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Ivica Marinová

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NOTE ON SEMIGROUP VALUED MEASURES

IVICA MARINOVÁ

In paper [2] the extension of measures defined on an algebra with values in partially ordered semigroups to a generated σ-algebra is done by transfinite induction. This paper is concerned with the extension of semigroup valued measures whose domain is a ring. We differ from [2] also by omitting transfinite induction although some other assumption is added. However all examples in [2] fulfil this added assumption.

Let $\mathcal{R}$ be a ring of subsets of a nonempty set X. Let $\mathcal{P}$ be a partially ordered semigroup with a binary operation $\oplus$, partial ordering $\leqq$ and let $\theta \in \mathcal{P}$ be such that $\theta \leqq a$ for all $a \in \mathcal{P}$. We shall write

$$x_n \uparrow x \text{ iff } x_n \leqq x_{n+1},\ x_n, x \in \mathcal{P}\ (n=1, 2, \ldots) \text{ and } x = \sup_n x_n$$

$$y_n \downarrow y \text{ iff } y_{n+1} \leqq y_n,\ y_n, y \in \mathcal{P}\ (n=1, 2, \ldots) \text{ and } y = \inf_n y_n$$

$$z_n \to z \text{ iff } z_n, z \in \mathcal{P} \text{ and there are } u_n, v_n \in \mathcal{P}\ (n=1, 2, \ldots)$$
$$\text{such that } u_n \leqq z_n \leqq v_n\ (n=1, 2, \ldots) \text{ and } u_n \uparrow z,\ v_n \downarrow z$$

$$A_n \uparrow A \text{ iff } A_n \in \mathcal{R},\ A_n \subset A_{n+1}\ (n=1, 2, \ldots) \text{ and } \bigcup_{n=1}^{\infty} A_n = A$$

$$B_n \downarrow B \text{ iff } B_n \in \mathcal{R},\ B_{n+1} \subset B_n\ (n=1, 2, \ldots) \text{ and } \bigcap_{n=1}^{\infty} B_n = B.$$

We shall denote by $\mathcal{P}^<$ the set of all functionals f: $\mathcal{P} \to \langle 0, \infty)$ satisfying the following properties:

(a) $f(\theta) = 0$
(b) $a \leqq b$ implies $f(a) \leqq f(b)$ for all $a, b \in \mathcal{P}$
(c) $f(a \oplus b) \leqq f(a) + f(b)$ for all $a, b \in \mathcal{P}$
(d) $a_n \to a$ implies $\lim_{n\to\infty} f(a_n) = f(a)$ for all $a_n, a \in \mathcal{P}$ $(n=1, 2, \ldots)$

Troughout the paper we shall assume that the semigroup $\mathcal{P}$ has the following properties:

(i) $a \oplus \theta = a$ for all $a \in \mathcal{P}$

(ii) $a \oplus b = b \oplus a$ for all $a, b \in \mathcal{P}$

(iii) $a \leqq b$ implies $a \oplus c \leqq b \oplus c$ for all $a, b, c \in \mathcal{P}$

(iv) $\mathcal{P}$ is relatively σ-complete (i.e. every increasing (decreasing) bounded sequence in $\mathcal{P}$ has the supremum (the infimum) in $\mathcal{P}$)

(v) $a_n \to a$, $b_n \to b$ implies $a_n \oplus b_n \to a \oplus b$ for all $a_n, b_n, a, b \in \mathcal{P}$ $(n = 1, 2, \ldots)$

(vi) $\mathcal{P}$ is separative (i.e. if $a, b \in \mathcal{P}$, $a \neq b$, then there is $f \in \mathcal{P}^<$ such that $f(a) \neq f(b)$)

(vii) $f(x) \leqq f(y)$ for all $f \in \mathcal{P}^<$ implies $x \leqq y$ (this is the assumption mentioned at the beginning).

Let m: $\mathcal{R} \to \mathcal{P}$ be such a set function that:

(1) $A \subset B \cup C$ implies $m(A) \leqq m(B) \oplus m(C)$ for all $A, B, C \in \mathcal{R}$ (i.e. m is monotone and subadditive)

(2) $A_n \downarrow \emptyset$, $A_n \in \mathcal{R}$ $(n = 1, 2, \ldots)$ implies $m(A_n) \downarrow \theta$ (i.e. m is continuous from above in $\emptyset$)

(3) $A_n \subset A_{n+1}$, $A_n \in \mathcal{R}$ $(n = 1, 2, \ldots)$ implies $m(A_{n+1} - A_n) \to \theta$ (i.e. m is exhausting)

(4) the range of m is bounded.

We shall call such a function m a submeasure.

Observe that when $\mathcal{R}$ is an algebra (4) holds. Notice further that a submeasure is continuous (i.e. $A_n \uparrow A$ $(B_n \downarrow B)$ implies $m(A_n) \uparrow m(A)$ $(m(B_n) \downarrow m(B))$ for all $A_n, B_n, A, B \in \mathcal{R}$ $(n = 1, 2, \ldots)$ and that $m(\emptyset) = \theta$.

The exhaustivity is a necessary condition of extension of a monotone, continuous and subadditive function. We can see it in the following lemma.

Lemma 1. *Let $\mathcal{S}$ be a σ-ring. Let m: $\mathcal{S} \to \mathcal{P}$ be a monotone, continuous and subadditive function. Then m is exhausting.*

Proof. Let $A_n \in \mathcal{S}$ $(n = 1, 2, \ldots)$, $A_n \uparrow A$. Then $A \in \mathcal{S}$, $(A - A_n) \downarrow \emptyset$ and $m(A - A_n) \downarrow \theta \cdot (A_{n+1} - A_n) \subset (A - A_n)$ for $n = 1, 2, \ldots$ and this implies $m(A_{n+1} - A_n) \leqq m(A - A_n)$. Thus $m(A_{n+1} - A_n) \downarrow \theta$.

But the exhaustivity need not be fulfilled automatically on a ring as we can see in the following example.

Example. Let $X = \langle 0, \infty)$. Let $\mathcal{R}$ be a ring containing finite unions of intervals $\langle n, n+1)$, $n = 0, 1, 2, \ldots$, complements of these unions and the empty set. Let $\mathcal{P} = (\langle 0, 1 \rangle, \oplus)$, where $a \oplus b = \dfrac{a+b}{1+ab}$ for $a, b \in \mathcal{P}$. One can easily find out that $\mathcal{P}$ with the usual ordering of real numbers is a semigroup satisfying the properties (i)—(vii). Define a set function m: $\mathcal{R} \to \mathcal{P}$ as follows:

$$m(\emptyset) = 0$$

$$m(\langle n, n+1)) = \frac{1}{2} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

$m\left(\bigcup_{j=1}^{n} \langle i_j, i_j + 1)\right) = \bigoplus_{j=1}^{n} a_j$, where i_j is an integer and

$$a_j = \frac{1}{2} \quad \text{for} \quad j = 1, 2, \ldots, n$$

$m(A) = 1$ for $\emptyset \neq A \neq \bigcup_{j=1}^{n} \langle i_j, i_j + 1)$.

It is not hard to see that such a function m is monotone, continuous and subadditive. It is obvious that m is bounded. Now take a sequence $\{A_n\}_{n=1}^{\infty}$, $A_n = \langle 0, n)$ for $n = 1, 2, \ldots$. Clearly $A_n \subset A_{n+1}$, but $\lim_{n\to\infty} m(A_{n+1} - A_n) = \lim_{n\to\infty} m(\langle n, n+1)) = \frac{1}{2}$. Hence m is not exhausting on $\mathcal{R}$. Thus a function m is an example of a monotone, continuous and subadditive function which cannot be extended to a generated σ-ring.

The following lemma is a consequence of [1, page 217].

Lemma 2. *Let $\mathcal{R}$ be a ring of subsets of a nonempty set X. Let μ: $\mathcal{R} \to \langle 0, \infty)$ be monotone, subadditive and continuous from above in the empty set function satisfying a condition $\lim_{n\to\infty} \mu(A_{n+1} - A_n) = 0$ for all $A_n \in \mathcal{R}$, $A_n \subset A_{n+1}$ $(n = 1, 2, \ldots)$ such that $\lim_{n\to\infty} \mu(A_n) < \infty$. Let $\mathcal{S}(\mathcal{R})$ be a σ-ring generated by $\mathcal{R}$. Then there is a ring $\mathcal{R} \subset \mathcal{L} \subset \mathcal{S}(\mathcal{R})$ and a unique extension ν: $\mathcal{L} \to \langle 0, \infty)$ of μ such that ν is monotone, subadditive and continuous from above in the empty set on $\mathcal{L}$. Moreover $\mathcal{L}$ is closed in the following sense: if $A_n \uparrow A$ $(A_n \downarrow A)$, $A_n \in \mathcal{L}$ $(n = 1, 2, \ldots)$ and $\{\nu(A_n)\}_{n=1}^{\infty}$ is bounded, then $A \in \mathcal{L}$ and $\lim_{n\to\infty} \nu(A_n) = \nu(A)$.*

Remark 3. If the range of μ in Lemma 2 is bounded, so is the range of ν. Then $\mathcal{L}$ is a monotone class and hence $\mathcal{S}(\mathcal{R}) \subset \mathcal{L}$.

Let m: $\mathcal{R} \to \mathcal{P}$ be a submeasure, $f \in \mathcal{P}^{<}$. Since the range of m is bounded, so is the range of $f \circ m$. Now from Lemma 2 and Remark 3 the following lemma is clear.

Lemma 4. *Let m: $\mathcal{R} \to \mathcal{P}$ be a submeasure, $f \in \mathcal{P}^{<}$. then a function $f \circ m$: $\mathcal{R} \to \langle 0, \infty)$ has a unique extension $(f \circ m)_1$: $\mathcal{S}(\mathcal{R}) \to \langle 0, \infty)$ which is monotone, subadditive and continuous on $\mathcal{S}(\mathcal{R})$.*

Theorem 5. *Let $\mathcal{R}$ be a ring of subsets of a nonempty set X. Let $\mathcal{P}$ be a semigroup satisfying the conditions (i)—(vii). Let m: $\mathcal{R} \to \mathcal{P}$ be a submeasure. Then there exists a unique submeasure $\bar{m}$: $\mathcal{S}(\mathcal{R}) \to \mathcal{P}$ so that $\bar{m}/\mathcal{R} = m$.*

Proof: Let $\mathcal{Q}=\{q\colon \mathcal{P}^{<}\to\langle 0,\infty)\}$. Let us assign $\mathcal{Q}$ the partial ordering $\prec$ in the following way: $q_1\prec q_2$ iff $q_1(f)\leqq q_2(f)$ for all $f\in\mathcal{P}^{<}$, $q_1, q_2\in\mathcal{Q}$. We consider a pointwise convergence on $\mathcal{Q}$, i.e., $q_n\xrightarrow{\mathcal{Q}}q$ iff $\lim_{n\to\infty} q_n(f)=q(f)$ for all $f\in\mathcal{P}^{<}$, $q_n, q\in\mathcal{Q}$, $n=1,2,\ldots$. Let $\tau\colon \mathcal{P}\to\mathcal{Q}$ be a mapping defined in the following way: for all $a\in\mathcal{P}$, $\tau(a)=q_a$ where $q_a\colon \mathcal{P}^{<}\to\langle 0,\infty)$ is such a function that $q_a(f)=f(a)$ for all $f\in\mathcal{P}^{<}$. Using the separativity of $\mathcal{P}$ one has that for $a, b\in\mathcal{P}$, $a\neq b$ there exists $f\in\mathcal{P}^{<}$ such that $q_a(f)=f(a)\neq f(b)=q_b(f)$. Hence τ is an injective mapping. For $a, b\in\mathcal{P}$, $a\leqq b$ iff $f(a)\leqq f(b)$ for all $f\in\mathcal{P}^{<}$ iff $q_a(f)\leqq q_b(f)$ for all $f\in\mathcal{P}^{<}$ iff $q_a\prec q_b$. Hence $a\leqq b$ iff $\tau(a)\prec\tau(b)$, $a, b\in\mathcal{P}$. Let $E\in\mathcal{R}$. Then $m(E)\in\mathcal{P}$. We put $q^E=q_{m(E)}$. Obviously $q^E\in\tau(\mathcal{P})$ and for all $f\in\mathcal{P}^{<}$ $q^E(f)=q_{m(E)}(f)=f(m(E))=(f\circ m)_1(E)$, where $(f\circ m)_1$ is the unique extension of $f\circ m$ to a generated σ-ring from Lemma 4. For $E\in\mathcal{S}(\mathcal{R})$ we put $q^E(f)=(f\circ m)_1(E)$ for all $f\in\mathcal{P}^{<}$. Let us denote $\bar{m}(E)=\tau^{-1}(q^E)$. We shall show that for all $E\in\mathcal{S}(\mathcal{R})$, q^E is an element of $\tau(\mathcal{P})$. Let $\mathcal{K}=\{E\in\mathcal{S}(\mathcal{R})\colon q^E\in\tau(\mathcal{P})\}$. Obviously $\mathcal{K}\supset\mathcal{R}$. We shall show that $\mathcal{K}$ is a monotone class. Let $A_n\uparrow A$, $A_n\in\mathcal{K}$, $n=1,2,\ldots$. Then the $\lim_{n\to\infty}(f\circ m)_1(A_n)=(f\circ m)_1(A)$ for all $f\in\mathcal{P}^{<}$, that is the $\lim_{n\to\infty} q^{A_n}(f)=q^A(f)$ for all $f\in\mathcal{P}^{<}$, hence $q^{A_n}\xrightarrow{\mathcal{Q}}q^A$. For $n=1,2,\ldots$, $q^{A_n}\prec q^{A_{n+1}}$. If it is false, a functional $f\in\mathcal{P}^{<}$ would exist such that $q^{A_n}(f)>q^{A_{n+1}}(f)$, i.e. $(f\circ m)_1(A_n)>(f\circ m)_1(A_{n+1})$. This contradicts the monotonicity of $(f\circ m)_1$. Hence $\{\tau^{-1}(q^{A_n})\}_{n=1}^{\infty}$ is an increasing sequence in $\mathcal{P}$. Observe that it is bounded. By relative σ-completeness of $\mathcal{P}$ there exists a $\sup_n\{\tau^{-1}(q^{A_n})\}=a\in\mathcal{P}$. For all $f\in\mathcal{P}^{<}$ $q_a(f)=f(a)=\lim_{n\to\infty} q^{A_n}(f)$, hence $q^{A_n}\xrightarrow{\mathcal{Q}}q_a$. It follows that $q^A=q_a\in\tau(\mathcal{P})$ and hence $A\in\mathcal{K}$. Further $\tau^{-1}(q^A)=\tau^{-1}(q_a)=a=\sup_n\{\tau^{-1}(q^{A_n})\}$ and so $\bar{m}(A)=\sup_n\bar{m}(A_n)$ in $\mathcal{P}$. In the same way one can prove that if $B_n\downarrow B$, $B_n\in\mathcal{K}$ $(n=1,2,\ldots)$, then $B\in\mathcal{K}$ and $\bar{m}(B)=\inf_n\bar{m}(B_n)$. Hence $\mathcal{S}(\mathcal{R})\subset\mathcal{K}$ and $\bar{m}$ is a continuous set function on $\mathcal{S}(\mathcal{R})$. Obviously $\bar{m}(E)=m(E)$ for all $E\in\mathcal{R}$. $\bar{m}$ is monotone on $\mathcal{S}(\mathcal{R})$ because if $A, B\in\mathcal{S}(\mathcal{R})$, $A\subset B$, then $(f\circ m)_1(A)\leqq(f\circ m)_1(B)$ for all $f\in\mathcal{P}^{<}$, that is iff $q^A(f)\leqq q^B(f)$ for all $f\in\mathcal{P}^{<}$ iff $q^A\prec q^B$ iff $\tau^{-1}(q^A)\leqq\tau^{-1}(q^B)$ iff $\bar{m}(A)\leqq\bar{m}(B)$. Now we shall claim the subadditivity of $\bar{m}$. Let us denote $\mathcal{L}_1=\{A\in\mathcal{S}(\mathcal{R})\colon \bar{m}(A\cup B)\leqq\bar{m}(A)\oplus\bar{m}(B)$ for all $B\in\mathcal{R}\}$. Obviously $\mathcal{R}\subset\mathcal{L}_1$. We shall show that $\mathcal{L}_1$ is a monotone class. If $A_n\in\mathcal{L}_1$, $n=1,2,\ldots$, $A_n\uparrow A(A_n\downarrow A)$, then for all $B\in\mathcal{R}$ $A_n\cup B\uparrow A\cup B$ $(A_n\cup B\downarrow\downarrow A\cup B)$. By continuity of $\bar{m}$ on $\mathcal{S}(\mathcal{R})$ one has

$$\bar{m}(A\cup B)=\sup_n\bar{m}(A_n\cup B)\leqq\sup_n(\bar{m}(A_n)\oplus\bar{m}(B))=\bar{m}(A)\oplus\bar{m}(B)$$

$$(\bar{m}(A \cup B) = \inf_n \bar{m}(A_n \cup B) \leqq \inf_n (\bar{m}(A_n) \oplus \bar{m}(B)) = \bar{m}(A) \oplus \bar{m}(B)).$$

Hence $\mathcal{S}(\mathcal{R}) \subset \mathcal{L}_1$. Further let us denote $\mathcal{L}_2 = \{A \in \mathcal{S}(\mathcal{R}): \bar{m}(A \cup B) \leqq \bar{m}(A) \oplus \bar{m}(B)$ for all $B \in \mathcal{S}(\mathcal{R})\}$. Then $\mathcal{R} \subset \mathcal{L}_2$. In the same way as for $\mathcal{L}_1$ one can prove that $\mathcal{L}_2$ is a monotone class. Then $\mathcal{S}(\mathcal{R}) \subset \mathcal{L}_2$ and hence the subadditivity of $\bar{m}$.

There remains to be proved the uniqueness of such an extension $\bar{m}$. Let $m_1: \mathcal{S}(\mathcal{R}) \to \mathcal{P}$, $m_2: \mathcal{S}(\mathcal{R}) \to \mathcal{P}$ be such submeasures that $m_1(E) = m_2(E) = m(E)$ for all $E \in \mathcal{R}$. Let $\mathcal{A}$ be a class of all sets $E \in \mathcal{S}(\mathcal{R})$ such that $m_1(E) = m_2(E)$. It will suffice to show that $\mathcal{S}(\mathcal{R}) \subset \mathcal{A}$. But this is clear since by continuity of m_1, m_2 $\mathcal{A}$ is a monotone class. Hence the theorem is proved.

If a submeasure $m: \mathcal{R} \to \mathcal{P}$ is additive, i.e. $m(A \cup B) \oplus m(A \cap B) = m(A) \oplus m(B)$ for all $A, B \in \mathcal{R}$, we shall call it a measure.

Theorem 6. *Let $\mathcal{R}$ be an arbitrary ring of subsets of $X \neq \emptyset$. Let $\mathcal{P}$ be a semigroup satisfying the properties* (i)—(vii). *Let $m: \mathcal{R} \to \mathcal{P}$ be a measure. Then there exists a unique measure $\bar{m}: \mathcal{S}(\mathcal{R}) \to \mathcal{P}$ extending m.*

Proof. From the preceding we know that a submeasure $\bar{m}: \mathcal{S}(\mathcal{R}) \to \mathcal{P}$ exists such that $\bar{m}/\mathcal{R} = m$. It suffices to show that $\bar{m}$ is additive. Denote $\mathcal{M}_1 = \{A \in \mathcal{S}(\mathcal{R}), \bar{m}(A \cup B) \oplus \bar{m}(A \cap B) = \bar{m}(A) \oplus \bar{m}(B)$ for each $B \in \mathcal{R}\}$. We shall show that $\mathcal{M}_1$ is a monotone class. Let $A_n \uparrow A$, $A_n \in \mathcal{M}_1$ $(n = 1, 2, \ldots)$. Then $\bar{m}(A) \oplus \bar{m}(B) =$

$$(\sup_n \bar{m}(A_n)) \oplus \bar{m}(B) = \sup_n (\bar{m}(A_n) \oplus \bar{m}(B)) =$$

$$\sup_n (\bar{m}(A_n \cup B) \oplus \bar{m}(A_n \cap B)) = \sup_n \bar{m}(A_n \cup B) \oplus \sup_n \bar{m}(A_n \cap B) =$$

$$= \bar{m}(A \cup B) \oplus \bar{m}(A \cap B).$$ Let $A_n \downarrow A$, $A_n \in \mathcal{M}_1$ $(n = 1, 2, \ldots)$. Then

$$\bar{m}(A) \oplus \bar{m}(B) = (\inf_n \bar{m}(A_n)) \oplus \bar{m}(B) = \inf_n (\bar{m}(A_n) \oplus \bar{m}(B)) =$$

$$= \inf_n (\bar{m}(A_n \cup B) \oplus \bar{m}(A_n \cap B)) = \inf_n \bar{m}(A_n \cup B) \oplus \inf_n \bar{m}(A_n \cap B) =$$

$= \bar{m}(A \cup B) \oplus \bar{m}(A \cap B)$. Hence $\mathcal{M}_1$ is a monotone class evidently containing $\mathcal{R}$ and so we have $\mathcal{S}(\mathcal{R}) \subset \mathcal{M}_1$. Further denote $\mathcal{M}_2 = \{A \in \mathcal{S}(\mathcal{R}), \bar{m}(A \cup B) \oplus \bar{m}(A \cap B) = \bar{m}(A) \oplus \bar{m}(B)$ for each $B \in \mathcal{S}(\mathcal{R})\}$. Then $\mathcal{R} \subset \mathcal{M}_2$. In the same way as for $\mathcal{M}_1$ one can prove that $\mathcal{M}_2$ is a monotone class. Then $\mathcal{S}(\mathcal{R}) \subset \mathcal{M}_2$ and the additivity of $\bar{m}$ is proved.

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Katedra matematiky
Elektrotechnickej fakulty SVŠT
Gottwaldovo nám. 19
812 19 Bratislava

ЗАМЕТКА О МЕРАХ С ЗНАЧЕНИЯМИ В ПОЛУГРУППАХ

Ivica Marinová

Резюме

В статье идет речь о разширении мер, определенных на кольце $\mathcal{R}$ подмножеств непустого множества X, с значениями в некоторых частично упорядоченных полугруппах на наименьшее σ-кольцо над $\mathcal{R}$.

В теореме 5 доказано, что монотонная, полуаддитивная, непрерывная функция $m: \mathcal{R} \to \mathcal{P}$ ($\mathcal{P}$ обозначает полугруппу, удовлетворяющую некоторым свойствам), для которой из $A_n \in \mathcal{R}$, $A_n \subset A_{n+1}$ $(n = 1, 2, \dots)$ следует $m(A_{n+1} - A_n) \to \theta$, имеет однозначное разширение на наименьшее σ-кольцо над $\mathcal{R}$. В работе показан и пример монотонной, полуаддитивной, непрерывной функции, которую невозможно разширить.