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1. Introduction

For a given real number α, $0<\alpha<2$ and a positive integer n let $G(n, \alpha)$ denote the graph whose vertices are points of the unit sphere $S_{n-1}=\{x=(x_1, x_2, \ldots, x_n)\in R^n\,;\, x_1^2+x_2^2+\ldots+x_n^2=1\}$ in the real n-dimensional space R^n, and two points of S_{n-1} are joined by an edge of $G(n, \alpha)$ if and only if their distance is α.

P. Erdős (see, e.g., Problems of the Sixth Hungarian Colloquium on Combinatorics, Eger, July 1981) asked whether the chromatic number $\chi(G(n, \alpha))$ of the graph $G(n, \alpha)$ tends to infinity while $n\to\infty$. (Since each S_{n-1} is a compact metric space, $\chi(G(n, \alpha))$ is always a finite number, in contrast to the fact that $G(n, \alpha)$ has uncountably many vertices.)

Our aim is to show that the answer to the above question is affirmative for all α, $0<\alpha<2$.

Theorem 1. *For any α, $0<\alpha<2$ we have $\lim_{n\to\infty}\chi(G(n, \alpha))=\infty$.*

The proof of Theorem 1 will be given in Sections 2 and 3, where the cases $0<\alpha\leqq\sqrt{2}$ and $\sqrt{2}<\alpha<2$ are handled separately. While in the first case it is relatively easy to prove that $\chi(G(n, \alpha))\geqq n$, the second requires to use the Kneser graphs $K_{2r+k}^{(r)}$ for a suitable family of numbers r, k to prove that $\chi(G(n, \alpha))\to\infty$ as $n\to\infty$. Recall that the Kneser graph $K_t^{(r)}$ has the vertex set $T^{(r)}$, the set of all r-subsets of a t-element set T, and two r-sets of T are joined in $K_t^{(r)}$ if and only if they are disjoint.

For all undefined concepts the reader is referred to Bollobás [1].

2. The case $0<\alpha\leqq\sqrt{2}$

Proposition 1. *If $0<\alpha\leqq\sqrt{2}$, then $\lim_{n\to\infty}\chi(G(n, \alpha))=\infty$.*

Proof. Let T_n be an n-dimensional simplex, $n\geqq 1$ such that the distance between any two of its $n+1$ vertices is α, $0<\alpha\leqq\sqrt{2}$. It is well known from the elementary geometry that the circumscribed $(n-1)$-sphere $S(T_n)$ of the simplex T_n has the radius r_n, where

$$r_n=\frac{\alpha}{\sqrt{2}}\sqrt{\frac{n}{n+1}}\,.$$

Obviously $r_n<1$ for each n and $0<\alpha\leqq\sqrt{2}$.

Without loss of generality we may suppose that the centre of $S(T_n)$ is the point $0=(0, 0, \ldots, 0)\in R^n$. Thus, each vertex $x_i=(x_{i,1}, x_{i,2}, \ldots, x_{i,n})$ of T_n, $1\leqq i\leqq n+1$ satisfies

$$x_{i,1}^2+x_{i,2}^2+\ldots+x_{i,n}^2=r_n^2.$$

Now let us assign to each x_i a point $f(x_i)\in R^{n+1}$, $f(x_i)=(x_{i,1}, x_{i,2}, \ldots, x_{i,n}, y_{n+1})$, where $y_{n+1}=\sqrt{1-r_n^2}$. It is easily seen that $f(x_i)\in S_n$ for each i, $1\leqq i\leqq n+1$. Moreover, for $i\neq j$ the distance between $f(x_i)$ and $f(x_j)$ in R^{n+1} is the same as the distance between x_i and x_j in R^n, namely, α. Therefore the set $\{f(x_i); 1\leqq i\leqq n+1\}$ induces a complete subgraph on $n+1$ vertices of $G(n+1, \alpha)$. Consequently, $\chi(G(n+1, \alpha))\geqq n+1$, whence

$$\lim_{n\to\infty}\chi(G(n, \alpha))=\infty$$

for $0<\alpha\leqq\sqrt{2}$, as desired.

3. The case $\sqrt{2}<\alpha<2$

Lemma 1. *Let $\sqrt{2}<\alpha<2$. There is a positive rational number m and a real number c such that*

$$(\alpha^2-2)c^2+4c+\alpha^2(1+m)=2. \tag{1}$$

Proof. Choose a positive rational m for which

$$(\alpha^2-2)m\leqq 4-\alpha^2;$$

this is possible because $\sqrt{2}<\alpha<2$. For the discriminant of (1) with the unknown c we then obtain

$$D=-4\alpha^2[(\alpha^2-2)m+\alpha^2-4]\geqq 0.$$

Lemma 1 follows.

Lemma 2. *Let $\sqrt{2} < \alpha < 2$. There are positive integers p, q and a real c such that for each positive integer t there is a real b_t satisfying both* (2) *and* (3):

$$tb_t^2(qc^2 + q + p) = 1, \tag{2}$$

$$2tqb_t^2(c-1)^2 = \alpha^2. \tag{3}$$

Proof. Let $m = p/q > 0$ and c satisfy (1). Then $c \neq 1$. Modifying (1) we easily obtain

$$\frac{c^2 + 1 + m}{2(c-1)^2} = \frac{1}{\alpha^2}. \tag{4}$$

From (4) for each positive integer t we have

$$\frac{t(qc^2 + q + p)}{2tq(c-1)^2} = \frac{1}{\alpha^2}. \tag{5}$$

It follows from (5) that putting

$$b_t^{-1} = \sqrt{t(qc^2 + q + p)}$$

we obtain the desired b_t satisfying both (2) and (3).

Proposition 2. *If $\sqrt{2} < \alpha < 2$, then $\lim_{n \to \infty} \chi(G(n, \alpha)) = \infty$.*

Proof. Let p, q, c, t and b_t be numbers as in Lemma 2 satisfying (2) and (3). Put $a_t = cb_t$, $tp = k$, $tq = r$, and $n = 2r + k$. Let M_t be the set of all ordered n-tuples composed of two numbers a_t, b_t such that a_t occurs in each n-tuple exactly r times. Clearly $M_t \subseteq R^n$ and $|M_t| = \binom{n}{r}$.

Choose a point $x = (x_1, x_2, \dots, x_n) \in M_t$. Then, according to (2) and the above relations,

$$x_1^2 + x_2^2 + \dots + x_n^2 = ra_t^2 + (r+k)b_t^2 = tb_t^2(qc^2 + q + p) = 1,$$

whence $M_t \subseteq S_{n-1}$. Further, if $y = (y_1, y_2, \dots, y_n) \in M_t$ such that $x_i = a_t$ implies $y_i = b_t$, $1 \leqq i \leqq n$, then, following (3), the distance $d(x, y)$ between x and y satisfies

$$\begin{aligned} d^2(x, y) &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2 = \\ &= 2r(a_t - b_t)^2 = 2tq(c-1)^2 b_t^2 = \alpha^2. \end{aligned}$$

Combining these facts we conclude that the subgraph H of $G(n, \alpha)$ induced by the set M_t contains a copy of the Kneser graph $K_{2r+k}^{(r)}$. Since $\chi(K_{2r+k}^{(r)}) = k + 2$ (cf. [1, p. 260, Theorem 4.4]), it follows that $\chi(G(n, \alpha)) \geqq \chi(H) \geqq k + 2$, or $\chi(G((2q + p)t, \alpha)) \geqq tp + 2$ for each positive integer t and $\sqrt{2} < \alpha < 2$. The proof of Proposition 2 (as well as that of Theorem 1) is complete.

4. Concluding remarks

Our proof of Theorem 1 yields the lower bound $\chi(G(n, \alpha)) \geqq c(\alpha)n$ with $c(\alpha) > 0$, $c(\alpha) = 1$ for $0 < \alpha \leqq \sqrt{2}$ and $\lim_{\alpha \to 2^-} c(\alpha) = 0$. Perhaps it is possible to show that $\chi(G(n, \alpha)) \geqq cn$ for an absolute constant $c > 0$, but we did not succeed in obtaining results along this line (i.e. bounds uniform in α).

Added in proof: The same problem has been solved independently by V. Rödl (to appear in Discrete Math.).

REFERENCES

[1] B. BOLLOBÁS: Extremal Graph Theory, Academic Press, London—New York—San Francisco, 1978.

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ОБ ОДНОЙ ПРОБЛЕМЕ П. ЭРДЕША

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Резюме

Пусть $G(n, \alpha)$ — граф, вершины которого суть точки единичной сферы в евклидовом пространстве размерности n, и две вершины соединены ребром в том случае, когда их расстояние равно α. В статье доказано, что

$$\lim_{n \to \infty} \chi(G(n, \alpha)) = \infty$$

для всех α, $0 < \alpha < 2$, где $\chi(G(n, \alpha))$ — хроматическое число графа $G(n, \alpha)$.