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FORMAL INTEGRATION IN COMPOSITION RINGS

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1. Introduction

Let $(A;+,\cdot,\circ)$ be a composition ring in the sense of Lausch and Nöbauer [1], that means, $(A;+,\cdot)$ is a ring, $(A;\circ)$ is a semigroup and there hold both right distributive laws $(x+y)\circ z=(x\circ z)+(y\circ z)$ and $(x\cdot y)\circ z=(x\circ z)\cdot(y\circ z)$ for all x, $y\in A$.

In composition rings A there can be defined a formal differential operator D: $A\to A$ by

(S) $$D(f+g)=D(f)+D(g),$$
(P) $$D(f\cdot g)=D(f)\cdot g+f\cdot D(g),$$
(C) $$D(f\circ g)=(D(f)\circ g)\cdot D(g)$$

for all f, $g\in A$ (cf. [2], [4]). By this definition in composition rings of functions a good characterization of the formal differentiation is obtained (cf. [3], [5]).

So far the problem of defining a formal integration operator for algebraic structures has not been investigated very much. In this paper a formal integration operator for composition rings is defined and studied in some well-known composition rings.

2. Definition and basic properties of formal integrations

An element c of a composition ring A is called a constant if $c\circ 0=0$ for the zero element 0 of A. From $0\circ 0=(0+0)\circ 0=(0\circ 0)+(0\circ 0)$ we deduce that 0 itself is always a constant of A. It can be shown easily that the set of all constants of A forms a subcomposition ring A_C of A. Since we are interested mainly in composition rings of functions, we assume in this paper that a possible identity 1 of the ring $(A;+,\cdot)$ also is a constant.

A formal integration I of A is defined as mapping I: $A\to A$, which satisfies for all g, $g\in A$ the following conditions:

(i) $$I(f+g)=I(f)+I(g),$$
(ii) $$I(c\cdot f)=c\cdot I(f), \quad c\in A_C,$$
(iii) $$I(I(f)\cdot g)=I(f)\cdot I(g)-I(f\cdot I(g)),$$
(iv) $$I((f\circ I(g))\cdot g)=I(f)\circ I(g).$$

As it can be seen, (iii) is an abstraction of the rule of partial integration and (iv) of the rule of integration for composed functions of analysis.

In each composition ring A there exists one trivial formal integration, namely, the zero mapping, which maps every element of A on the zero element 0.

Now we prove some basic properties of formal integrations.

Using (i) we have $I(0)=I(0+0)=I(0)+I(0)$. Therefore there holds

$$I(0)=0 \tag{1}$$

for any formal integration I.

For the identity 1 we deduce from (iv)

$$I(g)=I((1\circ I(g))\cdot g)=I(1)\circ I(g) \quad \text{for all} \quad g\in A \tag{2}$$

and in particular

$$I(1)=I(1)\circ I(1). \tag{3}$$

Lemma 1. *If $I(1)\in A_C$, then I is the zero mapping.*

Proof. $I(1)\in A_C$ implicates $I(1)\circ g=I(1)$ for all $g\in A$. Hence we obtain from (1) and (2) $0=I(0)=I(1)\circ I(0)=I(1)$ and $I(g)=0\circ I(g)=0$ for all $g\in A$.

The set of constants A_C is said to form a base of A, if for a, $b\in A$ from $a\circ c=b\circ c$ for all $c\in A_C$ there follows $a=b$.

Lemma 2. *If A_C forms a base of A and $(A_C\setminus\{0\};\cdot)$ is a group, then there exist non-trivial formal integrations I of A only if $(A;\circ)$ has a left neutral element x. If such an element x exists, then $I(1)=x$.*

Proof. Suppose $I(1)\notin A_C$. Then there exists at least one $a\in A_C$ such that $I(1)\circ a=b\neq 0$, $b\in A_C$. Therefore we have $I\left(\frac{c}{b}\right)\circ a = \left(\frac{c}{b}\cdot I(1)\right)\circ a=c$ for all $c\in A_C$. Hence $I\left(\frac{c}{b}\right)=I(1)\circ I\left(\frac{c}{b}\right)$ implies

$$c=I(1)\circ c \quad \text{for all} \quad c\in A_C. \tag{4}$$

As A_C forms a base of A, the element $I(1)$ is already uniquely determined by (4). If x is the left neutral element of $(A;\circ)$, then we have obviously $I(1)=x$.

For $g\in A$ let there be $I^2(g):=I(I(g))$ and recursively $I^n(g):=I(I^{n-1}(1))$, $n\in\mathbf{N}$, $n>1$. We write $n!:=1\cdot 2\ldots n$.

Lemma 3. *There holds $(n!)I^n(1)=(I(1))^n$.*

Proof. From (iii) we deduce $I(I(1)\cdot 1) = I(1)\cdot I(1) - I(1\cdot I(1))$. Hence we have $2I^2(1)=(I(1))^2$ and the assumption is true for $n=2$. Now we suppose

$$(m!)I^m(1)=(I(1))^m \quad \text{for} \quad 2\leqslant m<n. \tag{5}$$

Using (ii), (iii) and (5) we obtain $I(((n-1)!)I^{n-1}(1)\cdot 1) = ((n-1)!)I^{n-1}(1)\cdot I(1) - I(((n-1)!I^{n-2}(1)\cdot I(1))$, $((n-1)!)I^n(1) = (I(1))^n - I((n-1)(I(1))^{n-1})$ and finally $((n-1)!)I^n(1) + (n-1)((n-1)!)I^n(1) = (I(1))^n$, which completes the proof.

Now we determine all formal integrations in some composition rings.

3. The polynomial ring $K[x]$ over a field K

Let $K[x]$ be the polynomial ring in one indeterminate x over a field K. $K[x]$ is a composition ring with respect to the addition, multiplication and composition of polynomials. The identity 1 is a constant in this composition ring. Using the above results and comparing the degree of the polynomials on the left-hand and the right-hand side in (3) we get two cases:

a) Degree $I(1)$ equal to zero, that means $I(1)\in A_C$. Hence, by Lemma 1, I is the zero mapping.

b) Degree $I(1)$ equal to one, that means $I(1)=a_0+a_1x$, a_0, $a_1\in K$, $a_1\neq 0$. Then (3) implies $a_0+a_1x = a_0+a_1a_0+a_1a_1x$ and we obtain $a_0=0$, $a_1=1$. Hence

$$I(1)=x. \tag{6}$$

If the characteristic $K=p$ (p prime), then (6) and Lemma 3 imply $0=(p!)I^p(1) = (I(1))^p=x^p$, which is a contradiction.

If the characteristic $K=0$, then (6) and Lemma 3 imply $((n+1)!)I^{n+1}(1) = ((n+1)!)I^n(x)=x^{n+1}$. From this we obtain

$$I(x^n)=\frac{x^{n+1}}{n+1}, \quad n\in\mathbf{N}, \quad n\geqslant 1. \tag{7}$$

Now there follows by (i), (ii), (1), (6) and (7) that

$$I(g)=I(a_0+a_1x+\ldots+a_nx^n)=a_0x+a_1\frac{x^2}{2}+\ldots+a_n\frac{x^{n+1}}{n+1}$$

for all $g=a_0+a_1x+\ldots+a_nx^n\in K[x]$.

Conversely it is easy to verify that this mapping I is a formal integration of $K[x]$.

This yields

Theorem 1. *If K is a field of characteristic* 0, *then there exists in $K[x]$ exactly one formal integration I* besides *the zero mapping, namely,*

$$I(a_0+a_1x+\ldots+a_nx^n):=a_0x+a_1\frac{x^2}{2}+\ldots+a_n\frac{x^{n+1}}{n+1}$$

for all

$$a_0+a_1x+\ldots+a_nx^n\in K[x].$$

If K has the characteristic p (p prime), then the zero mapping is the only formal integration in $K[x]$.

Remark. If the characteristic $K=0$, (6) can be also derived from Lemma 2. In this case K forms a base of $K[x]$. But K does not form a base of $K[x]$ if the characteristic $K=p$ (p prime).

4. The composition ring K^K of all functions on a field K

Finally, we are going to investigate formal integrations of the composition ring K^K of all functions of K into K. The operations in K^K are the pointwise addition and multiplication of functions and the composition of functions. As we will expect from the analysis, there is only the trivial integration in K^K.

As the constants K form a base of K^K, there holds Lemma 2. Hence

$$I(1)=x. \tag{8}$$

If $I(1)\notin K$ and the characteristic $K=p$ (p prime), then (8) and Lemma 3 give a contradiction.

If $I(1)\notin K$ and the characteristic $K=0$, we conclude in the following way:

$$I\left(\left(\frac{1}{x}\circ I(x)\right)\cdot x\right)=I\left(\left(\frac{1}{x}\circ\frac{x^2}{2}\right)\cdot x\right)=I\left(\frac{2}{x^2}\cdot x\right)=I\left(\frac{2}{x}\right)=2I\left(\frac{1}{x}\right),$$

and also

$$I\left(\left(\frac{1}{x}\circ I(x)\right)\cdot x\right)=I\left(\frac{1}{x}\right)\circ I(x)=I\left(\frac{1}{x}\right)\circ\frac{x^2}{2},\quad \text{where}\quad \frac{1}{x}\circ 0:=0.$$

Therefore $2I\left(\frac{1}{x}\right)=I\left(\frac{1}{x}\right)\circ\frac{x^2}{2}$ and further on

$$2\left(I\left(\frac{1}{x}\right)\circ 2\right)=\left(2I\left(\frac{1}{x}\right)\right)\circ 2=\left(I\left(\frac{1}{x}\right)\circ\frac{x^2}{2}\right)\circ 2=I\left(\frac{1}{x}\right)\circ 2.$$

Hence we have

$$I\left(\frac{1}{x}\right)\circ 2=0. \tag{9}$$

Now, let $c \in K$, $c \neq 0$. Then

$$I\left(\left(\frac{1}{x} \circ I\left(\frac{1}{c}\right)\right) \cdot \frac{1}{c}\right) = I\left(\left(\frac{1}{x} \circ \frac{x}{c}\right) \cdot \frac{1}{c}\right) = I\left(\frac{1}{x}\right),$$

and also

$$I\left(\left(\frac{1}{x} \circ I\left(\frac{1}{c}\right)\right) \cdot \frac{1}{c}\right) = I\left(\frac{1}{x}\right) \circ I\left(\frac{1}{c}\right) = I\left(\frac{1}{x}\right) \circ \frac{x}{c}.$$

Therefore $I\left(\frac{1}{x}\right) = I\left(\frac{1}{x}\right) \circ \frac{x}{c}$ and further by (9)

$$I\left(\frac{1}{x}\right) \circ 2c = \left(I\left(\frac{1}{x}\right) \circ \frac{x}{c}\right) \circ 2c = I\left(\frac{1}{x}\right) \circ 2 = 0, \quad \text{for all} \quad c \in K, c \neq 0.$$

Since K forms a base of K^K, we obtain

$$I\left(\frac{1}{x}\right) = 0. \tag{10}$$

Finally we have $I\left(I\left(\frac{1}{x}\right) \cdot 1\right) = I\left(\frac{1}{x}\right) \cdot I(1) - I\left(\frac{1}{x} \cdot I(1)\right)$ and further $I\left(I\left(\frac{1}{x}\right)\right)$ $= I\left(\frac{1}{x}\right) \cdot x - I(1) = x \cdot \left(I\left(\frac{1}{x}\left(-1\right)\right)\right.$, which gives considering (10), $0 = I(0) = -x$, a contradiction.

Now we get

Theorem 2. *In the composition ring K^K of all functions on a field K there exists only the trivial formal integration, namely, the zero mapping.*

Remark. These results can be generalized for composition rings of higher dimension than one by defining formal integrations with respect to certain indeterminates.

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ФОРМАЛЬНОЕ ИНТЕГРИРОВАНИЕ В КОМПОЗИЦИОННЫХ КАЛЬЦАХ

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Резюме

В статье вводится понятие оператора формального интегрирования для композиционных колец. Указываются основные свойства формального интегрирования, а также описываются все такие операторы для двух обще известных классов композиционных колец.