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VARIETIES OF DIRECTED MULTILATTICES

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ABSTRACT. In the paper there is continued the study of the varieties of directed multilattices. It is proved, e.g., that the varieties of modular directed multilattices form a proper class.

In [3] infinitely many varieties of distributive directed multilattices covering the variety $\mathscr{D}$ of all distributive lattices in the lattice of varieties of directed multilattices have been described. In this paper there are investigated the varieties $\mathscr{V}_{\alpha,\beta}$ generated by the modular multilattices $M_{\alpha,\beta}$ shown in Figure 1 for different couples of cardinal numbers α, β, where $\alpha = \operatorname{card} A$, $\beta = \operatorname{card} B$. It is proved that for different couples α, β of positive integers, which are greater than or equal to two, the varieties $\mathscr{V}_{\alpha,\beta}$ are different and each of them covers $\mathscr{D}$ (Theorems 1.3 and 1.4). Further, the varieties $\mathscr{V}_{2,\beta}$ for infinite cardinal numbers β are studied. It is shown that for different infinite cardinal numbers β the varieties $\mathscr{V}_{2,\beta}$ are different, which implies that the varieties of modular directed multilattices form a proper class (Corollary 2.7, Theorem 2.8). In contrast with the case of a finite β, there exists no variety $\mathscr{V}$ covering $\mathscr{D}$ satisfying $\mathscr{V} \subseteq \mathscr{V}_{2,\beta}$, for any infinite cardinal number β (Theorem 2.11). Moreover, for every infinite cardinal number β there exists an infinite increasing sequence of cardinal numbers $\beta = \beta_0 < \beta_1 < \beta_2 < \dots$ such that $\mathscr{V}_{2,\beta} \supset \mathscr{V}_{2,\beta_1} \supset \mathscr{V}_{2,\beta_2} \supset \dots \supset \mathscr{D}$ (Theorem 2.10). In the last part of the paper there is described a variety containing only infinite multilattices, with the exception of those that are lattices, and covering $\mathscr{D}$.

We shall use the denotation introduced in [3]. By a multilattice always a directed multilattice is meant.

1. Varieties $\mathscr{V}_{\alpha,\beta}$

Let α, β be arbitrary cardinal numbers different from 0. Denote by $M_{\alpha,\beta}$ the multilattice shown in Figure 1, i.e. $M_{\alpha,\beta} = \{0, 1\} \cup A \cup B$, the order is defined by $0 < a < b < 1$ for every $a \in A$, $b \in B$, and $\alpha = \operatorname{card} A$, $\beta = \operatorname{card} B$.

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Evidently, assuming that $\alpha, \beta \geqslant 2$, the multilattice $M_{\alpha,\beta}$ is not a lattice, it is simple (i.e. card Con $M_{\alpha,\beta} = 2$) and all its proper subalgebras are lattices (even chains). If $\alpha = \beta = 2$, then $M_{\alpha,\beta}$ is distributive; if $\alpha > 2$ or $\beta > 2$, then $M_{\alpha,\beta}$ is modular, but not distributive.

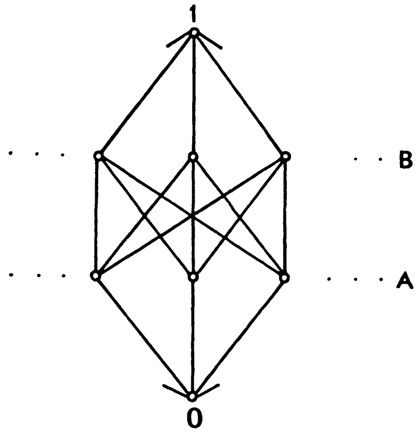

Fig. 1

Denote by $\mathscr{V}_{\alpha,\beta}$ the variety generated by $M_{\alpha,\beta}$. We shall investigate $\mathscr{V}_{\alpha,\beta}$ for some couples of the cardinal numbers α, β.

First we will show that if $\alpha, \beta \geqslant 2$, then $\mathscr{V}_{\alpha,\beta}$ contains no variety of lattices but the variety $\mathscr{D}$ of all distributive lattices and the variety of all one-element lattices.

Let M_3 denote the five-element modular non-distributive lattice.

1.1. Lemma. *If $\alpha, \beta \geqslant 2$, then $M_3 \notin \mathscr{V}_{\alpha,\beta}$.*

Proof. Suppose that $M_3 \in \mathscr{V}_{\alpha,\beta} = HSP\{M_{\alpha,\beta}\}$ (cf. 6.1 in [3]) for some α, $\beta \geqslant 2$. Since throughout the proof α, β will be fixed, let us denote $M_{\alpha,\beta} = M$. From $M_3 \in HSP\{M\}$ it follows that there exists a homomorphism φ of a subalgebra S of a direct product $\Pi(M_i \mid i \in I)$, where $M_i = M$ for every $i \in I$, onto M_3. Let x, y, z be elements of S such that $\varphi(x)$, $\varphi(y)$, $\varphi(z)$ are mutually incomparable. Let $u, v \in \{x, y, z\}$, $u \neq v$. We are going to describe a construction for finding $u_1, v_1 \in S$ such that $\varphi(u_1) = \varphi(u)$, $\varphi(v_1) = \varphi(v)$ and $u_1(i)$, $v_1(i)$ are comparable elements of M_i. Fix arbitrary different elements $b, b' \in B$. Let us take arbitrary $w \in u \vee v$ and define $r, s \in \Pi(M_i \mid i \in I)$ as follows:

$$r(i) = b,\ s(i) = b' \qquad \text{if } u(i), v(i) \in A,\ u(i) \neq v(i);$$
$$r(i) = s(i) = w(i) \qquad \text{in the opposite case.}$$

Evidently $r, s \in u \vee v$. Further, choose $r' \in (r \wedge s)_u$, $s' \in (r \wedge s)_v$, $t \in r' \wedge s'$,

$u' \in u \wedge t$, $v' \in v \wedge t$ (see Figure 2). It is easy to see that u', $v' \in S$, $\varphi(u') = \varphi(u)$, $\varphi(v') = \varphi(v)$ and that

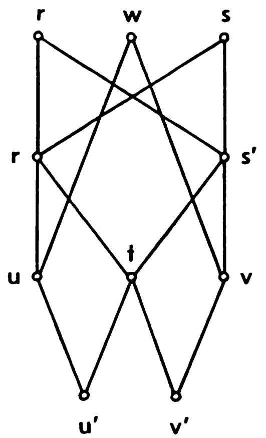

Fig. 2

$$u'(i) = v'(i) = 0 \qquad \text{if } u(i),\ v(i) \in A,\ u(i) \neq v(i);$$
$$u'(i) = u(i),\ v'(i) = v(i) \qquad \text{in the opposite case.}$$

Choosing two arbitrary different elements of A and using the dual procedure to the elements u', v', we can find u_1, $v_1 \in S$ such that $\varphi(u_1) = \varphi(u')$, $\varphi(v_1) = \varphi(v')$ and

$$u_1(i) = v_1(i) = 1 \qquad \text{if } u'(i),\ v'(i) \in B,\ u'(i) \neq v'(i);$$
$$u_1(i) = u'(i),\ v_1(i) = v'(i) \qquad \text{in the opposite case.}$$

For these elements u_1, $v_1 \in S$ we have $\varphi(u_1) = \varphi(u)$, $\varphi(v_1) = \varphi(v)$ and

$$u_1(i) = v_1(i) = 0 \qquad \text{if } u(i),\ v(i) \in A,\ u(i) \neq v(i);$$
$$u_1(i) = v_1(i) = 1 \qquad \text{if } u(i),\ v(i) \in B,\ u(i) \neq v(i);$$
$$u_1(i) = u(i),\ v_1(i) = v(i) \text{ otherwise.}$$

Hence, if $u(i)$, $v(i)$ are comparable for some $i \in I$, then $u_1(i) = u(i)$, $v_1(i) = v(i)$. If $u(i)$, $v(i)$ are incomparable, then either $u_1(i) = v_1(i) = 0$ or $u_1(i) = v_1(i) = 1$. For every $i \in I$ the elements $u_1(i)$, $v_1(i)$ are already comparable.

Now let us use the above construction to find u_1, v_1 to u, v first for the couple x, y. We obtain x_1, y_1. Then use the construction for the couple x_1, z; denote by $\bar{x}$, z_1 the obtained elements. Finally, applying the construction for the couple

$y_1 z_1$, we obtain $\bar{y}$, $\bar{z}$. It is easy to see that $\bar{x}, \bar{y}, \bar{z} \in S$, $\varphi(\bar{x}) = \varphi(x)$, $\varphi(\bar{y}) = \varphi(y)$, $\varphi(\bar{z}) = \varphi(z)$ and for every $i \in I$ the elements $\bar{x}(i)$, $\bar{y}(i)$, $\bar{z}(i)$ form a chain, which will be denoted by R_i. The subalgebra T of the multilattice S generated by $\{\bar{x}, \bar{y}, \bar{z}\}$ is a subalgebra of the product $\Pi(R_i | i \in I)$, which is a distributive lattice, hence T is also a distributive lattice. Then $\varphi(T) = M_3$ is a distributive lattice too, a contradiction.

1.2. Theorem. *The only varieties of lattices that are contained in $\mathscr{V}_{\alpha,\beta}$ for some $\alpha, \beta \geqslant 2$ are the variety $\mathscr{D}$ of all distributive lattices and the variety of all one-element lattices.*

Proof. If $\mathscr{V}_{\alpha,\beta}$ for some $\alpha, \beta \geqslant 2$ contains a variety of lattices different from $\mathscr{D}$ and from the least variety, then it contains also either the variety $HSP\{M_3\}$ or the variety $HSP\{N_5\}$ (N_5 is the five-element non-modular lattice) (cf., e.g., [2]). By the previous Lemma the first possibility cannot occur. As the variety $\mathscr{V}_{\alpha,\beta}$ contains only modular multilattices (see 5.4 of [3]), the second possibility is also excluded.

1.3 Theorem. *Let α, β be arbitrary finite cardinal numbers greated than 1. Then the variety $\mathscr{V}_{\alpha,\beta}$ covers the variety $\mathscr{D}$ in the lattice of all varieties of multilattices.*

Proof. Evidently $\mathscr{D} \subset \mathscr{V}_{\alpha,\beta}$. Let us suppose that $\mathscr{V}_1$ is a variety of multilattices satisfying $\mathscr{D} \subset \mathscr{V}_1 \subseteq \mathscr{V}_{\alpha,\beta}$. We will show that $\mathscr{V}_{\alpha,\beta} \subseteq \mathscr{V}_1$. By Theorem 1.2 $\mathscr{V}_1$ contains a multilattice C that is not a lattice. By a method analogous to that in the proof of 6.14 in [3] we can verify that $M_{\alpha,\beta} \in HSP\{C\}$. Thus $\mathscr{V}_{\alpha,\beta} \subseteq \mathscr{V}_1$.

1.4. Theorem. *For different couples (α, β) of finite cardinal numbers greater than 1 the varieties $\mathscr{V}_{\alpha,\beta}$ are different.*

The assertion is an immediate consequence of 6.12, [3].

2. The relations between $\mathscr{V}_{2,\beta}$ for various β

In this section we shall consider varieties $\mathscr{V}_{2,\beta}$ for various infinite cardinal numbers β. The symbol $\Pi_{\mathscr{F}}(M_i | i \in I)$ will denote the filter product of $(M_i | i \in I)$ by a filter $\mathscr{F}$ on I (see [3]). Let $A = \{a, a'\}$ (see the definition of $M_{\alpha,\beta}$).

2.1. Lemma. *Let $C \in \mathscr{V}_{2,\beta}$ and let C be generated by a four-element subset $\{r, s, t, u\}$, where $r, s \in t \wedge u$, $t, u \in r \vee s$. Then there exists a non-empty set I and a filter $\mathscr{F}$ on I different from the system of all subsets of I such that C is isomorphic to $\Pi_{\mathscr{F}}(M_i | i \in I)$ and $M_i = M_{2,\beta}$ for every $i \in I$.*

Proof. If $C \in \mathscr{V}_{2,\beta} = HSP\{M_{2,\beta}\}$, then there exists a homomorphism φ of a subalgebra A of a direct product $\Pi(M_i | i \in I_1)$ with $M_i = M_{2,\beta}$ for every $i \in I_1$, onto C. In view of 6.5 of [3] there exist elements $r', s', t', u' \in A$ such that $r', s' \in t' \wedge u'$, $t', u' \in r' \vee s'$ and $\varphi(r') = r$, $\varphi(s') = s$, $\varphi(t') = t$, $\varphi(u') = u$. We can suppose that A is generated by $\{r', s', t', u'\}$. Now just as in the proof of 6.6 in

[3] we can show that A is isomorphic to $\Pi(M_i|i\in I)$ with $I=\{i\in I_1: r'(i), s'(i), t'(i), u'(i)$ are mutually different$\}$. Using 4.5 of [3] we obtain that C is isomorphic to $\Pi(M_i|i\in I)/\Theta$ for a congruence relation Θ on $\Pi(M_i|i\in I)$. By 6.10 of [3], $\Theta=\Theta(\mathscr{F})$ for a filter $\mathscr{F}$ on I. Thus C is isomorphic to $\Pi_{\mathscr{F}}(M_i|i\in I)$.

2.2. Lemma. *Under the same assumptions and denotations as in the preceding Lemma C has only trivial congruence relations if and only if the filter $\mathscr{F}$ is an ultrafilter.*

Proof. First consider an arbitrary filter $\mathscr{F}_1\supseteq\mathscr{F}$. Then $\Theta(\mathscr{F}_1)\supseteq\Theta(\mathscr{F})$ and the congruence relation $\Theta(\mathscr{F}_1)/\Theta(\mathscr{F})$ on $\Pi(M_i|i\in I)/\Theta(\mathscr{F})$ defined by

$$[f]\,\Theta(\mathscr{F})\ \Theta(\mathscr{F}_1)/\Theta(\mathscr{F})[g]\,\Theta(\mathscr{F})\ \Leftrightarrow\ f\Theta(\mathscr{F}_1)g$$

(see 4.6 of [3]) is the least if and only if $\Theta(\mathscr{F}_1)=\Theta(\mathscr{F})$, which is equivalent to $\mathscr{F}_1\supseteq\mathscr{F}$, and the greatest in the case that $\Theta(\mathscr{F}_1)$ is the greatest, i.e. when $\mathscr{F}_1$ contains all subsets of the set I.

Now do not let $\mathscr{F}$ be an ultrafilter. Then there exists an ultrafilter $\mathscr{U}\supset\mathscr{F}$. The congruence relation $\Theta(\mathscr{U})/\Theta(\mathscr{F})$ on $\Pi_{\mathscr{F}}(M_i|i\in I)$ is neither the least, nor the greatest, hence also C has a non-trivial congruence relation.

Let there exist a non-trivial congruence relation on C. Then there exists a non-trivial congruence relation on $\Pi_{\mathscr{F}}(M_i|i\in I)=\Pi(M_i|i\in I)/\Theta(\mathscr{F})$, too. Take Φ to be such a one. The multilattice $\Pi_{\mathscr{F}}(M_i|i\in I)/\Phi$ is a homomorphic image of $\Pi(M_i|i\in I)$, so there exists a filter $\mathscr{F}_1$ on I such that $\Pi_{\mathscr{F}}(M_i|i\in I)\ \Phi\cong$ $\cong\Pi(M_i|i\in I)/\Theta(\mathscr{F}_1)$ (cf. 6.10 of [3]). Evidently, $\Theta(\mathscr{F}_1)\supseteq\Theta(\mathscr{F})$ and $\Theta(\mathscr{F}_1)$ $\Theta(\mathscr{F})=\Phi$. Since Φ is a non-trivial congruence relation, by the above there is $\mathscr{F}_1\neq\mathscr{F}$ and $\mathscr{F}_1$ is different from the system of all subsets of I. Hence $\mathscr{F}$ is not an ultrafilter.

Now let us investigate an ultraproduct $\Pi_{\mathscr{U}}(M_i|i\in I)$, where $M_i=M_{2,\beta}$ for every $i\in I$.

2.3. Theorem. *Let I be any nonempty set, $\mathscr{U}$ an ultrafilter on I and let $M_i=M_{2,\beta}$ for every $i\in I$. Then the ultraproduct $\Pi_{\mathscr{U}}(M_i|i\in I)$ is isomorphic to $M_{2,\gamma}$ for some $\gamma\geqslant\beta$.*

Proof. For any $c\in M_{2,\beta}$ the symbol $\boldsymbol{c}$ will denote such an element of $\Pi(M_i|i\in I)$ that $\boldsymbol{c}(i)=c$ for every $i\in I$. Throughout this proof we shall use the denotation $[f]$, $[g]$, ... for the elements of the factor multilattice $\Pi_{\mathscr{U}}(M_i|i\in I)=$ $=\Pi(M_i|i\in I)/\Theta(\mathscr{U})$, instead of $[f]\,\Theta(\mathscr{U})$, $[g]\,\Theta(\mathscr{U})$, ...

Let us fix the elements b, b' of B, $b\neq b'$, and introduce the denotation $U^{(0)}=\{a, a', b, b'\}$, $U^{(1)}=\cup\{x\vee y: x, y\in U^{(0)}\}$, $U^{(2)}=\cup\{x\wedge y: x, y\in U^{(1)}\}$. Evidently, $U^{(2)}=M_{2,\beta}$ and hence $\Pi(M_i|i\in I)=\Pi(U_i^{(0)}|i\in I)\cup\Pi(U_i^{(1)}|i\in I)\cup$ $\cup\Pi(U_i^{(2)}|i\in I)$, where $U_i^{(0)}$ and $U_i^{(1)}$ and $U_i^{(2)}$ means $U^{(0)}$ and $U^{(1)}$ and $U^{(2)}$, respectively, for every $i\in I$.

If $f\in\Pi(U_i^{(0)}|i\in I)$, then $f(i)$ is one of a, a', b, b' for every $i\in I$. Thus $I=$ $=I(f, \boldsymbol{a})\cup I(f, \boldsymbol{b})\cup I(f, \boldsymbol{a}')\cup I(f, \boldsymbol{b}')$ and using $\mathscr{U}$ as an ultrafilter we get that just one of the sets $I(f, \boldsymbol{a})$, $I(f, \boldsymbol{b})$, $I(f, \boldsymbol{a}')$, $I(f, \boldsymbol{b}')$ belongs to $\mathscr{U}$, since any two of these sets are disjoint. If e.g. $I(f, \boldsymbol{a}) \in \mathscr{U}$, *then* $f\Theta(\mathscr{U})\,\boldsymbol{a}$. We have proved that $\{[f]: f \in \Pi(U_i^{(0)}|i \in I)\} = \{[\boldsymbol{a}], [\boldsymbol{b}], [\boldsymbol{a}'], [\boldsymbol{b}']\}$. Evidently, the classes $[\boldsymbol{a}]$, $[\boldsymbol{b}]$, $[\boldsymbol{a}']$, $[\boldsymbol{b}']$ are different and there holds $[\boldsymbol{b}], [\boldsymbol{b}'] \in [\boldsymbol{a}] \vee [\boldsymbol{a}']$, $[\boldsymbol{a}], [\boldsymbol{a}'] \in [\boldsymbol{b}] \wedge [\boldsymbol{b}']$.

Now let $f \in \Pi(U_i^{(1)}|i \in I)$. Then for every $i \in I$ we have $f(i) \in x_i \vee y_i$ for some x_i, $y_i \in U_i^{(0)}$. Let us define $g, h \in \Pi(U_i^{(0)}|i \in I)$ by $g(i) = x_i$, $h(i) = y_i$ for every $i \in I$. Then $f \in g \vee h$, so $[f] \in [g] \vee [h]$. By the above $[f] \in \{[\boldsymbol{a}], [\boldsymbol{a}'], [\boldsymbol{1}]\}$ or $[f] = [f_1]$ for a mapping $f_1: I \to B$.

Finally, if $f \in \Pi(U_i^{(2)}|i \in I)$, then $f \in g \wedge h$ for some $g, h \in \Pi(U_i^{(1)}|i \in I)$. If g, h are mappings from I to B, then

$$\begin{array}{ll} f(i) = g(i) = h(i) \in B & \text{whenever } g(i) = h(i), \\ f(i) \in \{a, a'\} & \text{in the opposite case.} \end{array}$$

Hence $I = I(f, \boldsymbol{a}) \cup I(f, \boldsymbol{a}') \cup I'$, where $I' = \{i \in I: f(i) \in B\}$. Now if $[g] \neq [h]$, then $I(g, h) = I' \in \mathscr{U}$ and hence either $I(f, \boldsymbol{a}) \in \mathscr{U}$ or $I(f, a') \in \mathscr{U}$. In the first case $[f] = [\boldsymbol{a}]$, in the second $[f] = [\boldsymbol{a}']$. Evidently $[\boldsymbol{a}] \wedge [\boldsymbol{a}'] = [\boldsymbol{0}]$.

We have proved that $\Pi_{\mathscr{U}}(M_i|i \in I)$ is isomorphic to $M_{2,\gamma}$ for some cardinal number γ. As different constant mappings from I to B determine different classes, there is $\gamma \geqslant \beta$.

2.4. Corollary. *If $C \in \mathscr{V}_{2,\beta}$ and C is a multilattice generated by a four-element subset $\{r, s, t, u\}$ such that $r, s \in t \wedge u$, $t, u \in r \vee s$ and C has only trivial congruence relations, then C is isomorphic to $M_{2,\gamma}$ for some $\gamma \geqslant \beta$.*

The assertion is an immediate consequence of 2.1, 2.2 and 2.3.

2.5. Corollary. *If $M_{2,\delta} \in \mathscr{V}_{2,\beta}$ for some cardinal number $\delta \geqslant 2$, then $\delta \geqslant \beta$.*

Proof. If $M_{2,\delta} \in \mathscr{V}_{2,\beta}$, then using the fact that $M_{2,\delta}$ is generated by a four-element set $\{r, s, t, u\}$ such that $r, s \in t \wedge u$, $t, u \in r \vee s$ and that $M_{2,\delta}$ has only trivial congruence relations, by 2.4 we obtain that $M_{2,\delta}$ is isomorphic to $M_{2,\gamma}$ for some $\gamma \geqslant \beta$. But then the equality $\delta = \gamma$ holds true. Thus $\delta \geqslant \beta$.

2.6. Theorem. *If $\mathscr{V}$ is a variety such that $\mathscr{V}_{2,\beta} \supset \mathscr{V} \supset \mathscr{D}$, then there exists a cardinal number $\gamma > \beta$ such that $\mathscr{V}_{2,\beta} \supset \mathscr{V} \supseteq \mathscr{V}_{2,\gamma} \supset \mathscr{D}$.*

Proof. If $\mathscr{V}_{2,\beta} \supset \mathscr{V} \supset \mathscr{D}$, then by 1.2 there exists a multilattice $C_1 \in \mathscr{V}$ that is not a lattice. Then C_1 contains a four-element subset $\{r, s, t, u\}$ such that $r, s \in t \wedge u$, $t, u \in r \vee s$. Let C be the subalgebra of C_1 generated by $\{r, s, t, u\}$. Then $C \in \mathscr{V}$ and also $C \in \mathscr{V}_{2,\beta}$. By 2.1 C is isomorphic to $\Pi_{\mathscr{F}}(M_i|i \in I)$ for a non-empty set I and a filter $\mathscr{F}$ on I different from the system of all subsets of I, where $M_i = M_{2,\beta}$ for every $i \in I$. Let $\mathscr{U}$ be any ultrafilter on I containing $\mathscr{F}$. Using 4.6 of [3] we obtain $\Pi_{\mathscr{U}}(M_i|i \in I) \in H\{C\}$. By 2.3 there is $M_{2,\gamma} \in H\{C\}$ for some $\gamma \geqslant \beta$. Then $\mathscr{D} \subset \mathscr{V}_{2,\gamma} \subseteq HSP\{C\} \subseteq \mathscr{V} \subset \mathscr{V}_{2,\beta}$. The relation $\mathscr{V}_{2,\gamma} \subset \mathscr{V}_{2,\beta}$ eliminates the equality $\gamma = \beta$.

2.7. Corollary. *For different infinite cardinal numbers β the varieties $\mathscr{V}_{2,\beta}$ are different.*

Proof. If $\beta \neq \gamma$, then either $\beta < \gamma$ or $\beta > \gamma$. By 2.5 in the first case $M_{2,\beta} \notin \mathscr{V}_{2,\gamma}$ and in the second case $M_{2,\gamma} \notin \mathscr{V}_{2,\beta}$.

As an immediate consequence we obtain:

2.8. Theorem. *The varieties of modular multilattices form a proper class.*

Now we will prove that for any infinite cardinal number β there exists an infinite decreasing sequence of varieties $\mathscr{V}_0 = \mathscr{V}_{2,\beta} \supset \mathscr{V}_1 \supset \mathscr{V}_2 \supset \ldots \supset \mathscr{D}$.

If I is any nonempty set and $\mathscr{U}$ is an ultrafilter on I, then $\Pi_{\mathscr{U}}(M_i | i \in I) = \Pi(M_i | i \in I)/\Theta(\mathscr{U})$, where $M_i = M_{2,\beta}$ for every $i \in I$, belongs to $\mathscr{V}_{2,\beta}$. By 2.3 $\Pi_{\mathscr{U}}(M_i | i \in I)$ is isomorphic to $M_{2,\gamma}$ for some $\gamma \geqslant \beta$. What values of γ can be obtained for a given β, choosing index sets of various cardinalities and choosing various ultrafilters on the same index set? It is easy to see that

$$\gamma = \operatorname{card}\{[f]\,\Theta(\mathscr{U}) : f \text{ is a mapping } I \to B\} = \operatorname{card}\Pi_{\mathscr{U}}(B_i | i \in I),$$

where $B_i = B$ for every $i \in I$.

We will use the following assertion, which is a consequence of 6.1.14 and 6.3.21 of [1].

2.9. Theorem. *Let I be any infinite set of the cardinality λ, B a set of the cardinality β and let $B_i = B$ for every $i \in I$. Then there exists an ultrafilter $\mathscr{U}$ on I such that $\operatorname{card}\Pi_{\mathscr{U}}(B_i | i \in I) = \beta^{\lambda}$.*

Using 2.9 we obtain:

2.10. Theorem. *For every infinite cardinal number β there exists an increasing infinite sequence of cardinal numbers $\beta_0 < \beta_1 < \beta_2 < \ldots$ such that $\beta_0 = \beta$ and $\mathscr{V}_{2,\beta} = \mathscr{V}_{2,\beta_0} \supset \mathscr{V}_{2,\beta_1} \supset \mathscr{V}_{2,\beta_2} \supset \ldots \supset \mathscr{D}$.*

Proof. Define $\beta_0 = \beta$ and supposing that there is defined β_j for a nonnegative integer j, define $\beta_{j+1} = \beta_j^{\beta_j}$. Now let j be any fixed nonnegative integer. Take any set I of the cardinality β_j. In view of 2.9 there exists an ultrafilter $\mathscr{U}$ on I such that the ultraproduct $\Pi_{\mathscr{U}}(M_i | i \in I)$, where $M_i = M_{2,\beta_j}$ for every $i \in I$, is isomorphic to $M_{2,\beta_j^{\beta_j}} = M_{2,\beta_{j+1}}$. Since $\Pi_{\mathscr{U}}(M_i | i \in I) \in \mathscr{V}_{2,\beta_j}$, we have $M_{2,\beta_{j+1}} \in \mathscr{V}_{2,\beta_j}$. We have proved that $\mathscr{V}_{2,\beta_{j+1}} \subseteq \mathscr{V}_{2,\beta_j}$. As $\beta_{j+1} = \beta_j^{\beta_j} \geqslant 2^{\beta_j} > \beta_j$, by 2.5 $M_{2,\beta_j} \notin \mathscr{V}_{2,\beta_{j+1}}$. Hence $\mathscr{V}_{2,\beta_{j+1}} \subset \mathscr{V}_{2,\beta_j}$.

2.11. Theorem. *Let β be any infinite cardinal number. Then there exists no variety $\mathscr{V}$ of multilattices covering $\mathscr{D}$ in the lattice of all varieties of multilattices and satisfying $\mathscr{V}_{2,\beta} \supseteq \mathscr{V}$.*

Proof. Suppose that for an infinite cardinal number β there exists a variety $\mathscr{V}$ covering $\mathscr{D}$ and satisfying $\mathscr{V}_{2,\beta} \supseteq \mathscr{V}$. By 2.6 there is $\mathscr{V} = \mathscr{V}_{2,\gamma}$ for some cardinal number $\gamma \geqslant \beta$, but in view of 2.10 the variety $\mathscr{V}_{2,\gamma}$ does not cover $\mathscr{D}$. We have a contradiction.

3. Another variety covering $\mathcal{D}$

In the last part of the paper we will show that it can happen that a variety $\mathcal{V}^*$ covering $\mathcal{D}$ contains only infinite multilattices, with the exception of those that are lattices. The method applied in this section is analogous to that used in [3], Section 6.

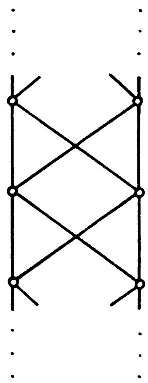

Fig. 3

Throughout this section we denote by M the multilattice shown in Figure 3 and by $\mathcal{V}^*$ we denote the variety generated by M. Evidently $\mathcal{V}^* \supset \mathcal{D}$.

3.1. Lemma. *The only varieties of lattices that are contained in $\mathcal{V}^*$ are the variety $\mathcal{D}$ and the variety of all one-element lattices.*

Proof. If $\mathcal{V}^*$ contained a variety of lattices different from the above mentioned, it would be $M_3 \in \mathcal{V}^*$ or $N_5 \in \mathcal{V}^*$. But since $\mathcal{V}^*$ contains only distributive multilattices, both these possibilities are excluded.

Having any subset U of a multilattice M' let us define the sets $U^{(k)}$ for nonnegative integers k as follows: $U^{(0)} = U$; if $U^{(l)}$ is defined for some nonnegative integer l, set $U^{(l+1)} = \cup\{x \vee y : x, y \in U^{(l)}\}$ for l even and $U^{(l+1)} = \cup\{x \wedge y : x, y \in U^{(l)}\}$ for l odd.

3.2. Lemma. *Let I be any non-empty set. Further, let A be a subalgebra of $\Pi(M_i | i \in I)$, where $M_i = M$ for every $i \in I$, generated by a four-element subset $\{r, s, t, u\}$ such that $r, s \in t \wedge u$, $t, u \in r \vee s$. Put $U_i = \{r(i), s(i), t(i), u(i)\}$ for every $i \in I$. Then A is isomorphic to the subalgebra $B = \bigcup_{k \geq 0} \Pi(U_i^{(k)} | i \in I_1)$ of $\Pi(M_i | i \in I_1)$, where $I_1 = \{i \in I : \operatorname{card} U_i = 4\}$.*

Proof. Consider the mapping assigning to every $f \in A$ its restriction to I_1, denoted by $f|I_1$. Let $U = \{r, s, t, u\}$. Since $A = \bigcup_{k \geq 0} U^{(k)}$, the relation $f \in A$ implies

$f \in U^{(k)}$ for some nonnegative integer k. Then $f(i) \in U_i^{(k)}$ for every $i \in I$ and we have $f \in \Pi(U_i^{(k)}|i \in I)$ and $f|I_1 \in \Pi(U_i^{(k)}|i \in I_1)$. Hence the mapping $f \mapsto f|I_1$ is a mapping from A to B and evidently it is order-preserving in both directions. It remains to show that this mapping is onto. It is easy to see, by induction on k, that every element of $\Pi(U_i^{(k)}|i \in I_1)$ has a pre-image in A (see the proof of 6.6 in [3]).

In 3.3—3.5 we shall assume that I is a non-empty set, $M_i = M$ for every $i \in I$, r, s, t, u are mutually different elements from $\Pi(M_i|i \in I)$ such that r, $s \in t \wedge u$, t, $u \in r \vee s$. Further, we shall suppose that for every $i \in I$ the set $U_i = \{r(i), s(i), t(i), u(i)\}$ has the cardinality 4. The aim is to prove that every congruence relation on $B = \bigcup_{k \geq 0} \Pi(U_i^{(k)}|i \in I)$ corresponds to a filter on I.

3.3. Lemma. *Let $f, g \in B$, $f \geq g$ and let $\Theta(f, g)$ be the corresponding principal congruence relation on B. Then the relation $p\Theta(f, g)q$ ($p, q \in B$) holds if and only if $I(f, g) \subseteq I(p, q)$.*

Proof. Let $p\Theta(f, g)q$ hold for some $p, q \in B$. By 3.4 of [3] there is $p(i)\,\Theta(f(i), g(i))\,q(i)$ for every $i \in I$. Since M has only trivial congruence relations, we have $I(f, g) \subseteq I(p, q)$.

Conversely let $I(f, g) \subseteq I(p, q)$. If $i \in I(f, g)$, then $i \in I(p, q)$ and hence evidently $p(i)\,\Theta(f(i), g(i))\,q(i)$. If $i \notin I(f, g)$, then $\Theta(f(i), g(i))$ is the greatest congruence relation on M and hence again $p(i)\,\Theta(f(i), g(i))\,q(i)$. Now we shall prove that $p\Theta(f, g)q$. Since $f, g, p, q \in B$, there exists a nonnegative integer k such that f, g, p, $q \in \Pi(U_i^{(k)}|i \in I)$. For every $i \in I$ take an arbitrary maximal chain $f_0^i > f_1^i > \ldots > f_{n_i}^i$ such that $f_0^i \in p(i) \vee q(i)$, $f_{n_i}^i \in p(i) \wedge q(i)$. If $p(i)$, $q(i)$ are comparable, then n_i is not greater than the length of $U_i^{(k)}$, which is $k + 1$. If $p(i)$, $q(i)$ are incomparable, then $n_i = 2$. Hence there exists a positive integer n and for every $i \in I$ a chain $e_0^i \geq e_1^i \geq \ldots \geq e_n^i$ such that $e_0^i \in p(i) \vee q(i)$, $e_n^i \in p(i) \wedge q(i)$ and for every $j \in \{0, \ldots, n - 1\}$ either $e_j^i = e_{j+1}^i$ or the quotient e_j^i/e_{j+1}^i is prime (i.e. e_j^i covers e_{j+1}^i). At that $\{e_0^i, \ldots, e_n^i\} \subseteq U_i^{(k+2)}$, too. Let us define $e_0, e_1, \ldots, e_n \in B$ in such a way that $e_j(i) = e_j^i$ for every $i \in I$, $j \in \{0, \ldots, n\}$. Then $e_0 \geq e_1 \geq \ldots \geq e_n$, $e_0 \in p \vee q$, $e_n \in p \wedge q$. It remains to show that for every $j \in \{0, \ldots, n - 1\}$ the quotient e_j/e_{j+1} is weakly projective into f/g. In M every two prime quotients are projective and for any $l \geq 0$ there exists a positive integer h_l such that any two prime quotients in $U_i^{(l)}$ are projective in no more than h_l steps. Now, if $i \notin I(f, g)$, i.e. $f(i) > g(i)$, then since $\{e_0^i, \ldots, e_n^i\} \subseteq U_i^{(k+2)}$ and $f(i), g(i) \in U_i^{(k)} \subset U_i^{(k+2)}$, every prime quotient e_j^i/e_{j+1}^i is projective with any prime subquestient of the quotient $f(i)/g(i)$ in no more than $h_{k+2} = h$ steps. Hence for every $i \notin I(f, g)$ every prime quotient e_j^i/e_{j+1}^i is weakly projective into $f(i)/g(i)$ in no more than h steps. The one-element quotient e_j^i/e_{j+1}^i is obviously also weakly projective into $f(i)/g(i)$ in no more than h steps. If $i \in I(f, g)$, then $i \in I(p, q)$, which implies $e_0^i = e_1^i = \ldots = e_n^i$. Hence again e_j^i/e_{j+1}^i is weakly projective into $f(i)/g(i)$ in no more than h steps, for every $j \in \{0, \ldots, n - 1\}$. Now it is easy to see that for every $j \in \{0,$

$\ldots, n-1\}$ the quotient e_j/e_{j+1} is weakly projective into f/g (see the proof of 6.7 in [3]). By 3.4 in [3] it means that $p\Theta(f, g)q$.

3.4. Lemma. *Let $\Theta \in \operatorname{Con} B$. Then $\Theta = \Theta(\mathscr{F})$ for some filter $\mathscr{F}$ on I.*

Proof. There holds $\Theta = \sup\{\Theta(f_\lambda, g_\lambda)\colon \lambda \in \Lambda\}$, where $\{(f_\lambda, g_\lambda)\colon \lambda \in \Lambda\} = \{(f, g) \in B \times B\colon f \geqslant g, f\Theta g\}$. Let $\mathscr{F}$ be the filter on I generated by the set $\{I(f_\lambda, g_\lambda)\colon \lambda \in \Lambda\}$. To prove that $\Theta = \Theta(\mathscr{F})$, it is sufficient to show that for $f, g \in B$, $f \geqslant g$ the relation $f\Theta g$ holds if and only if $I(f, g) \in \mathscr{F}$. Hence let $f, g \in B, f \geqslant g$. If $f\Theta g$, then $(f, g) = (f_\lambda, g_\lambda)$ for some $\lambda \in \Lambda$ and then $I(f, g) = I(f_\lambda, g_\lambda) \in \mathscr{F}$. Now let $I(f, g) \in \mathscr{F}$. Then $I(f, g) \supseteq I(f_{\lambda_1}, g_{\lambda_1}) \cap \ldots \cap I(f_{\lambda_r}, g_{\lambda_r})$ for a positive integer r. Define $f_0, f_1, \ldots, f_r$ as follows:

$$f_0 = g$$

$$f_1(i) = \begin{cases} f(i) & \text{if } i \notin I(f_{\lambda_1}, g_{\lambda_1}), \\ f_0(i) & \text{if } i \in I(f_{\lambda_1}, g_{\lambda_1}); \end{cases}$$

$$f_2(i) = \begin{cases} f(i) & \text{if } i \notin I(f_{\lambda_1}, g_{\lambda_1}) \cap I(f_{\lambda_2}, g_{\lambda_2}), \\ f_1(i) & \text{if } i \in I(f_{\lambda_1}, g_{\lambda_1}) \cap I(f_{\lambda_2}, g_{\lambda_2}); \end{cases}$$

$$\vdots$$

$$f_r(i) = \begin{cases} f(i) & \text{if } i \notin I(f_{\lambda_1}, g_{\lambda_1}) \cap \ldots \cap I(f_{\lambda_r}, g_{\lambda_r}), \\ f_{r-1}(i) & \text{if } i \in I(f_{\lambda_1}, g_{\lambda_1}) \cap \ldots \cap I(f_{\lambda_r}, g_{\lambda_r}). \end{cases}$$

Evidently $f_0, \ldots, f_r \in B$, because $f_j(i)$ is either $g(i)$ or $f(i)$ and since f, $g \in \Pi\,(U_i^{(k)} | i \in I)$ for some nonnegative integer k, also $f_j \in \Pi\,(U_i^{(k)} | i \in I)$ for the same k. Further, by 3.3 we have $g = f_0\Theta(f_{\lambda_1}, g_{\lambda_1}) f_1 \Theta(f_{\lambda_2}, g_{\lambda_2}) f_2 \ldots f_{r-1} \Theta(f_{\lambda_r}, g_{\lambda_r}) f_r = f$. Thus $g\Theta(f_{\lambda_1}, g_{\lambda_1}) \vee \ldots \vee \Theta(f_{\lambda_r}, g_{\lambda_r}) f$ and we have proved $g\Theta f$.

3.5. Lemma. *Let $\mathscr{U}$ be any ultrafilter on I. Then the factor multilattice $B/\Theta(\mathscr{U})$ is isomorphic to M.*

Proof. Given any $i \in I$ and $j \in \{0, 1, 2, \ldots\}$ let us define elements $r_j^i, s_j^i, t_j^i, u_j^i \in M_i$ in the way depicted in Figure 4. Further, define $r_j, s_j, t_j, u_j \in \Pi\,(M_i | i \in I)$ for $j \in \{0, 1, 2, \ldots\}$ by $r_j(i) = r_j^i, s_j^i = s_j^i, t_j(i) = t_j^i, u_j(i) = u_j^i$ for every $i \in I$. Obviously $r_0 = r$, $s_0 = s$, $t_0 = t$, $u_0 = u$ and $r_j, s_j, t_j, u_j \in B$ for every $j \in \{0, 1, 2, \ldots\}$.

Let for $f \in B$ the symbol $[f]$ denote the class $[f]\,\Theta(\mathscr{U})$. The classes $[r_j]$, $[s_j]$, $[t_j]$, $[u_j]$ for $j \in \{0, 1, 2, \ldots\}$ form a partially ordered set isomorphic to M (since $\mathscr{U}$ is an ultrafilter, there holds $\emptyset \notin \mathscr{U}$, which implies that these classes are mutually different). Now we are going to show that for any $f \in B$, $[f]$ is one of the above mentioned classes. If $f \in \Pi\,(U_i^{(0)} | i \in I)$, then $f(i) \in \{r(i), s(i), t(i), u(i)\} = \{r_0^i, s_0^i, t_0^i, u_0^i\}$ for every $i \in I$. Hence $I = I(f, r) \cup I(f, s) \cup I(f, t) \cup (I(f, u)$ and using the properties of an ultrafilter we obtain that just one of the sets $I(f, r)$, $I(f, s)$, $I(f, t)$, $I(f, u)$ belongs to $\mathscr{U}$. If, e.g., $I(f, r) \in \mathscr{U}$, then $[f] = [r] = [r_0]$. We have proved that if $f \in \Pi(U_i^{(0)} | i \in I)$, then $[f] \in \{[r_0], [s_0], [t_0], [u_0]\}$. Suppose that for some non-negative integer l, $[f] \in \{[r_j]: j \in \{0, \ldots, l\}\} \cup \{[s_j]: j \in \{0, \ldots, l\}\} \cup \cup \{[t_j]: j \in \{0, \ldots, l\}\} \cup \{[u_j]: j \in \{0, \ldots, l\}\}$ whenever $f \in \Pi(U_i^{(l)} | i \in I)$. We are going to prove that then for every $f \in \Pi(U_i^{(l+1)} | i \in I)$, $[f] \in \{[r_j]: j \in \{0, \ldots, l+1\}\} \cup \cup \{[s_j]: j \in \{0, \ldots, l+1\}\} \cup \{[t_j]: j \in \{0, \ldots, l+1\}\} \cup \{[u_j]: j \in \{0, \ldots, l+1\}\}$. Without

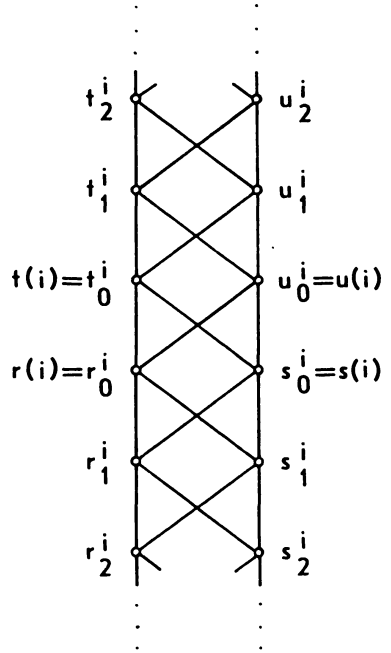

Fig. 4

loss of generality we can suppose that l is even. If $f \in \Pi(U_i^{(l+1)} | i \in I)$, then for every $i \in I$ there exist x_i, $y_i \in U_i^{(l)}$ such that $f(i) \in x_i \vee y_i$. Let us define g, $h \in \Pi(U_i^{(l)} | i \in I)$ by $g(i) = x_i$, $h(i) = y_i$ for every $i \in I$. Then $f \in g \vee h$, which gives $[f] \in [g] \vee [h]$. Using the induction hypothesis we obtain $[g], [h] \in \{[r_j]: j \in \{0, \ldots, l\}\} \cup \{[s_j]: j \in \{0, \ldots, l\}\} \cup \{[t_j]: j \in \{0, \ldots, l\}\} \cup \{[u_j]: j \in \{0, \ldots, l\}\}$. If $[g]$, $[h]$ are comparable, then $[f] \in \{[g], [h]\} \subseteq \{[r_j]: j \in \{0, \ldots, l\}\} \cup \{[s_j]: j \in \{0, \ldots, l\}\} \cup \{[t_j]: j \in \{0, \ldots, l\}\} \cup \{[u_j]: j \in \{0, \ldots, l\}\} \subset \{[r_j]: j \in \{0, \ldots, l+1\}\} \cup \{[s_j]: j \in \{0, \ldots, l+1\}\} \cup \{[t_j]: j \in \{0, \ldots, l+1\}\} \cup \{[u_j]: j \in \{0, \ldots, l+1\}\}$. If $[g]$, $[h]$ are incomparable, then either $\{[g], [h]\} = \{[t_j], [u_j]\}$ or $\{[g], [h]\} = \{[r_j], [s_j]\}$ for some $j \in \{0, \ldots, l\}$. Let e.g., the first possibility occur. Then $[f] \in [t_j] \vee [u_j]$ and hence there exists $f' \in t_j \vee u_j$ with $[f'] = [f]$. It follows that for every $i \in I$, $f'(i) \in t_j^i \vee u_j^i = \{t_{j+1}^i, u_{j+1}^i\}$. Again $I = I(f', t_{j+1}) \cup I(f', u_{j+1}) \in \mathcal{U}$, so either $I(f', t_{j+1}) \in \mathcal{U}$ or

$I(f', u_{j+1}) \in \mathscr{U}$. In the first case $[f] = [f'] = [t_{j+1}]$, in the second $[f] = [f'] = [u_{j+1}]$. If $\{[g], [h]\} = \{[r_j], [s_j]\}$, then $[f] \in \{[r_{j-1}], [s_{j-1}]\}$ whenever $j > 0$ and $[f] \in \{[t_0], [u_0]\}$ for $j = 0$. In all cases $[f] \in \{[r_j]: j \in \{0, \ldots, l+1\}\} \cup \cup \{[s_j]: j \in \{0, \ldots, l+1\}\} \cup \{[t_j]: j \in \{0, \ldots, l+1\}\} \cup \{[u_j]: j \in \{0, \ldots, l+1\}\}$.

3.6. Theorem. *The variety $\mathscr{V}^{\frown}$ generated by the multilattice M in Figure* 3 *covers the variety $\mathscr{D}$ in the lattice of varieties of multilattices and does not contain any finite multilattice that is not a lattice.*

Proof. Let $\mathscr{V}_1^{\frown}$ be a variety of multilattices such that $\mathscr{V}^{\frown} \supseteq \mathscr{V}_1^{\frown} \supset \mathscr{D}$. By 3.1 $\mathscr{V}_1^{\frown}$ contains a multilattice C' that is not a lattice. Then C' contains mutually different elements r', s', t', u' such that $t', u' \in r' \vee s'$, $r', s' \in t' \wedge u'$. Let C be the subalgebra of C' generated by the set $\{r', s', t', u'\}$. There holds $C \in \mathscr{V}_1^{\frown} \subseteq \subseteq \mathscr{V}^{\frown} = HSP\{M\}$, hence there exists a homomorphism φ of a subalgebra A of $\Pi(M_i | i \in I)$, where $M_i = M$ for every $i \in I$, onto C. By 6.5 of [3] there exist $r, s, t, u \in A$ with $r, s \in t \wedge u$, $t, u \in r \vee s$, $\varphi(r) = r'$, $\varphi(s) = s'$, $\varphi(t) = t'$, $\varphi(u) = u'$. We can suppose that A is generated by $\{r, s, t, u\}$. Using 3.2 and 3.4 we obtain that C is isomorphic to $B/\Theta(\mathscr{F})$, where $B = \bigcup_{k \geq 0} \Pi(U_i^{(k)} | i \in I_1)$, $U_i = \{r(i), s(i), t(i), u(i)\}$, $I_1 = \{i \in I: \text{card } U_i = 4\}$ and $\mathscr{F}$ is a filter on I_1. Since card $C > 1$, there exists an ultrafilter $\mathscr{U}$ on I_1 with $\mathscr{F} \subseteq \mathscr{U}$. Then $\Theta(\mathscr{U}) \supseteq \Theta(\mathscr{F})$ and by 4.6 of [3] we have $B/\Theta(\mathscr{U}) \in H\{C\}$. Using 3.5 we obtain $M \in H\{C\} \subseteq \mathscr{V}_1^{\frown}$, so $\mathscr{V}^{\frown} \subseteq \mathscr{V}_1^{\frown}$. We have proved that $\mathscr{V}^{\frown} = \mathscr{V}_1^{\frown}$.

If the variety $\mathscr{V}^{\frown}$ contained a finite multilattice which is not a lattice, then by the previous consideration, M would be the homomorphic image of a finite multilattice, which is a contradiction.

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