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Tibor Šalát

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CRITERION FOR UNIFORM DISTRIBUTION OF SEQUENCES AND A CLASS OF RIEMANN INTEGRABLE FUNCTIONS

TIBOR ŠALÁT

1. Introduction

In the paper [2] a criterion for the uniform distribution of sequences is introduced. We shall show that a condition of this criterion formulated by using the Lebesgue measure can be replaced by an analogous condition formulated by using the Jordan measure. Further it will be shown here that the set of all Riemann integrable functions which can be used in the mentioned criterion contains all Riemann integrable functions with the exception of a "little" set in a topological sense.

2. A remark about the Horbowicz criterion

The following criterion for the uniform distribution of sequences of real numbers is proved in [2]:

Let f be a complex Riemann integrable function on $\langle 0, 1 \rangle$. Let

$$\mu(Z(f)) = 0 \tag{1}$$

where

$$Z(f) = \{x \in \langle 0, 1 \rangle : f(x) = 0\} = f^{-1}(0)$$

and μ denotes the Lebesgue measure. Then a sequence $\{x_n\}_{n=1}^{\infty}$ of numbers of the interval $\langle 0, 1 \rangle$ is uniformly distrubuted if and only if for each interval $\langle a, b) \subset \langle 0, 1)$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \chi_{\langle a, b)}(x_n) = \int_a^b f\ (x)\,\mathrm{d}x$$

($\chi_{\langle a, b)}$ stands for the charateristic function of the interval $\langle a, b)$).

Š. Porubský posed the question whether the condition (1) can be replaced by the condition

(1′) $$m(Z(f)) = 0$$

where m denotes the Jordan measure.

Let us remark that each Riemann integrable function f: $\langle 0, 1\rangle \to R$ is also Lebesgue integrable and hence the set $Z(f)$ is Lebesgue measurable [1, p. 270].

In what follows $R(0, 1)$ denotes the set of all real Riemann integrable functions on $\langle 0, 1\rangle$. We restrict ourselves only to real functions, the results can be easily extended for complex functions.

The closure of a set $M \subset \langle 0, 1\rangle$ will be denoted by $\bar{M}$.

Theorem 1.1. *Let $f \in R(0, 1)$. Then we have*

$$\mu(\overline{Z(f)}) = \mu(Z(f)).$$

Corollary 1.1. *The condition* (1) *is equivalent to the condition* (1′) *(for Riemann integrable functions f).*

Proof of Corollary 1.1 Clearly (1′) implies (1). If (1) holds, then according to the theorem 1.1 we have $\mu(\overline{Z(f)}) = 0$. However, $\overline{Z(f)}$ is a compact set and therefore $m(\overline{Z(f)}) = 0$. From this we get $m(Z(f)) = 0$.

Proof of Theorem 1.1. Since each of the sets $Z(f)$, $\overline{Z(f)}$ is Lebesgue measurable and $Z(f) \subset \overline{Z(f)}$, it suffices to show that the inequality

(2) $$\mu(\overline{Z(f)}) > \mu(Z(f))$$

does not hold.

Let (2) hold. Choose an arbitrary $x_0 \in \overline{Z(f)} \setminus Z(f) = W$. Then $f(x_0) \neq 0$ and in every neighbourhood of x_0 there are points x with $f(x) = 0$. This implies that f is discontinuous at x_0. Since according to (2) we have $\mu(W) > 0$, the set of all discontinuity points of the function f has a positive Lebesgue measure. Therefore [1, p. 270] the function f is not Riemann integrable — a contradiction. This ends the proof.

2. The class of functions satisfying (1)

The question arises how "large" the class $H(0, 1)$ of all functions $f \in R(0, 1)$ is satisfying the condition (1). These functions can be used in the criterion of J. Horbowicz. The answer to this question is given in Theorem 2.1.

In what follows we shall consider $R(0, 1)$ as a linear normed space with the sup-norm:

$$\|f\| = \sup_{0 \leq t \leq 1} |f(t)|.$$

This space is evidently a Banach space since the convergence in this space coincides with the uniform convergence which preserves the Riemann integrability.

Theorem 2.1. *The set* $R(0,1)\setminus H(0,1)$ is a nowhere dense set in $R(0,1)$.
We shall use the following auxiliary result.

Lemma 2.1. *The set* $H(0,1)$ *is dense in* $R(0,1)$.
Proof. Let

$$K(f,\delta)=\{g\in R(0,1):\ \|f-g\|<\delta\}$$

be an arbitrary ball in $R(0,1)$ ($f\in R(0,1)$, $\delta>0$). It suffices to prove that

$$K(f,\delta)\cap H(0,1)\neq\emptyset. \tag{3}$$

Consider the fact that the sets $f^{-1}(\delta')$, $0\leqq\delta'<\delta$ are mutually disjoint and Lebesgue measurable. Therefore it is impossible for each of these sets to have a positive measure. Hence there is a δ_1, $0\leqq\delta_1<\delta$, such that

$$\mu(f^{-1}(\delta_1))=0.$$

Put $g(x)=f(x)-\delta_1$ for $x\in\langle 0,1\rangle$. Then evidently we have $g\in K(f,\delta)$ and

$$g(x)=0\Leftrightarrow f(x)=\delta_1.$$

Hence $\mu(Z(g))=0$, $g\in H(0,1)$ and so (3) holds.

Proof of Theorem 2.1. Let $K(f,\eta)$ ($f\in R(0,1)$, $\eta>0$) be an arbitrary ball in $R(0,1)$. It suffices to prove [3, p. 116, Theorem 8] that there exists a ball $G\subset K(f,\eta)$ such that

$$G\cap(R(0,1)\setminus H(0,1))=\emptyset. \tag{4}$$

According to Lemma 2.1 we can choose a function $g\in K(f,\eta)\cap H(0,1)$. Further an $\eta_1>0$ can be chosen in such a way that

$$K(g,\eta_1)\subset K(f,\eta). \tag{5}$$

Define the function h in the following way:

$$h(x)=g(x)+\frac{\eta_1}{2}\quad\text{if}\quad g(x)\geqq 0,$$

$$h(x)=g(x)-\frac{\eta_1}{2}\quad\text{if}\quad g(x)<0.$$

Clearly, h is a bounded function and

$$\forall_{x\in\langle 0,1\rangle}\ |h(x)|\geqslant\frac{\eta_1}{2}. \tag{6}$$

Denote by $C(\varphi)$ the set of all continuity points of the function φ. Then we have

(7) $$C(g) \cap (\langle 0, 1\rangle \setminus Z(g)) \subset C(h).$$

Indeed, let $x_0 \in C(g) \cap (\langle 0, 1\rangle \setminus Z(g))$. Then $g(x_0) \neq 0$. Let, e.g., $g(x_0) > 0$. If $x_n \to x$, then on the basis of continuity of g at x_0 there exists an n_0 such that for $n > n_0$ we have $g(x_n) > 0$. Thus

$$h(x_n) = g(x_n) + \frac{\eta_1}{2} \qquad \text{for } n > n_0.$$

Since $g(x_n) \to g(x_0)$, we get

$$h(x_n) \to g(x_0) + \frac{\eta_1}{2} = \mathrm{h}(\mathrm{x}_0).$$

Hence h is continuous at x_0.

We can similarly show also in the case of $g(x_0) < 0$ that the function h is continuous at x_0. Hence (7) holds.

Since $\mu(Z(g)) = 0$ and $\mu(C(g)) = 1$, according to (7) we get $\mu(C(h)) = 1$. Therefore $h \in R(0, 1)$,

(8) $$\|h - g\| \leqq \frac{\eta_1}{2}$$

and $h \in H(0, 1)$ (see (6)).

Using (8) it is easy to see that

(9) $$K\left(h, \frac{\eta_1}{2}\right) \subset K(g, \eta_1).$$

According to (5), (9) we get $K\left(h, \frac{\eta_1}{2}\right) \subset K(f, \eta)$. Further for each $w \in K\left(h, \frac{\eta_1}{2}\right)$ and each $x \in \langle 0, 1\rangle$ we obtain (see (6)):

$$|w(x)| \geqq |h(x)| - |w(x) - h(x)| \geqq \frac{\eta_1}{2} - \|w - h\| > \frac{\eta_1}{2} - \frac{\eta_1}{2} = 0.$$

Hence $w \notin R(0, 1) \setminus H(0, 1)$. Thus (4) holds if we put $G = K\left(h, \frac{\eta_1}{2}\right)$. This ends the proof.

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Katedra algebry a teórie čísel
Matematicko-fyzikálnej fakulty UK
Mlynská dolina
842 15 Bratislava

ОДИН ПРИЗНАК ДЛЯ РАВНОМЕРНОГО РАССПРЕДЕЛЕНИЯ ПОСЛЕДОВАТЕЛЬНОСТЕЙ И ОПРЕДЕЛЕННЫЙ КЛАСС ФУНКЦИЙ ИНТЕГРИРУЕМЫХ В СМЫСЛЕ РИМАНА

Tibor Šalát

Резюме

Пусть $R(0, 1)$ обозначает пространство всех функций, интегрируемых в смысле Римана на интервале $\langle 0, 1\rangle$ с метрикой $d(f, g) = \sup_{0 \leqq x \leqq 1} |f(x) - g(x)|$. В работе показано, что множество всех $f \in R(0, 1)$, которые можно использовать в признаке для равномерного расспределения последовательностей чз [2], имеет форму $R(0, 1)\backslash M$, где M нигде не плотное множество в $R(0, 1)$.