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SUPPORT AS AN INVARIANT FOR DYNAMICAL SYSTEMS

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In [3] H. Furstenberg introduced a notion of disjointness into ergodic theory, which was transferred in a way from elementary number theory. With respect to this connection we shall introduce a notion of support of a dynamical system by simulation of the fact that to every natural number n there corresponds the set $\pi(n)$ of those prime numbers which are necessary in the prime number representation of n. In the same way as the equality of two natural numbers n_1, n_2 requires the equality of $\pi(n_1)$ and $\pi(n_2)$, the isomorphism of dynamical systems requires the equality of support. Thus support turns out to be an isomorphy invariant.

Of course the situation for dynamical systems is much more complicated than for natural numbers. In most cases there is no representation of the given dynamical system as a product of prime systems (for some results with regard to this see [15]).

It is the purpose of this paper to define this notion of support and to apply it to some classes of dynamical systems with discrete and quasidiscrete spectra. Further, all K-automorphisms (and therefore Bernoulli systems also) have the same support but Halmos' invariant for totally ergodic systems with a quasidiscrete spectrum is of quite another kind.

1. Basic definitions and notations

Throughout this paper *measure spaces* $(X, \mathscr{S}, m)$, $(X', \mathscr{S}', m')$ are supposed to be Lebesgue (= normalized and separable, see e.g. [14]).

A *measure preserving transformation* from $(X, \mathscr{S}, m)$ to $(X' \mathscr{S}', m')$ is a mapping T: $X \to X'$ with $T^{-1}(E') \in \mathscr{S}$, $(E' \in \mathscr{S}')$ and $m'(E') = m(T^{-1}(E'))$, $(E' \in \mathscr{S}')$. If the two measure spaces are equal, then T maps X into itself and the quadruple $\mathbf{D} = (X, \mathscr{S}, m, T)$ is called a dynamical system. Let $\mathbf{D}_t = (X_t, \mathscr{S}_t, m_t, T_t)$ be the system with a normalized measure space $(X_t, \mathscr{S}_t, m_t)$ and $T_t = \mathrm{id}_{X_t}$.

A *measure algebra* (Σ, μ) is a pair, such that Σ is a (Boolean σ-) algebra and μ a measure on it. A (σ-) homomorphism τ from (Σ, μ) to (Σ', μ') is a mapping τ: $\Sigma \to \Sigma'$ that respects the operations of Σ:

$$\tau\left(\bigvee_{i=1}^{\infty} S_i\right) = \bigvee_{i=1}^{\infty} \tau(S_i), \quad \tau(S^c) = [\tau(S)]^c, \quad (S_i, S \in \Sigma),$$

and preserves the measure:

$$\mu'(\tau(S)) = \mu(S), \quad (S \in \Sigma).$$

The homomorphism τ is an *isomorphism* if τ is invertible. In case of $\Sigma = \Sigma'$, $\mu = \mu'$ a homomorphism is called *endomorphism* and an isomorphism is called an *automorphism.*

Given a measure space $(X, \mathscr{S}, m)$ and identifying sets $E, F \in \mathscr{S}$ if they differ only on a set of m-measure 0 (i.e.: $m(E \triangle F) = 0$), we obtain a set Σ of equivalence classes $\underline{E} = \{F \in \mathscr{S} | m(E \triangle F) = 0\}$, $(E \in \mathscr{S})$, being a (Boolean σ-) algebra. On Σ the mapping μ: $\Sigma \to R_+$ defined by $\mu(\underline{E}) = m(E)$, $(E \in \mathscr{S})$ is a measure. In this way every measure space $(X, \mathscr{S}, m)$ induces a measure algebra (Σ, μ). Furthermore, if $(X, \mathscr{S}, m)$, $(X', \mathscr{S}', m')$ are measure spaces and (Σ, μ), (Σ', μ') their induced measure algebras, then a measure preserving transformation T from $(X, \mathscr{S}, m)$ to $(X', \mathscr{S}', m')$ induces a homomorphism τ from (Σ', μ') to (Σ, μ) defined by

$$\tau(\underline{E'}) = \underline{T^{-1}(E')}, \quad (E' \in \mathscr{S}').$$

Therefore, a dynamical system $\mathbf{D} = (X, \mathscr{S}, m, T)$ induces a triple (Σ, μ, τ), where τ is a certain endomorphism on (Σ, μ). Further, T: $X \to X'$ is called a homo-, iso- endo- or automorphism, resp., if the induced τ has the corresponding property. Finally, $\mathbf{D} = (X, \mathscr{S}, m, T)$ is said to be automorphic if T is an automorphism.

For more details concerning the connection between measure spaces and measure algebras see [6], pp. 163.

In the following definitions all measure algebras and mappings between them are supposed to be induced (in the obvious manner).

$\mathbf{D}_1 = (X_1, \mathscr{S}_1, m_1, T_1)$ is a factor of $\mathbf{D}_2 = (X_2, \mathscr{S}_2, m_2, T_2)$ if there exists a homomorphism σ from (Σ_1, μ_1) to (Σ_2, μ_2) with $\tau_2 \circ \sigma = \sigma \circ \tau_1$. This relation will be denoted by $\mathbf{D}_1 \overset{\sigma}{\leqslant} \mathbf{D}_2$ (or: $\mathbf{D}_1 \leqslant \mathbf{D}_2$ if σ is just not important). The notion of factor fulfils the relation

$$\mathbf{D} \overset{\sigma_0}{\leqslant} \mathbf{D}, \quad \text{(reflexivity)},$$

with $\sigma_0 = \mathrm{id}_\Sigma$ and moreover

$$\mathbf{D}_1 \overset{\sigma}{\leqslant} \mathbf{D}_2, \quad \mathbf{D}_2 \overset{\sigma'}{\leqslant} \mathbf{D}_3 \Rightarrow \mathbf{D}_1 \overset{\sigma \circ \sigma'}{\leqslant} \mathbf{D}_3 \quad \text{(transitivity)}.$$

Two dynamical systems $\mathbf{D}_1$, $\mathbf{D}_2$ are said to be weakly isomorphic (Ja. G. Sinai [16], [17]) if

$$\mathbf{D}_1 \overset{\sigma_1}{\leqq} \mathbf{D}_2, \quad \mathbf{D}_2 \overset{\sigma_2}{\leqq} \mathbf{D}_1 \tag{1}$$

holds and isomorphic (=conjugate in Halmos' terminology [5]) if there is an isomorphism σ with

$$\mathbf{D}_1 \overset{\sigma}{\leqq} \mathbf{D}_2.$$

We shall denote weak isomorphy with $\mathbf{D}_1 \overset{\sigma_1,\sigma_2}{\approx} \mathbf{D}_2$ (or: $\mathbf{D}_1 \approx \mathbf{D}_2$) and isomorphy with $\mathbf{D}_1 \overset{\sigma}{\cong} \mathbf{D}_2$ (or: $\mathbf{D}_1 \cong \mathbf{D}_2$). Because of its definition and $\mathbf{D}_2 \overset{\sigma^{-1}}{\leqq} \mathbf{D}_1$ isomorphy implies weak isomorphy. A system $\mathbf{D}$ being isomorphic to $\mathbf{D}_t$ is called trivial. $\mathbf{D}_1$ is a proper factor of $\mathbf{D}_2$ iff $\mathbf{D}_1$ is a factor of $\mathbf{D}_2$ and nonisomorphic to $\mathbf{D}_2$. This will be denoted by $\mathbf{D}_1 < \mathbf{D}_2$ and will be used especially for the case $\mathbf{D}_1 = \mathbf{D}_t$.

2. Support and its simple properties

By the following two definitions we connect with every dynamical system a notion of support.

2.1. Definition. *For every dynamical system* $\mathbf{D}$ *let*

$$\mathcal{N}(\mathbf{D}) = \{\mathbf{D}' \mid \mathbf{D}_t < \mathbf{D}' \leqq \mathbf{D}\}$$

be the system of all nontrivial factors of $\mathbf{D}$. *Then for given* $\mathbf{D}_1$, $\mathbf{D}_2$ *a relation* $\mathbf{D}_1 \prec \mathbf{D}_2$ *is defined by*

$$\mathbf{D}_1 \prec \mathbf{D}_2 \Leftrightarrow (\mathbf{D}' \in \mathcal{N}(\mathbf{D}_1) \Rightarrow \mathcal{N}(\mathbf{D}') \cap \mathcal{N}(\mathbf{D}_2) \neq \emptyset) \vee (\mathbf{D}_1 \textit{ trivial}). \tag{2}$$

If this is the case, $\mathbf{D}_1$ *is said to have equal support as or less support than* $\mathbf{D}_2$.

2.2. Theorem. *Let* $\mathfrak{D}$ *be a class of dynamical systems. Then the support relation* $\prec$ *fulfils the properties*

(a) $\mathbf{D} \prec \mathbf{D}$,

(b) $\mathbf{D}_1 \prec \mathbf{D}_2$, $\mathbf{D}_2 \prec \mathbf{D}_3 \Rightarrow \mathbf{D}_1 \prec \mathbf{D}_3$

of a preorder (*according to* L. Fuchs [2], *p.* 1).

Proof. (a): If $\mathbf{D}$ is trivial, $\mathbf{D} \prec \mathbf{D}$ follows immediately from (2). Otherwise $\mathcal{N}(\mathbf{D})$ is nonvoid and therefore $\mathbf{D}' \in \mathcal{N}(\mathbf{D})$ implies $\mathcal{N}(\mathbf{D}') \subset \mathcal{N}(\mathbf{D})$ and $\mathcal{N}(\mathbf{D}') \cap \mathcal{N}(\mathbf{D}) = \mathcal{N}(\mathbf{D}') \neq \emptyset$.

(b): Let be $\mathbf{D}_1 \prec \mathbf{D}_2$, $\mathbf{D}_2 \prec \mathbf{D}_3$. If $\mathbf{D}_1$ is trivial, nothing has to be proved. Otherwise $\mathcal{N}(\mathbf{D}_1) \neq \emptyset$ and

$$\mathbf{D}' \in \mathcal{N}(\mathbf{D}_1) \Rightarrow \mathcal{N}(\mathbf{D}') \cap \mathcal{N}(\mathbf{D}_2) \neq \emptyset \tag{3}$$

hold. In this implication at least $\mathbf{D}' = \mathbf{D}_1$ is possible and therefore $\mathcal{N}(\mathbf{D}_2) \neq \emptyset$. Then $\mathbf{D}_2 < \mathbf{D}_3$ implies

$$\mathbf{D}'' \in \mathcal{N}(\mathbf{D}_2) \Rightarrow \mathcal{N}(\mathbf{D}'') \cap \mathcal{N}(\mathbf{D}_3) \neq \emptyset.$$

Choosing $\mathbf{D}'' \in \mathcal{N}(\mathbf{D}') \cap \mathcal{N}(\mathbf{D}_2)$, which is possible according to (3), we get $\mathcal{N}(\mathbf{D}'') \subset \mathcal{N}(\mathbf{D}')$ and $\emptyset \neq \mathcal{N}(\mathbf{D}'') \cap \mathcal{N}(\mathbf{D}_3) \subset \mathcal{N}(\mathbf{D}') \cap \mathcal{N}(\mathbf{D}_3)$, thus $\mathbf{D}_1 < \mathbf{D}_3$.

A preorder induces an equivalence relation, namely

2.3. Definition. *If for two dynamical systems* $\mathbf{D}_1$, $\mathbf{D}_2$ *the relation*

$$\mathbf{D}_1 \sim \mathbf{D}_2 :\Leftrightarrow \mathbf{D}_1 < \mathbf{D}_2 \quad \textit{and} \quad \mathbf{D}_2 < \mathbf{D}_1$$

holds, then $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are said to be of equal support.*

In a fixed class $\mathcal{D}$ of dynamical systems the support of any $\mathbf{D} \in \mathcal{D}$ can be interpreted as the name of the equivalence class (modulo $\sim$) containing $\mathbf{D}$. As simple relations concerning this notion we have for any two $\mathbf{D}_1$, $\mathbf{D}_2$

$$\mathbf{D}_1 \leqslant \mathbf{D}_2 \Rightarrow \mathbf{D}_1 < \mathbf{D}_2, \tag{4}$$

$$\mathbf{D}_1 \approx \mathbf{D}_2 \Rightarrow \mathbf{D}_1 \sim \mathbf{D}_2. \tag{5}$$

The first of them follows from $\mathcal{N}(\mathbf{D}') \subset \mathcal{N}(\mathbf{D}_1) \subset \mathcal{N}(\mathbf{D}_2)$ for all $\mathbf{D}' \in \mathcal{N}(\mathbf{D}_1)$ and the second is a corollary of the first. It should be remarked that at present examples of nonisomorphic but weakly isomorphic dynamical systems still seem to fail. From this there depends, of course, the question whether (5) is a sharper statement than

$$\mathbf{D}_1 \cong \mathbf{D}_2 \Rightarrow \mathbf{D}_1 \sim \mathbf{D}_2.$$

3. Support invariants

Let $\mathcal{D}$ be a class of dynamical systems, Y a set and Φ: $\mathcal{D} \to Y$ a mapping. As it is well known Y is said to be a system of isomorphy invariants of $\mathcal{D}$, if the implication

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{D} \quad \text{and} \quad \mathbf{D}_1 \cong \mathbf{D}_2 \Rightarrow \Phi(\mathbf{D}_1) = \Phi(\mathbf{D}_2) \tag{6}$$

holds. Further, Y is called complete if moreover $\Phi(\mathcal{D}) = Y$ and the inverse direction of (6), namely

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{D} \quad \text{and} \quad \Phi(\mathbf{D}_1) = \Phi(\mathbf{D}_2) \Rightarrow \mathbf{D}_1 \cong \mathbf{D}_2$$

are valid. Well known examples of complete systems of isomorphy invariants are, e.g.,

(a) the set $\overline{R_+}$ of the extended nonnegative reals for the class of all automorphic

Bernoulli systems (A. N. Kolmogorov [8], D. S. Ornstein [12]) with Φ being the entropy mapping,

(b) the set M of all countable subgroups of K (=group of all complex numbers with modulus 1) for the class of all ergodic systems with a discrete spectrum (J. V. Neumann [11], P. R. Halmos [5]) with Φ defined by $\mathbf{D}\mapsto H(\mathbf{D})$, where $H(\mathbf{D})$ denotes the group of all proper values of $\mathbf{D}$ (i.e. of the isometric operator induced by T in $\mathbf{D}=(X, \mathcal{S}, m, T)$).

In accordance with these facts we define

3.1. Definition. *For a given class $\mathcal{D}$ of dynamical systems, a certain set Y and a mapping $\Phi: \mathcal{D}\to Y$, the set Y is said to be a system of support invariants of $\mathcal{D}$, if*

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{D} \quad \text{and} \quad \mathbf{D}_1 \sim \mathbf{D}_2 \Rightarrow \Phi(\mathbf{D}_1) = \Phi(\mathbf{D}_2). \tag{7}$$

Y is called a complete system of support invariants (or: supportic) if moreover the conditions

$$\Phi(\mathcal{D}) = Y, \tag{8}$$

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{D} \quad \text{and} \quad \Phi(\mathbf{D}_1) = \Phi(\mathbf{D}_2) \Rightarrow \mathbf{D}_1 \sim \mathbf{D}_2 \tag{9}$$

are valid. A system Y of isomorphy invariants is called nonsupportic if (7) *doesn't hold.*

If in the example (b) presented above only the subclass of all totally ergodic systems (i.e. in every $\mathbf{D}=(X, \mathcal{S}, m, T)$ not only T but also T^n, ($n=2, 3, \dots$) are ergodic) is considered, then the corresponding invariants in M are constituting the set M_t of all torsionfree countable subgroups of K (see e.g. L. M. Abramov [1]). The following theorem concerns this class:

3.2. Theorem. *Let $\mathcal{D}$ be the class of all totally ergodic dynamical systems with discrete spectrum and $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{D}$. Then the following relations are valid:*

$$\mathbf{D}_1 < \mathbf{D}_2 \Leftrightarrow H(\mathbf{D}_1) \subset \overline{H(\mathbf{D}_2)}, \tag{10}$$

$$\mathbf{D}_1 \sim \mathbf{D}_2 \Leftrightarrow \overline{H(\mathbf{D}_1)} = \overline{H(\mathbf{D}_2)}, \tag{11}$$

where $H(\mathbf{D}_i)$ denotes the group of proper values of $\mathbf{D}_i$ and $\overline{H(\mathbf{D}_i)}$ is the completion of $H(\mathbf{D}_i)$ in the following group-theoretical sense:

$$\overline{H(\mathbf{D}_i)} = \{h \in K \mid \exists n \in N: h^n \in H(\mathbf{D}_i)\}.^{*)}$$

*) For further completions of such types see, e.g., A. G. Kuroš [9]. Concerning this theorem the author is indebted to his collaborator M. Franke for some improvement.

Proof. We use the fact that in the supposed class $\mathscr{D}$ the factor relation $\mathbf{D}_1 \leqslant \mathbf{D}_2$ is equivalent to $H(\mathbf{D}_1) \subset H(\mathbf{D}_2)$. If, therefore, we call the subgroup $H = \{1\}$ of K trivial, we have triviality of $\mathbf{D} \in \mathscr{D}$ iff $H(\mathbf{D})$ is trivial. If $\mathbf{D}_1$ is trivial in one of the relations (10) and (11), these assertions are obvious. 1. Let be $\mathbf{D}_1 \prec \mathbf{D}_2$ and $\mathbf{D}_1$ nontrivial. Then we conclude from (2): for every nontrivial subgroup H' of $H(\mathbf{D}_1)$ there exists a nontrivial $H^* \subset H' \cap H(\mathbf{D}_2)$. Let be $c \in H(\mathbf{D}_1)$ and $c \neq 1$. Then c generates a nontrivial subgroup $H' := \{c\} \subset H(\mathbf{D}_1)$. H^* must be generated by a certain $c^p \neq 1$ (because of the nontriviality of the existing H^*) and we have $\{c\} \subset H(\mathbf{D}_2)$. Further, $c \in \overline{H(\mathbf{D}_2)}$ since every solution of $x^p = c^p$ is in $\overline{H(\mathbf{D}_2)}$ and one direction is proved.

If on the other hand $H(\mathbf{D}_1) \subset \overline{H(\mathbf{D}_2)}$ is valid and H' is an arbitrary nontrivial subgroup of $H(\mathbf{D}_1)$, then $H^* = H' \cap H(\mathbf{D}_2)$ is a subgroup of $H(\mathbf{D}_2)$ and of H' and only the nontriviality of H^* has to be shown. The possible choice $c \in H'$ with $c \neq 1$ yields with the supposed inclusion $c \in \overline{H(\mathbf{D}_2)}$ and therefore $c^q \in H(\mathbf{D}_2)$ for some q because of the above mentioned definitions of completion and moreover $c^q \neq 1$ (H' as a subgroup of the torsionfree $H(\mathbf{D}_1)$ has the same property!). Thus we have $1 \neq c^q \in H' \cap H(\mathbf{D}_2)$ and the nontriviality of H^* is shown.

2. (11) is an easy corollary of (10) if we consider definition 2.3. and the completion property $\bar{\bar{H}} = \bar{H}$ fulfilled for every torsionfree abelian group H.

3.3. Corollary. *Let $\mathscr{D}$ be the class of all totally ergodic dynamical systems with a discrete spectrum. Then the set M_t of all countable complete torsionfree subgroups of K is a complete system of support invariants of $\mathscr{D}$.*

Proof. Firstly the mapping

$$\Phi : \mathbf{D} \mapsto \overline{H(\mathbf{D})}, \quad (\mathbf{D} \in \mathscr{D})$$

shows together with (11) the validity of (7) and the general assumption of separability of all considered systems yields $H(\mathbf{D})$ and $\overline{H(\mathbf{D})}$ countable. Therefore we have Φ: $\mathscr{D} \to M_t$.

On the other hand (8) is fulfilled: every element in M_t is possible as a proper value group of some $\mathbf{D} \in \mathscr{D}$, see e.g. [5], p. 48. Finally (9) follows from $\overline{H(\mathbf{D}_1)} = \overline{H(\mathbf{D}_2)}$ and (11).

4. Support and dynamical systems of positive entropy

As we have seen support simulates the set of those prime numbers (see introduction) that occur in a certain integer. Of course, it is understandable if we also ask for invariants of quite another kind. Entropy, e.g., seems to be (in suitable classes) an invariant simulating the power of a prime number in the correlation just mentioned. If such a conjecture can be proved to hold, the support of all these systems is constant.

In the following we will examine this for the class $\mathscr{P}$ (in Furstenbergs terminology [3]) of all K-systems, i.e. the class of all dynamical systems $\mathbf{D}$ such that every nontrivial factor has a positive entropy.

In his paper [16] Ja. G. Sinai formulated the following results: if $\mathscr{E}$ denotes the class of all automorphic ergodic dynamical systems and if $\mathscr{E}_+$ denotes the subclass of those elements of $\mathscr{E}$ having a positive entropy, then

$$\mathbf{D}_1 \in \mathscr{E}, \quad \mathbf{D}_2 \in \mathscr{B}, \quad h(\mathbf{D}_1) \geq h(\mathbf{D}_2) \Rightarrow \mathbf{D}_2 \leq \mathbf{D}_1, \tag{12}$$

$$\mathbf{D}_1 \in \mathscr{E}_+ \Rightarrow \exists \mathbf{D}_2 \in \mathscr{P} \quad \text{with} \quad h(\mathbf{D}_2) = h(\mathbf{D}_1) \tag{13}$$

are valid, where $h(\mathbf{D})$ denotes the entropy of T in $\mathbf{D} = (X, \mathscr{S}, m, T)$. This leads to the following

4.1. Theorem. *For any $\mathbf{D}_1 \in \mathscr{P}$ and $\mathbf{D}_2 \in \mathscr{E}_+ \mathbf{D}_1$ has equal support as or less support than $\mathbf{D}_2$:*

$$\mathbf{D}_1 \in \mathscr{P}, \quad \mathbf{D}_2 \in \mathscr{E}_+ \Rightarrow \mathbf{D}_1 < \mathbf{D}_2.$$

Proof. Let $\mathbf{D}_1$ be nontrivial and $\mathbf{D}_t < \mathbf{D}' \leq \mathbf{D}_1$. Then $\mathbf{D}' \in \mathscr{P}$ with $h(\mathbf{D}') > 0$. Now choosing $\mathbf{D}^* \in \mathscr{B}$ in such a way that

$$h(\mathbf{D}^*) = \min\,(h(\mathbf{D}'), h(\mathbf{D}_2)),$$

we have with (12) $\mathbf{D}^* \leq \mathbf{D}'$, $\mathbf{D}_t < \mathbf{D}^* \leq \mathbf{D}_2$ and therefore $\mathbf{D}_1 < \mathbf{D}_2$, q.e.d.

Since $\mathscr{P} \subset \mathscr{E}_+$, we have by changing $\mathbf{D}_1$ and $\mathbf{D}_2$ the following

4.2. Corollary. *Any two K-systems have equal support:*

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathscr{P} \Rightarrow \mathbf{D}_1 \sim \mathbf{D}_2.$$

This especially (because of $\mathscr{B} \subset \mathscr{P}$) implies

4.3. Corollary. *Any two automorphic Bernoulli systems have equal support:*

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathscr{B} \Rightarrow \mathbf{D}_1 \sim \mathbf{D}_2.$$

5. Non-supportic invariants

Suggested by the last properties in sect. 4 we give the following

5.1. Definition. *Let $\mathscr{D}$ be a class of dynamical systems, Y a set, Φ: $\mathscr{D} \to Y$ a mapping and Y a system of isomorphy invariants of $\mathscr{D}$. Then Y is called a strongly non-supportic system of isomorphy invariants if the implication*

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathscr{D} \Rightarrow \exists \mathbf{D}_2' \in \mathscr{D} \text{ with } \mathbf{D}_2' \sim \mathbf{D}_2 \text{ and } \Phi(\mathbf{D}_2') = \Phi(\mathbf{D}_1) \tag{14}$$

is valid.

Since $\mathscr{B}$, resp. $\mathscr{P}$, consist of only one support class, it is obvious that $h: \mathscr{B} \to \overline{R_+}$, resp. $h: \mathscr{P} \to \overline{R_+}$, produces with $h(\mathscr{B})$, resp. $h(\mathscr{P})$, a strongly non-supportic system of isomorphy invariants for $\mathscr{B}$, resp. $\mathscr{P}$. We will consider in the following less obvious cases. They are possible even in the class of dynamical systems with entropy 0. As already remarked above, $\mathbf{D} = (X, \mathscr{S}, m, T)$ is called totally ergodic (as well as T) if all powers T^n, $(n \in N^*)$ of T are ergodic. The proper value mapping $R_{\mathbf{D}}$ defined by

$$R_{\mathbf{D}} f = \frac{f \circ T}{f}, \quad (f \in L^2(X), |f| = 1 m - \text{a.e.}),$$

where $L^2(X)$ denotes the Hilbert space $L^2(X, \mathscr{S}, m)$, induces a group $G(\mathbf{D})$ of the so-called quasiproper vectors defined by

$$G_0(\mathbf{D}) = K; \quad G_{n+1}(\mathbf{D}) = R_{\mathbf{D}}^{-1} G_n(\mathbf{D}), \quad (n \in N);$$

$$G(\mathbf{D}) = \bigcup_{n=1}^{\infty} G_n(\mathbf{D}).$$

If the elements of $G(\mathbf{D})$ (they are pairwise orthogonal if $\mathbf{D}$ is totally ergodic) span $L^2(X)$, $\mathbf{D}$ is said to have a quasidiscrete spectrum. The class of all totally ergodic systems with quasidiscrete spectra are denoted by $\mathscr{K}^*$. In [1] it was proved that the group

$$H(\mathbf{D}) = \bigcup_{n=1}^{\infty} H_n(\mathbf{D}) = \bigcup_{n=1}^{\infty} R_{\mathbf{D}}(G_n(\mathbf{D}))$$

of quasiproper values together with $R_{\mathbf{D}}$ carry all information for a complete system of isomorphy invariants of $\mathscr{K}^*$. $H(\mathbf{D})$ is commutative and torsionfree. The proper value mapping $R_{\mathbf{D}}$ works on $H(\mathbf{D})$ as a group-theoretical endomorphism, being locally nilpotent: if $h \in H(\mathbf{D})$, then there is an integer n with $R_{\mathbf{D}}^n h = 1$. Further, it may happen that there is at least $n \in N^*$ with $H_n(\mathbf{D}) = H_{n+1}(\mathbf{D})$. Then $H_n(\mathbf{D}) = H_{n+k}(\mathbf{D})$, $(k \in N^*)$ is valid and this n (denoted by $n(\mathbf{D})$) is an isomorphy invariant of $\mathscr{K}^*$, called Halmos' invariant. For a detailed representation of the theory of $\mathscr{K}^*$ see L. M. Abramov [1] or K. Jacobs [7].

To consider $n(\mathbf{D})$ in relation to support we need a theorem on factors in $\mathscr{K}^*$ proved by K. Häsler [4]: Let $\mathbf{D}_1$, $\mathbf{D}_2$ be two dynamical systems in $\mathscr{K}^*$. Then $\mathbf{D}_1 \leq \mathbf{D}_2$ holds iff there exists an injective (group-theoretical) homomorphism V: $H(\mathbf{D}_1) \to H(\mathbf{D}_2)$ with

$$Vf = f, \quad (f \in H_1(\mathbf{D}_1)),$$
$$U_{T_2} V = V U_{T_1},$$

where $U_{T_i} = f \circ T_i$, $(f \in L^2(X_i), i = 1, 2)$.

5.2. Theorem. *In the class $\mathcal{K}^*$ Halmos' invariant $n(\mathbf{D})$ is non-supportic.*

Proof. To violate (7) in 3.1 it is enough to find two $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{K}^*$ with equal support and $n(\mathbf{D}_1) \neq n(\mathbf{D}_2)$. We choose $\mathbf{D}_1 = (X_1, \mathcal{S}_1, m_1, T_1)$ with $X_1 = K$, $\mathcal{S}_1$ the Borel sets in K, m_1 the normalized rotation-invariant measure, $T_1(x) = cx$ $(c, x \in K)$ and c being no root of unity. Then (see [5], p. 58) $H(\mathbf{D}_1) = \{c^m \mid m \in Z\}$, $R_{\mathbf{D}_1}(c^m) = 1$, $(m \in Z)$ and $n(\mathbf{D}_1) = 1$ are valid. Let $\mathbf{D}_2 = (X_2, \mathcal{S}_2, m_2, T_2)$ be the system with $X_2 = K^2$, $\mathcal{S}_2$ the Borel sets in K^2, m_2 the product measure $m_1 \otimes m_1$,

$$T_2(x, y) = (cx, xy), \ (c, x, y \in K)$$

and c like above. Then ([5], p. 59) we have with $f_n(x, y) = x^n$

$$H(\mathbf{D}_2) = \{c^m f_n \mid m, n \in Z\},$$

$R_{\mathbf{D}_2}(c^m f_n) = c^n$, $(m, n \in Z)$ and $n(\mathbf{D}_2) = 2$. However, the two dynamical systems have equal support. $\mathbf{D}_1 \prec \mathbf{D}_2$ follows from (4) and the factor condition in $\mathcal{K}^*$, mentioned above, if we define the homomorphism V: $H(\mathbf{D}_1) \to H(\mathbf{D}_2)$ as the canonical embedding:

$$V(c^m) = c^m, \quad (m \in Z).$$

To prove $\mathbf{D}_2 \prec \mathbf{D}_1$

$$\mathbf{D}' \in \mathcal{N}(\mathbf{D}_2) \Rightarrow \mathcal{N}(\mathbf{D}') \cap \mathcal{N}(\mathbf{D}_1) \neq \emptyset \tag{15}$$

has to be shown. By Abramov's representation theory every nontrivial factor of $\mathbf{D}_2$ may be represented by a subgroup

$$H'_{kl} = \{c^{km} f_{ln} \mid m, n \in Z, \ k, l \in N^*, \ k \mid l\}$$

or a subgroup

$$H'_k = \{c^{km} \mid m \in Z, \ k \in N^*\}$$

of $H(\mathbf{D}_2)$. But every such subgroup has with $H(\mathbf{D}_1)$ the (sub)group H'_k in common. By this (15) is proved.

The groups $H(\mathbf{D})$ possible as groups of quasiproper values of $\mathbf{D} \in \mathcal{K}^*$ can be very complicated. It therefore seems difficult to prove $n(\mathbf{D})$ strongly non-supportic in the whole class $\mathcal{K}^*$. But if we consider only the subclass of those $\mathbf{D} \in \mathcal{K}^*$ for which $H(\mathbf{D})$ is $R_{\mathbf{D}}$-direct decomposable, this can be shown.

5.3. Definition. *Let $\mathbf{D}$ be in $\mathcal{K}^*$, $H(\mathbf{D})$ the group of all quasiproper values and $R_{\mathbf{D}}$ the proper value mapping on $H(\mathbf{D})$. $H(\mathbf{D})$ is called $R_{\mathbf{D}}$-direct decomposable, iff*

$$H(\mathbf{D}) = \bigcup_{n=1}^{\infty} H_n(\mathbf{D}) \quad \text{with} \quad H_n(\mathbf{D}) = \bigotimes_{m=1}^{n} K_m, \quad (n \in N^*)$$

and

$$R_{\mathbf{D}} K_{m+1} = K_m, \quad (m \in N, \ K_0 := \{1\}).$$

The subclass of all such $\mathbf{D} \in \mathcal{K}^*$ is denoted by $\mathcal{K}^{**}$. $\mathcal{K}^* \backslash \mathcal{K}^{**}$ is nonvoid (see e.g. [10], example 2.3.2). In $\mathcal{K}^{**}$ we have the following

5.4. Theorem. *In the class $\mathcal{K}^{**}$ Halmos' invariant is strongly nonsupportic.*

Proof. For the two $\mathbf{D}_1, \mathbf{D}_2$ chosen in the proof of 5.2 we have $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{K}^{**}$ and therefore Halmos' invariant is non-supportic on $\mathcal{K}^{**}$, in the same way as in $\mathcal{K}^*$.

Now let $\mathbf{D}_1, \mathbf{D}_2$ be two elements in $\mathcal{K}^{**}$ with $n(\mathbf{D}_1) \leqslant (\mathbf{D}_2)$. $H(\mathbf{D}_2)$ is representable in the Form

$$H(\mathbf{D}_2) = \bigotimes_{m=1}^{n(\mathbf{D}_2)} K_m \quad \text{with} \quad R_{\mathbf{D}_2} K_m = K_{m-1}, \quad (m = 1, 2, \ldots, n(\mathbf{D}_2)) \tag{16}$$

and the case $n(\mathbf{D}_2) = \infty$ is not excluded. Obviously $R_{\mathbf{D}_2}$ can be restricted to the subgroup

$$H_2' = \bigotimes_{m=1}^{n(\mathbf{D}_1)} K_m$$

and Abramov's theory yields a $\mathbf{D}_2' \in \mathcal{K}^{**}$ with $H(\mathbf{D}_2') = H_2'$, $R_{\mathbf{D}_2'} = R_{\mathbf{D}_2}$, $n(\mathbf{D}_2') = n(\mathbf{D}_1)$ and with the factor theorem in $\mathcal{K}^*$ cited above we have $\mathbf{D}_2' \leqslant \mathbf{D}_2$ and therefore $\mathbf{D}_2' < \mathbf{D}_2$.

To prove $\mathbf{D}_2 < \mathbf{D}_2'$ (we can suppose $\mathbf{D}_2$ non trivial) we have to verify

$$\mathbf{D}' \in \mathcal{N}(\mathbf{D}_2) \Rightarrow \mathcal{N}(\mathbf{D}') \cap \mathcal{N}(\mathbf{D}_2') \neq \emptyset \tag{17}$$

Again with Abramov's theory $\mathbf{D}' \in \mathcal{N}(\mathbf{D}_2)$ may be represented by a subgroup of the form

$$H' = \bigotimes_{m=1}^{n'} K_m'$$

with $n' \leqslant n(\mathbf{D}_2)$ and subgroups $K_m' \subset K_m$, $(m = 1, \ldots, n')$ and with $R_{\mathbf{D}_2} K_m' \subset K_{m-1}'$, $(m = 1, \ldots, n')$. Of course, among these groups there are those belonging to $\mathcal{N}(\mathbf{D}_2')$ if n' is chosen sufficiently small. This proves (17) and therefore $\mathbf{D}_2 < \mathbf{D}_2'$ and altogether $\mathbf{D}_2' \sim \mathbf{D}_2$.

Recalling the fact that (14) is not symmetric in $\mathbf{D}_1, \mathbf{D}_2$ we have now to consider the case $n(\mathbf{D}_2) < n(\mathbf{D}_1)$. Again $H(\mathbf{D}_2)$ is of the form (16) with finite $n(\mathbf{D}_2)$. Here we construct a group

$$H_2' = \bigotimes_{m=1}^{n(\mathbf{D}_2)} K_m \otimes \bigotimes_{m=n(\mathbf{D}_2)+1}^{n(\mathbf{D}_1)} K_m$$

with $K_m = K_{n(\mathbf{D}_2)}$, $(m = n(\mathbf{D}_2) + 1, \ldots, n(\mathbf{D}_1)$ and R_2' being the extension of $R_{\mathbf{D}_2}$ that maps K_m isomorphically onto K_{m-1} $(m = n(\mathbf{D}_2) + 1, \ldots, n(\mathbf{D}_1))$. In this case Abramov's results yield a $\mathbf{D}_2'$ with $\mathbf{D}_2' \sim \mathbf{D}_2$ and $n(\mathbf{D}_2') = n(\mathbf{D}_1)$ is fulfilled, q.e.d.

Finally we remark that in the first part of theorem 5.4, namely

$$\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{K}^{**} \quad \text{and} \quad n(\mathbf{D}_1) \leqslant n(\mathbf{D}_2) \Rightarrow \mathbf{D}_2' \in \mathcal{K}^{**} \quad \text{with}$$
$$\mathbf{D}_2' \sim \mathbf{D}_2 \quad \text{and} \quad n(\mathbf{D}_2') = n(\mathbf{D}_1),$$

$\mathcal{K}^{**}$ may be replaced by $\mathcal{K}^{*}$.

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ОПОРА В КАЧЕСТВЕ ИНВАРИАНТА ДЛЯ ДИНАМИЧЕСКИХ СИСТЕМ

Горст Михель

Резюме

Перенос понятия «множество всех простых чисел встречающихся в некотором натуральном числе» из элементарной теории чисел на энтродическую теорию проводит к понятию опоры. Это понятие может быть использовано для сравнения динамических систем на изоморфность. Если две динамические системы изоморфны, то они имеют равные опоры. В классе всех вполне эргодических систем с дискретным спектром опора идентифицируема пополнением группы всех собственных чисел системы. Все автоморфизмы Колмогорова, а потому, и все автоморфизмы Бернулли, имеют равные опоры.

Кроме того, определено понятие неопорного и сильно неопорного инвариантов и пременено на инвариант Хальмоша.