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Mathematica Slovaca, Vol. 34 (1984), No. 1, 97--102

Persistent URL: http://dml.cz/dmlcz/136350

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TRANSFERABLE PRINCIPIAL CONGRUENCES AND REGULAR ALGEBRAS

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A variety $\mathcal{V}$ is *regular* if any two congruences on each $\mathfrak{A} \in \mathcal{V}$ coincide whenever they have a congruence class in common. B. Csákány [2] gave the following characterization of regular varieties:

Theorem A. *For a variety $\mathcal{V}$, the following conditions are equivalent:*
(1) *$\mathcal{V}$ is regular;*
(2) *there exist ternary polynomials $p_1, \ldots, p_n$ over $\mathcal{V}$ such that*

$(p_1(x, y, z) = z$ *and* $\ldots$ *and* $p_n(x, y, z) = z)$ *if and only if* $x = y$.

If we apply this theorem in the case of known regular varieties: *groups, quasigroups, rings, modules, Boolean algebras,* etc., we have $n = 1$ in the condition (2) above. Since all of the quoted variaties are *permutable* ones, we can ask about the dependence of these properties. The aim of this paper is to characterize such varieties with $n = 1$ in (2).

Definition. *An algebra $\mathfrak{A}$ has Transferable Principal Congruences* (briefly TPC) if for any elements a, b, c of $\mathfrak{A}$ there exists an element d of $\mathfrak{A}$ such that $\theta(a, b) = \theta(c, d)$. *A variety $\mathcal{V}$ has* TPC if each $\mathfrak{A} \in \mathcal{V}$ has TPC.

Theorem 1. *Let $\mathcal{V}$ be a variety. The following conditions are equivalent:*
(1) *$\mathcal{V}$ has TPC;*
(2) *there exist a ternary polynomial p and 5-ary polynomials $q_1, \ldots, q_m$ such that*

$$p(x, x, z) = z$$

$$q_1(z, p(x, y, z), x, y, z) = x$$

$$q_m(p(x, y, z), z, x, y, z) = y$$

$$q_{j-1}(p(x, y, z), z, x, y, z) = q_j(z, p(x, y, z), x, y, z) \quad \textit{for } j = 2, \ldots, m.$$

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{V}$ have TPC and let $F_3(x, y, z)$ be a free algebra of $\mathcal{V}$ with free generators x, y, z. Then there exists $w \in F_3(x, y, z)$ such that

$$\theta(x, y) = \theta(z, w).$$

Then $w=p(x, y, z)$ for some ternary polynomial p and $x=y$ implies $z=w$, i.e. $p(x, x, z)=z$.

We have $\langle x, y\rangle \in \theta(z, w)$, and according to [1] there exist binary algebraic functions $\varphi_1, \ldots, \varphi_m$ such that

$$x=\varphi_1(z, w)$$

$$\varphi_{j-1}(w, z)=\varphi_j(z, w) \quad (j=2, \ldots, m)$$

$$y=\varphi_m(w, z).$$

Therefore, there exist 5-ary polynomials $q_1, \ldots, q_m$ with $q_j(u, v, x, y, z)=\varphi_j(u, v)$ and (2) is proved.

(2) $\Rightarrow$ (1): Let $\mathfrak{A} \in \mathcal{V}$ and a, b, c be elements of $\mathfrak{A}$. Put $d=p(a, b, c)$ and prove $\theta(a, b)=\theta(c, d)$. Clearly

$$\langle c, d\rangle = \langle p(a, a, c), p(a, b, c)\rangle \in \theta(a, b).$$

Conversely, (2) implies

$$a=q_1(c, d, a, b, c), \quad b=q_m(d, c, b, c) \quad \text{and}$$

$$q_{j-1}(d, c, a, b, c)=q_j(c, d, a, b, c) \quad \text{for} \quad j=2, \ldots, m,$$

i.e. $\langle a, b\rangle \in \theta(c, d)$ proving $\theta(a, b)=\theta(c, d)$.

Example. For *groups*, we can put $m=1$ and

$$p(x, y, z)=x \,.\, y^{-1} \,.\, z$$

$$q_1(v, w, x, y, z)=w \,.\, z^{-1} \,.\, y.$$

For *Boolean algebras*, we have $m=1$ and

$$p(x, y, z)=x \oplus y \oplus z$$

$$q_1(v, w, x, y, z)=w \oplus y \oplus z,$$

where

$$a \oplus b=(a' \wedge b) \vee (a \wedge b').$$

It can be a reasonable conjecture that $m=1$ in (2) of Theorem 1 if $\mathcal{V}$ is permutable. However, also the converse assertion is valid:

Theorem 2. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) *$\mathcal{V}$ is permutable and has TPC;*

(2) *there exist a 3-ary polynomial p and 4-ary polynomial q such that*

$$p(x, x, z)=z$$

$$q(z, x, y, z) = x$$

$$q(p(x, y, z), x, y, z) = y.$$

Proof. (1) $\Rightarrow$ (2): Denote by $D(a, b)$ the least diagonal subalgebra of $F_3(x, y, z) \times F_3(x, y, z)$ containing the pair $\langle a, b \rangle$ of elements a, b of $F_3(x, y, z)$, where $F_3(x, y, z)$ is a free alebra of $\mathcal{V}$ with free generators x, y, z. By the Theorem of Werner [4], $\theta(a, b) = D(a, b)$ for each a, b of $F_3(x, y, z)$ because of the permutability of $\mathcal{V}$. Since $\mathcal{V}$. Since $\mathcal{V}$ has TPC, there exists $w \in F_3(x, y, z)$ such that

$$D(x, y) = D(z, w),$$

i.e. $\langle x, y \rangle \in D(z, w)$. It implies that there exists a unary algebraic function φ over $F_3(x, y, z)$ such that

$$x = \varphi(z), \quad y = \varphi(w),$$

i.e. $x = q(z, x, y, z)$, $y = q(w, x, y, z)$ for some 4-ary polynomial q. Analogously to the proof of Theorem 1, we have $w = p(x, y, z)$ with $p(x, x, z) = z$.

(2) $\Rightarrow$ (1): By Theorem 1, (2) implies TPC. Put

$$t(x, y, z) = q(p(y, z, y), x, z, y).$$

Then clearly $t(x, x, z) = z$ and $t(x, z, z) = x$ proving the permutability of $\mathcal{V}$.

Theorem 3. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) *$\mathcal{V}$ has TPC;*

(2) *there exists a ternary polynomial p such that*

$$p(x, y, z) = z \text{ if and only if } x = y.$$

Proof. (1) $\Rightarrow$ (2): By Theorem 1, there exists a ternary polynomial p with $p(x, x, z) = z$. Further, if $p(x, y, z) = z$, then (2) of Theorem 1 implies

$$x = q_1(z, z, x, y, z) = \ldots = q_m(z, z, y, z) = y$$

proving (2).

(2) $\Rightarrow$ (1): Let $\mathfrak{A} \in \mathcal{V}$ and a, b, c be elements of $\mathfrak{A}$. Put $d = p(a, b, c)$. Then clearly $\langle c, d \rangle \in \theta(a, b)$, i.e.

$$\theta(c, d) \subseteq \theta(a, b).$$

In the factor algebra $\mathfrak{A}/\theta(c, d)$ we have

$$[c]_{\theta(c,d)} = [d]_{\theta(c,d)} = [p, (a, b, c)]_{\theta(c,d)} =$$

$$= p([a]_{\theta(c,d)}, [b]_{\theta(c,d)}, [c]_{\theta(c,d)}).$$

Since $\mathfrak{A}/\theta(c, d) \in \mathcal{V}$, it implies

$$[a]_{\theta(c,d)} = [b]_{\theta(c,d)}$$

proving the converse inclusion $\theta(a, b) \subseteq \theta(c, d)$.

Corollary. *If a variety $\mathcal{V}$ has TPC, it is regular.*

The concept of regularity can be weakened in the case of varieties with nullary operations, see [3]:

A variety $\mathcal{V}$ with nullary operation 0 is *weakly regular* (*with respect to* 0) if every congruence θ on $\mathfrak{A} \in \mathcal{V}$ is uniquely determined by its congruence class $[0]_\theta$.

As mentioned in [3] if we tend from regularity to weak regularity (with respect to 0), we need only to replace z by 0 in the conditions and proofs. Hence the Theorem A (B. Csákány [2]) gives immediately:

Theorem B. *Let $\mathcal{V}$ be a variety with nullary operation* 0. *The following conditions are equivalent:*

(1) *$\mathcal{V}$ is weakly regular;*

(2) *there exist binary polynomials $b_1, \ldots, b_n$ such that ($b_1(x, y) = 0$ and ... and $b_n(x, y) = 0$) if and only if $x = y$.*

We have $n = 1$ in (2) of Theorem B for some "nice" varieties. E.g. for *groups* or *Boolean algebras* we have $n = 1$ and $b_1(x, y) = x - y$ or $b_1(x, y) = x \oplus y$, respectively. On the contrary, there are weakly regular "nice" varieties with $n > 1$, e.g. the *variety of all implicative semilattices.* Recall that it is a variety $\mathcal{V}$ with one nullary operation 1 and one binary operation (denoted by juxtaposition) fulfilling the axioms:

$$\begin{aligned} (ab)a &= a \\ a(bc) &= b(ac) \\ (ab)b &= (ba)a \\ aa &= 1 . \end{aligned}$$

In this case we have

$$(ab = 1 \text{ and } ba = 1) \text{ if and only if } a = b ,$$

thus $\mathcal{V}$ is weakly regular but $n = 2$. One can easily see on $F_2(a, b) \in \mathcal{V}$, that there cannot be $n = 1$. Thus it is also reasonable to ask when $n = 1$. Introduce:

An algebra $\mathfrak{A}$ with nullary operation 0 has 0-*Transferable Principal Congruences* (briefly 0-TPC) if for each a, b of $\mathfrak{A}$ there exists c of $\mathfrak{A}$ such that $\theta(a, b) = \theta(0, c)$. A variety $\mathcal{V}$ with nullary operation 0 has 0-TPC if each $\mathfrak{A} \in \mathcal{V}$ has 0-TPC.

The proofs of the following theorems are quite similar to those of Theorems 1, 2, 3 and hence omitted:

Theorem 4. *For a variety $\mathcal{V}$ with nullary operation* 0, *the following conditions are equivalent:*

(1) *$\mathcal{V}$ has 0-TPC;*

(2) *there exist binary polynomial b and 4-ary polynomials $r_1, \ldots, r_m$ such that*

$$b(x, x)=0$$

$$r_1(0, b(x, y), x, y)=x$$

$$r_m(b(x, y), 0, x, y)=y$$

$$r_{j-1}(b(x, y), 0, x, y)=r_j(0, b(x, y), x, y) \quad \text{for} \quad j=2, \ldots, m.$$

Theorem 5. *For a variety $\mathcal{V}$ with nullary operation* 0, *the following conditions are equivalent:*

(1) *$\mathcal{V}$ is permutable and has 0-TPC;*

(2) *there exist a binary polynomial b and ternary polynomial t such that*

$$b(x, x)=0$$

$$t(0, x, y)=x$$

$$t(b(x, y), x, y)=y.$$

Theorem 6. *For a variety $\mathcal{V}$ with nullary operation* 0, *the following conditions are equivalent:*

(1) *$\mathcal{V}$ has 0-TPC;*

(2) *there exists a binary polynomial b such that*

$$b(x, y)=0 \quad \text{if and only if} \quad x=y.$$

Corollary. *If a variety $\mathcal{V}$ with nullary operation* 0 *has 0-TPC, it is weakly regular (with respect to* 0).

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Received January 7, 1982

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ПЕРЕМЕСТИТЕЬНЫЕ ГЛАВНЫЕ КОНГРУЭНЦИИ И РЕГУЛЯРНЫЕ АЛГЕБРЫ

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Резюме

Многообразие $\mathcal{V}$ имеет Переместительные главные конгруэнции, если для каждои алгебры $\mathfrak{A} \in \mathcal{V}$ и любых элементов a, b, $c \in \mathfrak{A}$ существует элемент $d \in \mathfrak{A}$ такой, что $\theta(a, b) = \theta(c, d)$ Такие многообразия, очевидно, регулярны. Мы даем условие Мальцева, характеризирующее многообразие с Переместительными главными конгруэнциямы и тоже специальные случаи и обобщения таких многообразий.