u^+, 0)$, $(u^-, 0)$, respectively, and $\Omega^+ = \Omega^- = R^2$, which implies the assertion. For the sake of symmetry we shall only consider the case "plus".

As $\int_0^x g_+(s)\,\mathrm{d}s \to +\infty$ for $x \to \pm\infty$, all the nonstationary trajectories of (S_+) are closed — see [19], p. 95 — and symmetrical with respect to the lines $v = 0$, $u = u^+$. If (η_1, η_2) is a solution of the adjoint system (7) corresponding to $u(t)$, then

$$\eta_2'' + g'[u(t)]\,\eta_2 = 0\,.$$

This equation has a solution $u'(t)$ (the adjoint equation (28) coincides in this case with the equation in variations of (25)). Let the period of $u(t)$ be $2d > 0$. Then due to the symmetry $u(t)$ is even, half-periodic with the period d, i.e. $u(t + d) = -u(t)$, $t \in R$. Evidently the zeros of $u(t)$ are equidistant with the distance d. Denote σ a zero lying in $\langle 0, d \rangle$. In [21] it is shown that in this situation all the solutions of (28) are bounded iff

$$\int_0^d \left(\frac{1}{u^2(t)} - \frac{\pi^2}{d^2 (u'(\sigma))^2 \sin^2\left(\dfrac{\pi}{d}(t - \sigma)\right)} \right) \mathrm{d}t = 0\,.$$

But this condition is equivalent to the fact that all the nontrivial solutions of (28) have equidistant zeros with the distance d — see [20]. Now, if $C = (u(t), v(t))$, $t \in R$ is a cycle of (S_+), then $T^+(u_0, v_0) = -d$ for each $(u_0, v_0) \in C$ and the symmetry of C implies that Λ^+ is the above mentioned rotation.

If the assumptions of lemma 15 are satisfied, then theorem 7 can be applied to (25) but neither the results of Boltyanskii and, if $g'(u) > 0$ does not hold for all $u \in R$, nor those of Lee and Markus — see [13; p. 471; Th. 6] — are applicable (except the linear case).

In [13], p. 474 the following assumption for the equation (1) is proposed to guarantee that V is a graph of a function: All the solutions of $(S_\pm)$ are periodic with the same period $2d > 0$. This assumption is not quite correct because, as the proof of lemma 15 shows, it does not ensure the distance of the neighbour zeros of solutions of (28) or (7) to be d, which is needed in the proof in [13]. Moreover, it is questionable whether any nonlinear equation (1) with this property exists. For example, using Theorem 3.1.2., p. 97 in [19] we can easily show that the only equation of the type (25) fulfilling this assumption and (26) is the linear one.

In [22] a numerical analysis of the equation (25) with

$$g(u) = u + \varepsilon \sin nu\,, \qquad \varepsilon \in R,\ n \in N_0,\ K = \pi$$

can be found.

REFERENCES

[1] BARBANTI, L.: Liénard equations and control. Functional differential equations and bifurcations (Proceedings, Sao Carlos, Brasil 1979), Springer-Verlag, 1—22.

[2] BOLTYANSKII, V. G.: Mathematical Methods of Optimal Control (Russian). Nauka, Moscow 1969 (Second edition).

[3] BOLTYANSKII, V. G.: Sufficient conditions of optimality and the justification of the method of dynamic programming. (Russian). Izv. Akad. Nauk SSSR, Seria Mat., 28, 1964, 481—514.

[4] BOLTYANSKII, V. G.: Sufficient conditions of optimality (Russian). DAN SSSR 140, No. 5, 1961, 994—997.

[5] BRUNOVSKÝ, P.: Every normal linear system has a regular time-optimal synthesis. Math. Slovaca 28, 1978, 81—100.

[6] BRUNOVSKÝ, P.: Existence of regular synthesis for general control problems. J. Diff. Eq. 38, 1980, 317—343.

[7] BRUNOVSKÝ, P.: Regular synthesis for the linear-quadratic optimal control problem with linear control constraints. J. Diff. Eq. 38, 1980, 344—360.

[8] CONTI, R.: Equazione di Van der Pol e controllo in tempo minimo. Rapporti del Ist. Mat. "U. Dini", 13 1976 77.

[9] DAVIS, M. J.: A property of the switching curve for certain systems. Int. J. Control 12, 1970, 457—463.

[10] JAMES, E. M.: Time optimal control and the Van der Pol oscillator. J. Inst. Math. Applics. 13, 1974, 67—81.

[11] KUBEN, J.: Time-optimal control of two-dimensional systems. CSc. — thesis, UJEP Brno 1985, (Czech).

[12] KUBEN, J.: Global controllability of two-dimensional systems and time-optimal control (Czech). Sbornik VAAZ Brno, řada B, 2, 1986.

[13] LEE, E. B., MARKUS, L.: Foundations of Optimal Control Theory. J. Wiley and Sons 1967. In Russian Nauka, Moscow 1972.

[14] LEE, E. B., MARKUS, L.: Optimal Control for Nonlinear Processes. Archive for Rational Mech. and Anal. 8, 1961, 36—58.

[15] LEE, E. B., MARKUS, L.: On the existence of optimal controls. Trans. ASME, series D, J. of Basic Engin. 84, 1962, No 1, 13—23.

[16] LEE, E. B., MARKUS, L.: Synthesis of optimal control for nonlinear processes with one degree of freedom. Proceedings of Inter. Sympos. on Nonlin. Vibrations, t. III, Izdat. AN USSR, Kijev 1963, 200—218.

[17] LEE, E. B., MARKUS, L.: On necessary and sufficient conditions of time-optimality for nonlinear second order systems (Russian). Proceedings of 2nd Congress IFAC, Basel 1963, Nauka, Moscow 1965, 155—167.

[18] VILLARI, G.: Ciclo limite di Liénard e controllabilità. Bol. Univ. Math. Ital. (5) 17-A, 1980, 406—413.

[19] REISSIG, R., SANSONE, G., CONTI, R.: Qualitative Theorie nichtlinearer Differentialgleichungen. Edizioni Cremonese, Roma 1963. In Russian Nauka, Moscow 1974.

[20] NEUMAN, F., Sur les équations différentielles linéaires oscillatoires du deuxième ordre avec la dispersion fondamentale $\varphi(t) = t - \pi$. Buletinul Institutuliu Politehnic Din Iasi, X(XIV), Fasc. 1—2, 1964, 37—42.

[21] NEUMAN, F.: Criterion of periodicity of solutions of a certain differential equation with a periodic coefficient. Annali di Mat. pura et app., (IV), vol. LXXV, 1967, 385—396.

[22] KUBEN, J.: Establishing of Locus of Switching of Optimal Feedback Control (Czech). Sbornik VAAZ Brno, řada B, 4, 1988.

Received May 21, 1986

Marie Kudeříkové 13
636 00 Brno