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Neighbourhood tournaments

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NEIGHBOURHOOD TOURNAMENTS

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A tournament is a directed graph in which any two distinct vertices are joined by exactly one edge.

Let T be a tournament, let v be its vertex. By the symbol $N_T(v)$ we denote the subtournament of T induced by the set of all vertices of T which are terminal vertices of edges outgoing from v; this tournament will be called the neighbourhood tournament of v in T.

At the Symposium on Graph Theory in Smolenice in 1963 A. A. Zykov [1] has suggested a problem concerning neighbourhood graphs in undirected graphs. We shall study the tournament variant of this problem:

Characterize the tournaments T_0 with the property that there exists a tournament T such that $N_T(v) \cong T_0$ for each vertex v of T.

We shall give a partial solution of this problem.

Theorem 1. *Let a tournament T_0 with n vertices, where n is a positive integer, have the property that there exists a tournament T such that $N_T(v) \cong T_0$ for each vertex v of T. Then T has $2n+1$ vertices.*

Proof. Let p be the number of vertices of T. As each tournament $N_T(v)$ for a vertex v of T has n vertices, the outdegree of any vertex of T is n and its indegree is $p-n-1$. The number of edges of T is the sum of outdegrees of all vertices of T, namely np. However it is equal also to the sum of indegrees of all vertices of T, namely $p(p-n-1)$. Hence $np = p(p-n-1)$; as evidently $p \neq 0$, this implies $p = 2n+1$.

For any positive integer n by $\mathcal{T}(n)$ we denote the class of all tournaments with the following structure. For any tournament $T_0 \in \mathcal{T}(n)$ there exists a subset $A(T_0)$ of the number set $\{1, 2, \ldots, 2n\}$ which has n elements and the property that $x + y \neq 2n+1$ for any two elements x, y of $A(T_0)$. The vertices of T_0 can be labelled by the elements of $A(T_0)$ in such a way that for each edge of T_0 the difference of the label of the terminal vertex and the label of the initial vertex is congruent to an element of $A(T_0)$ modulo $2n+1$.

Theorem 2. *Let $T_0 \in \mathcal{T}(n)$ for a positive integer n. Then there exists a tournament T such that $N_T(v) \cong T_0$ for each vertex v of T.*

Proof. Let the vertex set of T be $V(T)=\{v_1, v_2, \ldots, v_{2n+1}\}$. There exists an edge $\overline{v_iv_j}$ in T if and only if $j-i$ is congruent to an element of $A(T_0)$ modulo $2n+1$. Evidently if i and j are distinct numbers from the set $\{1, 2, \ldots, 2n+1\}$, then $j-i$ is not congruent to 0 modulo $2n+1$. As $|A(T_0)|=n$, any number $x\in\{1, 2, \ldots, 2n\}$ has the property that exactly one of the numbers x, $2n+1-x$ belongs to $A(T_0)$. Hence exactly one of the numbers $j-i$, $i-j$ is congruent modulo $2n+1$ to an element of $A(T_0)$ and T is a tournament. For each vertex v_i the vertex set of $N_T(v_i)$ consists of exactly all vertices v_j such that $j-i$ is congruent to an element of $A(T_0)$ modulo $2n+1$. If we label each vertex v_j of $N_T(v_i)$ by $j-i$ if $j>i$ and by $j-i+2n+1$ if $j<i$, we see that $N_T(v_i)\cong T_0$.

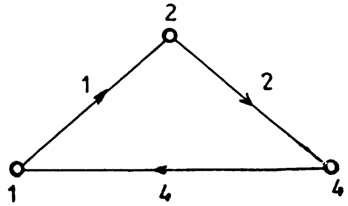

Fig. 1

A tournament $T_0\in\mathcal{T}(n)$ can be given by a $2n$-dimensional Boolean vector $v(T_0)=(a_1, a_2, \ldots, a_{2n})$ such that $a_i=1$ for $i\in A(T_0)$ and $a_i=0$ for $i\notin A(T_0)$. Obviously $v(T_0)$ must contain n coordinates equal to one and n coordinates equal to zero.

For an acyclic tournament with n vertices we may put $A(T_0)=\{1, 2, \ldots, n\}$ and label its vertices in such a way that the label of the terminal vertex of any edge is greater than the label of its initial vertex. Thus we have a corollary.

Corollary. *Each finite acyclic tournament T_0 has the property that there exists a tournament T such that $N_T(v)\cong T_0$ for each vertex v of T.*

A cycle with three vertices is given by the vector (1, 1, 0, 1, 0, 0), the tournament obtained from it by adding a source is given by (1, 0, 1, 1, 0, 0, 1, 0), the tournament with four vertices having a Hamiltonian cycle is given by (1, 1, 1, 0, 1, 0, 0, 0). The labelling of vertices of these tournaments is shown in Figs. 1, 2, 3.

Theorem 3. *Let T_0 be a non-acyclic tournament having a sink. Then no tournament T has the property that $N_T(v)\cong T_0$ for each vertex v of T.*

Remark. A sink of a directed graph is a vertex which is an initial vertex of no edge.

Proof. Let u be the sink of T_0. Suppose that the tournament T with the required property exists. Let v_1 be a vertex of T. Consider the tournament $N_T(v_1)$; let v_2 be the vertex of $N_T(v_1)$ which is the image of u in an isomorphic mapping of T_0 onto

$N_T(v_1)$. To the vertex v_2 the edges go from v_1 and from all vertices of $N_T(v_1)$ which are distinct from v_2. Thus the vertex set of $N_T(v_2)$ is disjoint with the vertex set of $N_T(v_1)$ and does not contain v_1. As the number of vertices of T is $2n+1$ (see Theorem 1), the vertex sets of $N_T(v_1)$ and $N_T(v_2)$ and the one-element set $\{v_1\}$ form a partition of the vertex set of T. Let v_3 be the vertex of $N_T(v_2)$ which is the image of u in an isomorphic mapping of T_0 onto $N_T(v_2)$. To the vertex v_3 the edges go from v_2 and from all vertices of $N_T(v_2)$ distinct from v_3. As the outdegree of v_3 must be n (see the proof of Theorem 1), the vertex set of $N_T(v_3)$ consists of v_1 and all vertices of $N_T(v_1)$ which are distinct from v_2. Hence $N_T(v_3)$ is isomorphic to the graph T_0' obtained from T_0 by reversing the orientations of all edges incident with

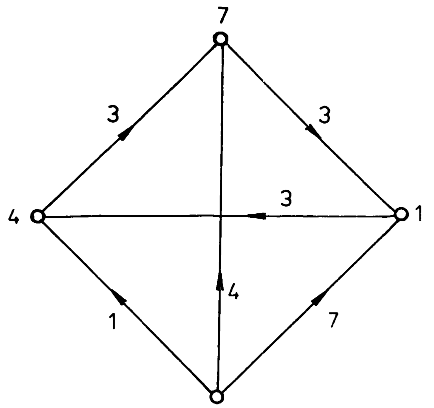

Fig. 2

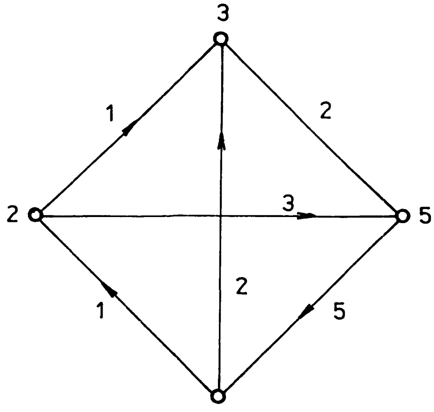

Fig. 3

u. For an integer $k \in \{0, 1, \ldots, n-1\}$ let $d(k)$ (or $d'(k)$) be the number of vertices which have the outdegree k in T_0 (or in T_0' respectively). Evidently for each $k \neq 0$ we have $d'(k-1) = d(k)$ and $d'(n-1) = 1$. If $T_0' \cong T_0$, then evidently $d'(k) = d(k)$ for any $k \in \{0, 1, \ldots, n-1\}$. This is possible only if $d(k) = 1$ for all $k \in \{0, 1, \ldots, n-1\}$. However in this case T_0 is acyclic, which is a contradiction with the assumption of the theorem.

REFERENCE

[1] ZYKOV, A. A.: Problem 30. In: Theory of graphs and its applications. Proc. Symp. Smolenice 1963 (ed. M. Fiedler), Prague 1964, 164—165.

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ТУРНИРЫ ОКРЕСТНОСТЕЙ

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Резюме

Если T есть турнир и v есть его вершина, тогда $N_T(v)$ есть подтурнир турнира T, порожденный множеством всех концевых вершин ребер, выходящих из v в T. Статья исследует турниры T_0, обладающее тем свойством, что существует турнир T такой, что $N_T(v) \cong T_0$ для каждой вершины v из T. Описан определенный класс турниров (содержащий также все ациклические турниры), все элементы которого обладают требуемым свойством. Доказано, что неациклический турнир, цодержащий вершину с полустепенью выхода равной нулю, не обладает требуемым свойством.