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COMPARING MEASURE THEORETIC ENTROPY FOR σ-FINITE MEASURES WITH TOPOLOGICAL ENTROPY

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This paper follows the collection of works dealing with comparing topological and measure theoretic entropy (cf. [3], [5], [6], [7], [8]). The famous variational principle (cf. [4]) asserts that topological entropy of a continuous transformation of a compact space is the supremum of measure theoretic entropies of regular invariant probability measures.

We wanted to find an extension of this result for noncompact spaces. The original definition of topological entropy does not require compactness (cf. [1]). The class of invariant probability measures for a continuous transformations of a noncompact space can be empty (cf. [4]). Therefore we introduce the measure theoretic entropy for a σ-finite measure as the supremum for entropies of conditional probabilities determined by this measure on sets with finite positive measure.

If the considered measure is finite the new invariant is not smaller then the standard entropy. On the other hand, if the σ-finite measure μ is regular invariant this entropy is not greater than the topological entropy.

I. Entropy for σ-finite measure

Definition 1. *Let μ be a σ-finite measure on a measurable space $(X, \mathcal{S})$ and T be a measure preserving transformation of X. Let $\mathcal{P}$ be the system of all finite measurable partitions of X. For $\xi \in \mathcal{P}$ and $A \in \mathcal{S}$, $0<\mu(A)<\infty$ we put*

$$\xi_A = \{B \cap A : B \in \xi\}, \tag{1}$$

$$H(\xi_A) = \sum_{B \in \xi} \eta[\mu(B|A)]$$

where

$$\eta(t) = \begin{cases} 0 & \text{for} \quad t=0 \\ -t \cdot \ln t & \text{for} \quad t \in (0, 1\rangle \end{cases}$$

Let N be the set of natural numbers. For any $n \in N$ we put

$$\xi^n = \bigvee_{i=0}^{n-1} T^{-i}(\xi).$$

Further we put

$$h_\mu(T, A, \xi) = \overline{\lim_n} \frac{1}{n} \cdot H([\xi^n]_A). \tag{2}$$

Finally we define

$$h_\mu(T, A) = \sup \{h_\mu(T, A, \xi) : \xi \in \mathcal{P}\} \tag{3}$$

and

$$\bar{h}_\mu(T) = \sup \{h_\mu(T, A) : 0 < \mu(A) < \infty\}. \tag{4}$$

Remark 1. Analogical topological notions were introduced in [7].

Remark 2. It is obvious that the function $\bar{h}_\mu(T)$ is an invariant with respect to the class of the measure preserving isomorphisms commuting with the considered transformations (cf. [8]).

Proposition 1. *For any $m \in N$ and $A \in \mathcal{S}$ with $0 < \mu(A) < \infty$ we have*

$$h_\mu(T^m, A) = m \cdot h_\mu(T, A). \tag{5}$$

Proof. For any $m < n \in N$ there exists $q \in N$ such that

$$m \cdot q \leqslant n \leqslant (q+1) \cdot m - 1.$$

This yelds that for any $\xi \in \mathcal{P}$ we have

$$\bigvee_{j=0}^{q} T^{-m \cdot j}(\xi^m) = \bigvee_{i=0}^{(q+1) \cdot m - 1} T^{-i}(\xi) \geqslant \xi^{n+1} \geqslant \xi^n \geqslant \bigvee_{j=0}^{q-1} T^{-n \cdot j}(\xi^m).$$

Therefore

$$\overline{\lim_q} \frac{1}{m \cdot (q+1) - 1} \cdot H\left(\left[\bigvee_{j=0}^{q-1} T^{-m \cdot j}(\xi^m)\right]_A\right) \leqslant \overline{\lim_n} \frac{1}{n} \cdot H([\xi^n]_A) \leqslant$$

$$\leqslant \overline{\lim_q} \frac{1}{m \cdot q} \cdot H\left(\left[\bigvee_{j=0}^{q} T^{-m \cdot j}(\xi^m)\right]_A\right).$$

The first and the last terms of the above chain of unequalities are equal to

$$\frac{1}{m} \cdot h(T^m, A, \xi^m).$$

Hence

$$h(T^m, A, \xi) \leqslant h(T^m, A, \xi^m) \leqslant m \cdot h(T, A, \xi) \leqslant h(T^m, A, \xi^m)$$

and

$$h_\mu(T^m, A) = m \cdot h_\mu(T, A).$$

II. Comparing with topological entropy

Definition 2. *Let $\mathcal{B}(X)$ be the σ-algebra generated by open subsets of a topological space X. We say that a σ-finite measure μ defined on $\mathcal{B}(X)$ is regular if for any $B \in \mathcal{B}(X)$*

$$\mu(X) = \inf \{\mu(U) \colon B \subset U,\ U \text{ open}\}.$$

We show that this definition is equivalent to another one, which seems to be stronger.

Proposition 2. *Let μ be a regular σ-finite measure on a topological space X. Then for any $B \in \mathcal{B}(X)$ we have*

$$\inf \{\mu(U-B) \colon B \subset U,\ U \text{ open}\} = \inf \{\mu(B-C) \colon C \subset B,\ C \text{ closed}\} = 0.$$

Proof. Consider a fixed measurable partition $\{A_n\}_{n=1}^{\infty}$ of X composed by disjoint sets of finite measure. For any $B \in \mathcal{B}(X)$ put

$$B_n = B \cap A_n, \quad n \in N.$$

We have

$$\infty > \mu(B_n) = \inf \{\mu(U) \colon B_n \subset U,\ U \text{ open}\}.$$

For every $\varepsilon > 0$ and $n \in N$ there exists an open set $U_n \supset B_n$ such that

$$\mu(U_n) - \mu(B_n) = \mu(U_n - B_n) < \frac{\varepsilon}{2^n}.$$

Put

$$U = \bigcup_{n=1}^{\infty} U_n.$$

We have

$$\mu(U-B) = \mu\left(\bigcup_{n=1}^{\infty} U_n - \bigcup_{n=1}^{\infty} B_n\right) \leqslant \sum_{n=1}^{\infty} \mu(U_n - B_n) < \varepsilon.$$

Let T be a continuous transformation of a Hausdorff space X. We consider the topological entropy $h(T)$ introduced in [1] by means of open coverings containing finite subcoverings (see also [8], ch. 6).

Theorem. *Let μ be an invariant regular σ-finite measure on $\mathcal{B}(X)$. Then the inequality*

$$\bar{h}_\mu(T) \leqslant h(T) \tag{6}$$

holds.

Proof. First we repeat an important fact (cf. [2] th. 6.1). For any natural number m there exists $\varepsilon_m > 0$ such that for any two measurable partitions (of any probability space)

$$\alpha=\{A_1, \ldots, A_m\} \quad \text{and} \quad \beta=\{B_0, B_1, \ldots, B_m\}$$

the inequalities

$$\mu(A_i \triangle B_i)<\varepsilon_m \quad \text{for} \quad i=1, \ldots, m$$

imply

$$H_\mu(\alpha|\beta)<1. \tag{7}$$

Consider

$$A \in \mathscr{B}(X) \quad \text{with} \quad 0<\mu(A)<\infty.$$

Note that for any $B_1, B_2 \in \mathscr{B}(X)$ we have

$$\mu(B_1 \triangle B_2|A) \leqslant \frac{\mu(B_1 \triangle B_2)}{\mu(A)}. \tag{8}$$

Let $\alpha=\{A_1, \ldots, A_m\}$. We can choose closed sets $C_i \subset A_i$ with

$$\mu(A_i-C_i)<\varepsilon_m \cdot \mu(A) \quad \text{for} \quad i=1, \ldots, m.$$

Put

$$C_0=X-\bigcup_{i-1}^{m} C_i \quad \text{and} \quad U_i=C_0 \cup C_i \quad \text{for} \quad i=1, \ldots, m.$$

Let $\beta=\{C_0, C_1, \ldots, C_m\}$ and $\gamma=\{U_1, \ldots, U_m\}$. Note that γ is an open covering of X and that the topological conditional entropy

$$H(\beta|\gamma)=\ln 2 \quad \text{(cf. [7])}.$$

Consider the measure space $(A, \mathscr{S} \cap A, \mu(\cdot|A))$. According to the considerations giving (7) and (8) for any $j=0, 1, \ldots$ we have

$$H([T^{-j}\alpha]_A|[T^{-j}\beta]_A)<1.$$

Hence for any $n \geqslant 1$ we have

$$H([\alpha^n]_A)-H([\beta^n]_A) \leqslant \sum_{j=0}^{n-1} H([T^{-j}\alpha]_A|[T^{-j}\beta]_A)$$

(cf. [8], th. 4.5).

Therefore

$$h_\mu(T, A, \alpha) \leqslant h_\mu(T, A, \beta)+1. \tag{9}$$

Note that

$$h_\mu(T, A, \beta) \leqslant h(T, A, \beta) \leqslant h(T, \beta) \tag{10}$$

where $h(T, A, \beta)$ and $h(T, \beta)$ were introduced in [7] in a similar way by topological means.

We have

$$h(T, \beta) \leqslant h(T, \gamma)+H(\beta|\gamma) \leqslant \ln 2+h(T). \tag{11}$$

From (9), (10) and (11) we get for any α and A inequality

$$h_\mu(T, A, \alpha) \leq h(T) + 1 + \ln 2. \tag{12}$$

Hence the relation

$$\bar{h}_\mu(T) \leq h(T) + \ln 2 + 1 \tag{13}$$

holds.

Considering T^n instead of T and applying (5) we get

$$\bar{h}_\mu(T) = \frac{1}{n} \cdot \bar{h}_\mu(T^n) \leq \frac{1}{n} \cdot h(T^n) + \frac{(\ln 2 + 1)}{n} = h(T) + \frac{(\ln 2 + 1)}{n}$$

for arbitrary n.

Hence the inequality (6) is fulfilled.

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СРАВНЕНИЕ ЭНТРОПИИ σ-КОНЕЧНЫХ МЕР И ТОПОЛОГИЧЕСКОЙ ЭНТРОПИИ

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Резюме

Вводится энтропия для σ-конечных мер как супремум из энтропий для индуцированных условных вероятностных мер. В случае, когда рассматриваемая мера регулярна на нормальном пространстве Гаусдорффа и инвариянтна относительно непрерывной трансформации, ее энтропия не больше топологической энтропии.