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COMPLEXITY OF DECOMPOSING GRAPHS INTO FACTORS WITH GIVEN DIAMETERS OR RADII

JÁN PLESNÍK

1. Introduction

We shall consider the following problem. *Given a graph (or a digraph) G and integers* $d_1, d_2, \ldots, d_k$, *decompose G into k factors* (i.e. edge-disjoint spanning subgraphs) $F_1, F_2, \ldots, F_k$ *with diameters not exceeding* $d_1, d_2, \ldots, d_k$, *respectively.* The problem will be referred to as the *diameter decomposition problem* and denoted by DD. If bounds $r_1, r_2, \ldots, r_k$ for radii (instead for diameters) of factors are prescribed, we have the *radius decomposition problem* RD.

Both problems appear as strengthened versions of the problem of decomposing a graph into k connected factors. The last problem was investigated by Tutte [29] and Nash-Williams [15], who gave a necessary and sufficient condition. An algorithmic approach can be found in Kameda [12]. On the other hand, the problems DD and RD have been considered only for special cases. In their pioneering work [7] Bosák, Rosa and Znám have treated a modification of DD, which could be called the *strict* DD, where the factors $F_1, \ldots, F_k$ are required to have exactly the given diameters $d_1, \ldots, d_k$, respectively. However, they consider only the decompositions of complete graphs. The same can be said about all the succeeding papers, where mostly strict DD, or RD, or related problems were studied [1, 4, 5, 6, 7, 8, 10, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32]. These papers include decompositions of complete graphs, complete bipartite graphs, complete digraphs, and complete hypergraphs.

Such problems can be interpreted in terms of communication networks [7] and therefore an algorithmic approach would be appreciated. In this connection let us mention the following problem of "small" size [7]: Can the complete graph with 12 vertices be decomposed into 3 factors of diameter 2? (It is known [7] that this is possible for 13 vertices and impossible for 11 vertices.) In spite of using a computer [8] the problem remains open. This indicates that DD is a difficult problem and it is the purpose of this paper to support such a conviction. In fact, we shall show that DD and RD belong to such difficult problems as finding the chromatic number or a hamiltonian cycle of a graph. Mo e precis y, we shall show that DD and RD are NP-hard [11].

2. Definitions and notations

Graphs and digraphs are understood in the sense of [2]; for hypergraphs see [3]. If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. If G is a digraph, then $E(G)$ is the set of its arcs. The edge joining two vertices u and v is denoted by uv or vu but the arcs uv and vu must be distinguished. A digraph G is called *symmetric* if for any arc $uv \in E(G)$ also $vu \in E(G)$. We let $XY = \{xv | x \in X,\ y \in Y,\ x \neq y\}$ and instead of $\{x\}Y$ we write only xY. Further, $N_G(v)$ denotes the set of adjacent vertices to v in the graph G and we let $N_G(S) = \{x | x \in N\ (v),\ v \in S\}$ The distance from a vertex x to a vertex y in the graph (or digraph) G is denoted by $d_G(x, y)$. The diameter of G is $d(G) = \max\{d_G(x, y) | x,\ y \in V(G)\}$ and the radius of G is $r(G) - \min_x \max_y \max\{d_G(x, y),\ d_G(y, x)\}$. As for the NP hardness and other computational notions, the reader is referred to [11] One usual way to prove the NP-hardness of a problem P is to choose an appropriate NP-hard problem Q and then to find a polynomial transformation from Q to P. In the proofs below, we use an idea of Chvátal and Thomassen [9] and take for Q the following NP hard problem [14]. The *hypergraph 2-colourability problem* (H2C): *Given a hypergraph H with a vertex set A and an edge set B, is H 2-colourable* (i.e., is there a partition of A into two colour classes A_1, A_2 such that every edge has vertices of both colours)? (Note that each edge $b \in B$ is a subset of A and we can suppose that $|b| \geq 2$ and that A and B are sufficiently large. Otherwise, without changing the 2-colourability, new vertices and 2-element edges can be added.)

3. Decompositions of graphs

Lemma 1. *Let K_p denote the complete graph with p vertices, $p \geq 6$, and let $V(K_p)$ be partitioned into subsets X, Y with $|X| > 3$, $|Y| \geq 3$. Then K_p can be decomposed into two factors G_1, G_2 of diameter 2 such that*
(a) *for every $x \in X$ there are $y_1, y_2 \in Y$ with $d_G(x\ y)\ \ 1 = d_{G_2}(x, y_2)$,*
(b) *for every $y \in Y$ there are $x_1, x \in X$ with $d_G(x_1\ y)\ \ 1\ \ d_{G_2}(x_2, y)$.*

Proof. If $p = 6$, then $|X| = |Y| = 3$ and the required decomposition is shown in Fig. 1 (black vertices form X) If $p > 6$, then we can assume that $|X| \geq 4$. Let us choose $u \in X$. By the induction hypothesis he complete graph $K_p - u$ can be decomposed into two factors G'_1 and G'_2 of diameter 2 fulfilling (a) and (b) for

$X' = X - \{u\}$ and $Y' = Y$. Let $u' \in X'$; then we put $E(G_1) = E(G_1') \cup \{uv \mid u'v \in E(G_1')\}$. One sees that G_1 and $G_2 = K_p - G_1$ have the desired properties.

Theorem 1. *The diameter decomposition problem for graphs is* NP-*hard even in the case of two factors with diameter bound* 2.

Proof. Given a hypergraph H with a vertex set A and an edge set B, we shall construct in a polynomial number of steps (in the size of H) a graph G with the property: H is 2-colourable iff G can be decomposed into two factors of diameter 2.

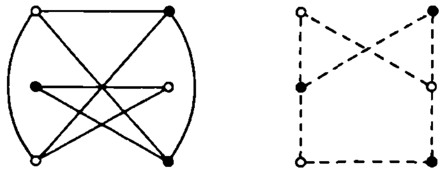

Fig 1. Decomposition of K_6 for the proof of Lemma 1

Let w, u_i, v_i $(1 \leq i \leq 4)$ be new elements. Let us denote

$$U_1 = \{u_1, u_2\}, \quad U_2 = \{u_3, u_4\}, \quad S = U_1 \cup U_2 \cup A,$$
$$T = \{v_1, v_2, v_3, v_4\}.$$

Then G is given as follows.

$$V(G) = \{w\} \cup S \cup T \cup B,$$
$$E(G) = wS \cup ST \cup TB \cup SS \cup TT \cup D$$

with $D = \{ab \mid a \in A, b \in B$ and $a \in b$ (in H)$\}$. A rough construction of G is given in Fig. 2, where $A = A_1 \cup A_2$ and G consists of both solid and dash edges.

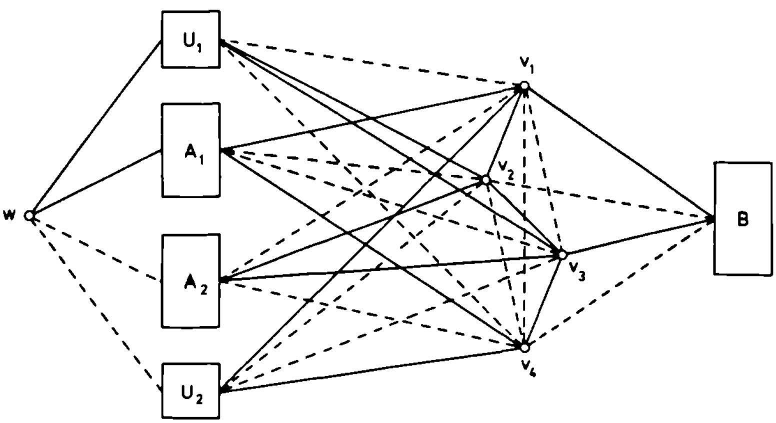

Fig. 2. Illustration for the proof of Theorem 1

Assume that H is 2-colourable. Let A_1 be the set of vertices with colour 1 and $A_2 = A - A_1$ be the set of vertices with colour 2. Then one can take for factor F_1 the subgraph consisting of solid edges in Fig. 2 and $F_2 = G - F_1$ (dash edges). More precisely

$$E(F_1) = wU_1 \cup \{wa \mid a \in A_1\} \cup U_1\{v_2, v_3\} \cup U_2\{v_1, v_4\} \cup$$
$$\{v_1v_2, v_2v_3, v_3v_4\} \cup \{av_1, av_4 \mid a \in A_1\} \cup \{av_2, av_3 \mid a \in A_2\} \cup$$
$$\{v_1b, v_3b \mid b \in B\} \cup \{ab \mid a \in A_1, b \in B, a \in b \text{ (in } H)\} \cup E_1,$$

where $E_1 = E(G_1)$ and G_1 is a subgraph of G by Lemma 1 for $X = A_1 \cup U_1$ and $Y = A_2 \cup U_2$.

The proof that F_1 and F_2 have diameter 2 is easy but long and therefore it is left to the reader.

Conversely, suppose that G is decomposable into two factors F_1, F_2 with diameter 2. Then we can colour H with two colours 1 and 2 as follows. A vertex $a \in A$ gets colour 1 whenever there exists in F_1 a path wab (of length two) with $b \in B$; otherwise a gets colour 2. Since $d_{F_1}(w, b) = 2 = d_{F_2}(w, b)$ for every $b \in B$, each edge b of H contains a vertex of colour 1 and also a vertex of colour 2. Hence H is 2-colourable.

The constructions employed in the proof are obviously all of polynomial time complexity. This completes the proof.

An analogy with Theorem 1 can be stated also for bipartite graphs. However, the diameter bound 2 must be altered because any incomplete bipartite graph has the diameter at least 3.

Lemma 2. *Let $K_{6,6}$ denote the complete bipartite graph with the bipartition (P, Q) where $|P| = |Q| = 6$ and with the edge set PQ. Let the set P be partitioned into subsets P_1, P_2 with $|P_1| = |P_2| = 3$. Then the graph $K_{6,6}$ can be decomposed into two factors G_1, G_2 of diameter 3 with $N_{G_i}(P_j) = Q$ for $i, j = 1, 2$.*

Proof. We can take for G_1 the graph of Fig. 3.

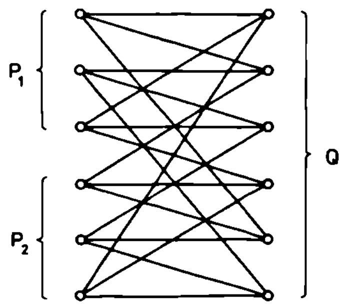

Fig. 3. The graph $G_1(P_1, P_2, Q)$ and also the graph $G_1(P_1 \cup P_2, Q)$

Theorem 2. *The diameter decomposition problem is* NP-*hard even for bipartite graphs in the case of two factors with diameter bound* 3.

Proof. Again, given a hypergraph H with a vertex set A and an edge set B, we shall construct a bipartite graph G such that H is 2-colourable iff G can be decomposed into two factors of diameter 3.

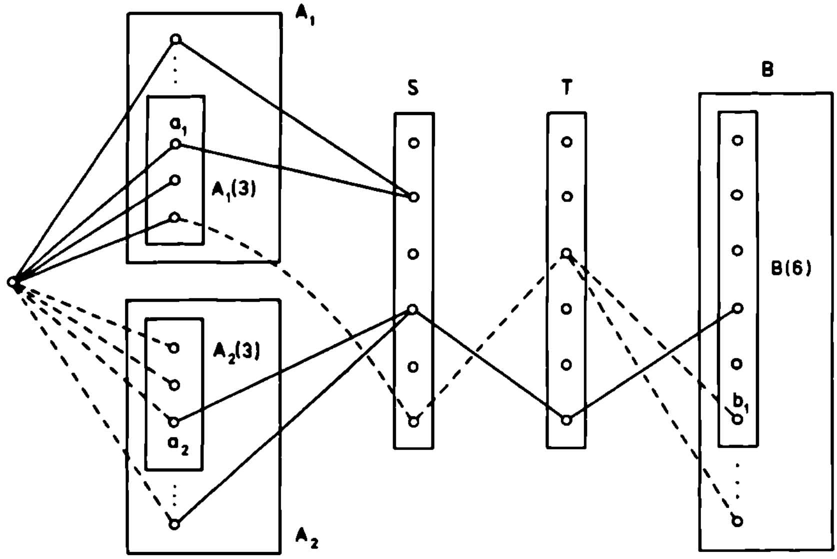

Fig. 4. Illustration for the proof of Theorem 2

Let $S=\{s_1, s_2, \ldots, s_6\}$ and $T=\{t_1, t_2, \ldots, t_6\}$ be two sets of new vertices. Then we construct G in accordance with Fig. 4 (where $A_1 \cup A_2 = A$):

$$V(G)=\{w\} \cup A \cup S \cup T \cup B,$$
$$E(G)=wA \cup AS \cup ST \cup TB \cup \{ab \mid a \in A,\ b \in B, \text{and } a \in b \text{ (in } H)\}.$$

Assume that H is 2-colourable. Let A_1 be the set of vertices with colour 1 and $A_2 = A - A_1$ be the set of vertices with colour 2. We shall use the following notation: If M is a set and r is a nonnegative integer, then $M(r)$ denotes some (arbitrarily chosen) r-element subset of M.

Fix some $A_1(3)$, $A_2(3)$ and $B(6)$, and then some $a_1 \in A_1(3)$, $a_2 \in A_2(3)$ and $b_1 \in B(6)$. The required factor F_1 can be described as follows.

$$E(F_1)=wA_1 \cup \{ab \mid a \in A_1,\ b \in B,\ a \in b \text{ (in } H)\} \cup E_1 \cup E_2,$$

where E_1 consists of the edges of the following three graphs (defined in Fig. 3):

$$G_1(A_1(3) \cup A_2(3), S),\ G_1(T, S),\ G_1(T, B(6)),$$

and

$$E_2 = \{as \mid a \in A_1 - A_1(3),\ s \in S,\ a_1s \in E_1\} \cup$$
$$\{as \mid a \in A_2 - A_2(3),\ s \in S,\ a_2s \in E_1\} \cup$$
$$\{tb \mid tb_1 \in E_1,\ t \in T,\ b \in B - B(6)\}.$$

Note that the introduction of E_2 is an extension of G_1 from Fig. 3. It is a long but easy task to verify that both F_1 and $F_2 = G - F_1$ are of diameter 3, and therefore the proof is omitted.

Now suppose that G is decomposable into two factors F_1, F_2 with diameter 3. Then $a \in A$ gets colour 1 iff there is in F_1 a path wab for some $b \in B$; otherwise a gets colour 2. Since for each $b \in B$ we have $d_{F_i}(w, b) = 2$, the edge b of H must contain a vertex of colour 1 and also a vertex of colour 2. Thus H is 2-colourable, which completes the proof.

We have mentioned that a bipartite graph cannot be decomposed into two (or more) factors of diameter 2. Analogously [28], no decomposition into factors of radius 2 is possible. Therefore, we first consider the general case, where this is possible.

Theorem 3. *The radius decomposition problem for graphs is* NP *hard even in the case of two factors with radius bound* 2.

Proof. We can proceed analogously as above. Therefore only a brief description is given. Given a hypergraph H, we construct a graph G, which is decomposable into two factors of radius 2 iff H is 2-colourable. We set

$$V(G) = \{w, v_1, v_2, v_3, v_4\} \cup A \cup B,$$
$$E(G) = \{wv_1, wv_2, wv_3, wv_4, v_1v_3, v_1v_4, v_2v_3, v_2v_4\} \cup$$
$$wA \cup AA \cup \{ab \mid a \in A,\ b \in B,\ a \in b \text{ (in } H)\}.$$

Graph G is outlined in Fig. 5 ($A = A_1 \cup A_2$).

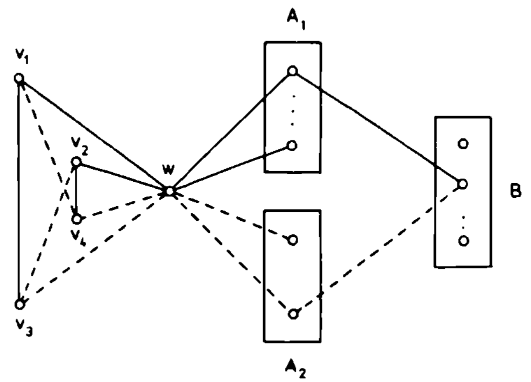

Fig 5 Illustration for the proof of Theorem 3

Clearly, G has radius 2 and w is a central vertex of G. Perhaps it should be said how to define F_1 and F_2 if H is 2-colourable with the colour sets A_1 and A_2. We set (see Fig. 5)

$$E(F_1)=\{wv_1, wv_2, v_1v_3, v_2v_4\}\cup wA_1\cup\{ab|a\in A_1,\\ b\in B, a\in b \text{ (in } H)\}\cup E_1,$$

where $E_1=E(G_1)$ and G_1 is determined as follows. The complete bipartite subgraph of G with the parts A_1 and A_2 ($|A_1\geqslant 2$ and $|A_2|\geqslant 2$) can be easily decomposed into two factors G_1, G_2 with the properties: $N_{G_1}(A_1)=A_2$ and $N_{G_2}(A_2)=A_1$.

Now the reader should be able to prove an analogy with Theorem 2 for the decomposition of bipartite graphs into two factors with radius bound 3.

4. Digraphs

There are similar results for digraphs and they have similar proofs.

Theorem 4. *The diameter decomposition problem for digraphs is* NP-*hard even in the case of symmetric digraphs and two factors with diameter bound* 2.

Proof. We can follow the proof of Theorem 1. However, we alter the graph G to a digraph $\hat{G}$ by replacing every edge $uv\in E(G)$ by two arcs uv and vu. Clearly, $\hat{G}$ is a symmetric digraph.

If H is 2-colourable, the desired factors can be obtained from the graphs F_1 and F_2 by changing them to symmetric digraphs. If $\hat{G}$ is decomposable into two (not necessarily symmetric) factors F_1, F_2 then a 2-colouring of H can be given as follows. A vertex $a\in A$ gets colour 1 iff there is in F_1 a directed path wab with $b\in B$. The theorem is proved.

For symmetric digraphs, one could define also the so-called *perfect decomposition into two factors,* when no pair of arcs uv and vu lies in the same factor. And again, we have NP-hard problems even for diameter or radius 2. In fact such results are due to Chvátal and Thomassen [9]. Namely, any perfect decomposition of a symmetric digraph $\hat{G}$ corresponds to an orientation of the underlying graph G and conversely, and they have shown that the problem of deciding whether a graph has an orientation of diameter (or radius) 2 is NP-hard.

5. Concluding remarks

The reader has certainly observed that all the above results hold also for the strict DD or RD. The same can be asserted about the following remarks.

Note that we have established the NP-hardness for the smallest possible diameter or radius bound. However, it is not difficult to extend the diameter or the radius. As for the number of factors similar proofs are possible also by using certain hypergraph k-colourability problem with $k \geq 3$, whose NP-hardness can be proved from the NP-hardness of H2C.

Further, note that one can easily see that the decision versions of our problems are in the class NP, hence they are NP-complete.

Finally it should be mentioned that the above problems can be formulated also as optimization problems. E.g.: Given a graph (or digraph) G, decompose it into two factors F_1, F_2 with minimum equal diameters (or radii). Then our results show that even to find a decomposition with the value less than 3/2 of the optimum is an NP-hard problem. If the maximum or the sum of diameters (or radii) is minimized, then we can take 3 2 or 5 4, respectively.

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СЛОЖНОСТЬ РАЗ ЮЖЕНИЯ ГРАФОВ НА ФАКТОРЫ С ДАННЫМИ ДИАМЕТРАМИ ИЛИ РАДИУСАМИ

Ян Плесник

Резюм

Пока ывается, что проблема разложения графов на факторы с заданными диаметрами NP-трудная. То самое показано для двудольных графов и для ориентированных графов Эти проблемы с радиусами тоже NP трудные