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OSCILLATORY AND NONOSCILLATORY PROPERTIES OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y^{(4)}+P(t)y''+Q(t)y=0$

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1. Introduction

The purpose of this paper is to study some properties of solutions of the linear differential equation of the fourth order

$$\text{(L)} \qquad L[y] \equiv y^{(4)}+P(t)y''+Q(t)y=0,$$

where $P(t)$, $Q(t)$ are real-valued continuous functions on the interval $I=[a, \infty)$, $-\infty<a<\infty$. It is assumed throughout that

$$\text{(A)} \qquad P(t)\leqq 0, \quad Q(t)\leqq 0 \quad \text{for all} \quad t\in I \quad \text{and} \quad Q(t)$$

not identically zero in any subinterval of I.

The equation (L) has been studied by V. Pudei [8, 9]. W. Leighton and Zeev Nehari [7] have studied a slightly more general class of self-adjoint linear differential equations of the fourth order and have given a number of results concerning the existence of oscillatory and nonoscillatory solutions.

So far the results of papers dealing with the oscillation of solutions of the differential equations of the fourth order were based on the distribution of the zeros of nontrivial solutions. These methods are extremely difficult. This paper deals with the oscillation of solutions but the method of deriving the results will be based on the behaviour of nonoscillatory solutions. New results and another view of the behaviour of solutions will be obtained. The method that has been used in this paper has been used only in the equations of the third order and in the equation of the fourth order $y^{(4)}+Q(t)y=0$ [6].

A necessary and sufficient condition is given for the oscillation of the differential equation (L) in terms of the behaviour of nonoscillatory solutions. At the same time necessary and sufficient conditions are derived for the nonoscillation of the differential equation (L). It is shown that if (L) is oscillatory, then it has two linearly independent oscillatory solutions such that the zeros of any two independent linear combinations of these solutions separate on (t_0, ∞), $t_0 \in I$.

A nontrivial solution of a differential equation of the n-th order is called *oscillatory* if its set of zeros is not bounded from above. Otherwise, it is called *nonoscillatory.* A differential equation of the n-th order will be called nonoscillatory, when all its solutions are nonoscillatory; oscillatory, when at least one of its solutions is oscillatory. A differential equation of the n-th order is said to be *disconjugate* in an interval I iff every nontrivial solution has at most $n-1$ zeros in I.

Let $C^n(I)$ denote the set of all real-valued functions such that its n-th derivatives are continuous on I.

2. Preliminary results

Lemma 1, [1]. *Let $c(t)$, $f(t)$ be functions of class $C[t_0, \infty)$, let the differential equation*

$$w'' + c(t)w = 0$$

be nonoscillatory and $f(t)$ does not change the sign in $[t_0, \infty)$. Then also the differential equation

$$w'' + c(t)w = f(t)$$

is nonoscillatory in $[t_0, \infty)$.

Lemma 2. *Suppose that* (A) *holds. Then for every nonoscillatory solution y of the equation* (L) *there exists a number $t_0 \geqq a$ such that either*

$$y(t)y'(t) > 0, \quad y(t)y''(t) > 0,$$

or

$$y(t)y'(t) < 0, \quad y(t)y''(t) > 0,$$

or

$$y(t)y'(t) > 0, \quad y(t)y''(t) < 0$$

for all $t \geqq t_0$.

Proof. Let $y(t)$ be a nonoscillatory solution of (L). Then there exists a number $t_1 \geqq a$ *such that* $y(t) \neq 0$ in $[t_1, \infty)$. Assume, without loss of generality that $y(t) > 0$ on $[t_1, \infty)$. Substitution $y''(t) = z(t)$ into (L) leads to the following differential equation for z

$$z'' + P(t)z = -Q(t)y. \tag{1}$$

Since the equation

$$z'' + P(t)z = 0$$

is nonoscillatory in $[t_1, \infty)$ and $Q(t)y(t)$ does not change the sign in $[t_1, \infty)$, it follows that equation (1) is nonoscillatory in $[t_1, \infty)$, by Lemma 1. Hence there exists a number $t_2 \geqq t_1$ such that $z(t) \neq 0$, i.e. $y'' \neq 0$ in $[t_2, \infty)$. From this it follows further that there exists a number $t_0 \geqq t_2$ such that $y' \neq 0$ for $t \geqq t_0$. Four cases may occur for $t \geqq t_0$:

a) $$y(t)y'(t)>0, \quad y(t)y''(t)>0,$$
b) $$y(t)y'(t)<0, \quad y(t)y''(t)>0,$$
c) $$y(t)y'(t)>0, \quad y(t)y''(t)<0,$$
d) $$y(t)y'(t)<0, \quad y(t)y''(t)<0.$$

The case d) is easily seen to be impossible. Thus there are possible only the cases a), b), c). This completes the proof of the Lemma.

Lemma 3. *Let $A(t, s)$ be nonnegative and continuous function for $t_0 \leqq s \leqq t$ (nonpositive for $a \leqq t \leqq s \leqq t_0$). If $g(t)$, $\varphi(t)$ ($\psi(t)$) are continuous functions in the interval $[t_0, \infty)$ ($[a, t_0]$) and*

$$\varphi(t) \leqq g(t) + \int_{t_0}^{t} A(t, s)\varphi(s)\,\mathrm{d}s\,, \quad \text{for} \quad t \in [t_0, \infty)$$

$$\left(\psi(t) \geqq g(t) + \int_{t_0}^{t} A(t, s)\psi(s)\,\mathrm{d}s\,, \quad \text{for} \quad t \in [a, t_0]\right),$$

then every solution $y(t)$ of the integral equation

$$y(t) = g(t) + \int_{t_0}^{t} A(t, s)y(s)\,\mathrm{d}s \tag{2}$$

satisfies the inequality

$$y(t) \geqq \varphi(t) \quad \text{in} \quad [t_0, \infty)$$
$$(y(t) \leqq \psi(t) \quad \text{in} \quad [a, t_0]).$$

The assertion of this Lemma may be proved by the fact that the resolvent of the equation (2) under the assumptions is nonnegative function for $t_0 \leqq s \leqq t$ (nonpositive function for $a \leqq s \leqq t \leqq t_0$). If we suppose in addition that $g(t) \geqq 0$ for $t \in [t_0, \infty)$ ($g(t) \leqq 0$ for $t \in [a, t_0]$), then the solution $y(t)$ of (2) satisfies the inequality

$$y(t) \geqq g(t) \geqq 0 \quad \text{for} \quad t \in [t_0, \infty)$$
$$(y(t) \leqq g(t) \leqq 0 \quad \text{for} \quad t \in [a, t_0]).$$

Lemma 4. *Suppose that* (A) *holds and let $y(t)$ be a nontrivial solution of* (L) *satisfying the initial conditions*

$$y(t_0) = y_0 \geqq 0, \quad y'(t_0) = y_0' \geqq 0,$$
$$y''(t_0) = y_0'' \geqq 0, \quad y'''(t_0) = y_0''' \geqq 0$$

($t_0 \in I$ *arbitrary*, $y_0 + y_0' + y_0'' + y_0''' > 0$).
Then

$$y(t) > 0, \quad y'(t) > 0, \quad y''(t) > 0, \quad y'''(t) > 0$$

for all $t > t_0$.

Proof. The initial-value problem

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \\ y''(t_0) = y_0'', \quad y'''(t_0) = y_0'''$$

is equivalent to Voltera's following integral equation,

$$y'''(t) = g(t) + \int_{t_0}^{t} A(t, s) y'''(s) \, ds, \tag{3}$$

where

$$g(t) = y_0''' - y_0'' \int_{t_0}^{t} \left[P(s) + \frac{(s-t)^2}{2} Q(s) \right] ds - \\ - y_0' \int_{t_0}^{t} (s - t_0) Q(s) \, ds - y_0 \int_{t_0}^{t} Q(s) \, ds$$

and

$$A(t, s) = - \int_{t_0}^{t} \left[P(\xi) + \frac{(\xi - s)^2}{2} Q(\xi) \right] d\xi.$$

The hypotheses of the Lemma imply that $g(t) > 0$ and $A(t, s) \geqq 0$ for $t \in (t_0, \infty)$. Then by Lemma 3

$$y'''(t) \geqq g(t) > 0 \quad \text{for all} \quad t \in (t_0, \infty).$$

Hence there follows the assertion of Lemma 4.

Lemma 5. *Suppose that* (A) *holds and let* $y(t)$ *be a nontrivial solution of* (L) *satisfying the initial conditions*

$$y(t_0) = y_0 \geqq 0, \quad y'(t_0) = y_0' \leqq 0, \\ y''(t_0) = y_0'' \geqq 0, \quad y'''(t_0) = y_0''' \leqq 0,$$

($t_0 \in I$ *arbitrary*, $y_0^2 + y_0'^2 + y_0''^2 + y_0'''^2 > 0$).

Then

$$y(t) > 0, \quad y'(t) < 0, \quad y''(t) > 0, \quad y'''(t) < 0$$

for all $t \in [a, t_0)$.

Proof. The initial-value problem

$$L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \\ y''(t_0) = y_0'', \quad y'''(t_0) = y_0'''$$

is equivalent to the integral equation (3). The hypotheses of the Lemma imply that $g(t)<0$, $A(t,s)\leqq 0$ for $a\leqq t\leqq s\leqq t_0$. Then by Lemma 3 there is $y'''(t)<0$ for $t\in[a,t_0)$. Hence the assertion of the Lemma follows.

Let $W(w_i, w_k; t)$ denote the Wronskian determinant of the functions w_i, w_k at the point t:

$$W(w_i, w_k; t) = w_i(t)w_k'(t) - w_i'(t)w_k(t).$$

Lemma 6. *Let there be functions* $w_i(t)\in C^4[t_0,\infty)$, $i=1,2,3$, $t_0\in I$ *with the properties*

$$\begin{gathered} w_2>0,\quad w_3>0,\\ W(w_1,w_2;t)>0,\quad W(w_1,w_3;t)>0,\quad W(w_2,w_3;t)>0,\\ W(w_1,w_2;w_3;t)>0 \quad \textit{for} \quad t\in(t_0,\infty) \quad \textit{and}\\ L[w_1]\leqq 0,\quad L[w_2]\geqq 0,\quad L[w_3]\leqq 0 \quad \textit{for} \quad t\in(t_0,\infty). \end{gathered}$$

Then equation (L) *is disconjugate in the interval* $[t_0,\infty)$ ([5], pp. 77, 80).

We note if y is a solution of (L), then so is $-y$. Hence it follows from Lemma 4 that $y(t_0)\leqq 0$, $y'(t_0)\leqq 0$, $y''(t_0)\leqq 0$, $y'''(t_0)\leqq 0$ (but not all zero) implies $y(t)<0$, $y'(t)<0$, $y''(t)<0$, $y'''(t)<0$ for all $t>t_0$. Similarly, it follows from Lemma 5 that if y is a nontrivial solution such that $y(t_0)\leqq 0$, $y'(t_0)\geqq 0$, $y''(t_0)\leqq 0$, $y'''(t_0)\geqq 0$, then $y(t)<0$, $y'(t)>0$, $y''(t)<0$, $y'''(t)>0$ for all $t\in[a,t_0)$.

3. The existence of monotonic solutions

Throughout the remainder of this paper let z_0, z_1, z_2, z_3 denote solutions of (L) defined on I by the initial conditions

$$z_i^{(j)}(a)=\delta_{ij}=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases},$$

for $i,j=0,1,2,3$.

We will show the existence of solutions $y(t)$ and $z(t)$ such that $y(t)>0$, $y'(t)>0$, $y''(t)>0$, $y'''(t)>0$ for all $t\in I$ and $z(t)>0$, $z'(t)<0$, $z''(t)>0$, $z'''(t)<0$ for all $t\in I$.

Theorem 1. *Suppose that* (A) *holds. There exists a solution* $y(t)$ *of* (L) *such that* $y(t)>0$, $y'(t)>0$, $y''(t)>0$, $y'''(t)>0$ *for all* $t\in I$.

Proof. Let $y(t)$ be a solution of (L) which satisfies the initial conditions $y^{(i)}(a)>0$, $i=0,1,2,3$. Then by Lemma 4 $y^{(i)}(t)>0$ for all $t\in I$ and $i=0,1,2,3$.

Theorem 2. *Suppose that* (A) *holds. There exists a solution* $z(t)$ *of* (L) *such that* $(-1)^i z^{(i)}(t)>0$ *for all* $t\in I$ *and* $i=0,1,2,3$.

Proof. For each natural number $n>a$, let c_{0n}, c_{1n}, c_{2n} and c_{3n} be numbers satisfying

$$
(4)\qquad
\begin{aligned}
c_{0n}z_0(n)+c_{1n}z_1(n)+c_{2n}z_2(n)+c_{3n}z_3(n)&=0\\
c_{0n}z_0'(n)+c_{1n}z_1'(n)+c_{2n}z_2'(n)+c_{3n}z_3'(n)&=0\\
c_{0n}z_0''(n)+c_{1n}z_1''(n)+c_{2n}z_2''(n)+c_{3n}z_3''(n)&=0\\
c_{0n}z_0'''(n)+c_{1n}z_1'''(n)+c_{2n}z_2'''(n)+c_{3n}z_3'''(n)&<0\\
c_{0n}^2+c_{1n}^2+c_{2n}^2+c_{3n}^2&=1\,.
\end{aligned}
$$

Let $z_n(t)=c_{0n}z_0(t)+c_{1n}z_1(t)+c_{2n}z_2(t)+c_{3n}z_3(t)$, The existence of numbers c_{0n}, c_{1n}, c_{2n} and c_{3n}, satisfying the above conditions, is easy to verify. Since z_0, z_1, z_2 and z_3 are linearly independent, $z_n(t)$ is a nontrivial solution of (L). Since for each natural number n, the sequences $\{c_{in}\}$, $i=0, 1, 2, 3$ are bounded, there exists a sequence of integers $\{n_j\}$ such that the subsequences $\{c_{in_j}\}$ converge to numbers c_i, $i=0, 1, 2, 3$. From (4) we see that

$$
(5)\qquad c_0^2+c_1^2+c_2^2+c_3^2=1\,.
$$

The sequences $\{z_{n_j}(t)\}$, $\{z_{n_j}'(t)\}$, $\{z_{n_j}''(t)\}$, $\{z_{n_j}'''(t)\}$ converge uniformly on any finite subinterval of $[a, \infty)$ to the functions $z(t)$, $z'(t)$, $z''(t)$, $z'''(t)$, respectively, where $z(t)$ is a nontrivial solution of (L). By Lemma 5 $(-1)^i z^{(i)}(t)\geqq 0$ for all $t\in I$ and $i=0, 1, 2, 3$. Further, since $z(t)$ is a nontrivial solution and $Q(t)\leqq 0$ and not identically zero in any subinterval of I, it is easy to see that there is no number $\tau\in I$ such that $z^{(i)}(\tau)=0$ for some $i=0, 1, 2, 3$. Hence $(-1)^i z^{(i)}(t)>0$ on I.

Theorem 3. *Suppose that* (A) *holds and let*

$$
\int_{t_0}^{t} t^{2+\alpha}Q(t)\,\mathrm{d}t=-\infty,\quad t_0\geqq\max\{a, 0\},\quad 0\leqq\alpha<1\,.
$$

Then for every solution $y(t)$ of (L) *such that $y(t)y'(t)\leqq 0$, $y(t)y''(t)\geqq 0$ and $y(t)y'''(t)\leqq 0$ for $t\geqq t_0$ there holds*

$$
\lim_{t\to\infty} y(t)=\lim_{t\to\infty} y'(t)=\lim_{t\to\infty} y''(t)=\lim_{t\to\infty} y'''(t)=0\,.
$$

Proof. Suppose that $y(t)>0$. Then by the above conditions it follows that $y'(t)\leqq 0$, $y''(t)\geqq 0$, $y'''(t)\leqq 0$ for $t\geqq t_0$. From this and equation (L) we obtain $y^{(4)}(t)\geqq 0$ for $t\geqq t_0$. From the above inequalities it follows easily that

$$
\lim_{t\to\infty} y'(t)=\lim_{t\to\infty} y''(t)=\lim_{t\to\infty} y'''(t)=0\,.
$$

Suppose that $\lim y(t)=B>0$.

Multiplying (L) by $t^{2+\alpha}$, $0\leqq\alpha<1$, integrating from t_0 to t, we obtain

$$
[y'''(s)s^{2+\alpha}]_{t_0}^{t}-[(2+\alpha)s^{1+\alpha}y''(s)]_{t_0}^{t}+[(2+\alpha)(1+\alpha)s^{\alpha}y'(s)]_{t_0}^{t}-
$$

$$(6)\quad -[(2+\alpha)(1+\alpha)\alpha s^{\alpha-1}y(s)]_{t_0}^{t}+(2+\alpha)(1+\alpha)\alpha(\alpha-1)\int_{t_0}^{t} s^{\alpha-2}y(s)\,ds+ + \int_{t_0}^{t} s^{2+\alpha}P(s)y''(s)\,ds+\int_{t_0}^{t} s^{2+\alpha}Q(s)y(s)\,ds=0.$$

Since $y(t)$ has a finite limit and $0\leqq\alpha<1$ from (6) it follows that

$$t^{2+\alpha}y'''(t)\geqq K-B\int_{t_0}^{t} s^{2+\alpha}Q(s)\,ds,$$

where K is a constant. Hence it follows that $y'''(t)>0$ for sufficiently large t. But this is a contradiction and the proof is complete.

Remark. Later we shall show the uniqueness (except for a constant factor) of the solution $y(t)$.

4. Conditions for disconugation

Theorem 4. *Let there be functions $w_i(t)\in C^4[t_0,\infty)$, $i=1,2,3$, $t_0\in I$ such that*

$$(7)\quad \begin{gathered} w_1(t)>0,\quad w_1'(t)<0,\quad w_1''(t)>0 \quad \textit{for}\quad t\in[t_0,\infty),\\ w_2(t)>0,\quad w_2'(t)>0,\quad w_2''(t)\leqq 0 \quad \textit{for}\quad t\in[t_0,\infty),\\ w_3(t)>0,\quad w_3'(t)>0,\quad w_3''(t)>0 \quad \textit{for}\quad t\in(t_0,\infty),\\ w_3(t_0)=0 \end{gathered}$$

and

$$L[w_1]\leqq 0,\quad L[w_2]\geqq 0,\quad L[w_3]\leqq 0 \quad \textit{for}\quad t\in(t_0,\infty).$$

Then equation (L) *is disconjugate on* $[t_0,\infty)$.

Proof. Conditions (7) imply that $W(w_1,w_2;t)>0$, $W(w_1,w_3;t)>0$ on $[t_0,\infty)$. We will show that $W(w_2,w_3;t)>0$ and $W(w_1,w_2,w_3;t)>0$ on (t_0,∞).

Indeed, since

$$W(w_2,w_3;t_0)=w_2(t_0)w_3'(t_0)-w_2'(t_0)w_3(t_0)\geqq 0$$

and

$$W'(w_2,w_3;t)=w_2(t)w_3''(t)-w_2''(t)w_3(t)>0 \text{ on } (t_0,\infty),$$

$$(8)\quad W(w_2,w_3;t)>0 \quad\text{on}\quad (t_0,\infty).$$

$$\begin{aligned} W(w_1,w_2,w_3;t)=w_1(t)[w_2'(t)w_3''(t)-w_2''(t)w_3'(t)]-\\ -w_1'(t)[w_2(t)w_3''(t)-w_2''(t)w_3(t)]+\\ +w_1''(t)[w_2(t)w_3'(t)-w_2'(t)w_3(t)]. \end{aligned}$$

It is clear that the first and second term on the right-hand side is positive for $t>t_0$. Since $w_1''(t)>0$ in $[t_0,\infty)$ by hypothesis, it follows from (8) that the last term is also positive for $t>t_0$. Hence $W(w_1,w_2,w_3;t)>0$ on (t_0,∞).

Since the conditions of Lemma 6 are satisfied, equation (L) is disconugate on $[t_0, \infty)$. This completes the proof of Theorem 4.

Theorem 5. *Let there be functions* $w_i(t) \in C^4[t_0, \infty)$, $i=1, 2, 3$, $t_0 \in I$ *such that*

$$
\begin{aligned}
&w_1(t)>0, \quad w_1'(t)<0, \quad w_1''(t)>0, w_1'''(t)<0 \quad \textit{for} \quad t\in[t_0, \infty),\\
(9) \qquad &w_2(t)>0, \quad w_2'(t)>0, \quad w_2''(t)>0, w_2'''(t)\leqq 0 \quad \textit{for} \quad t\in[t_0, \infty),\\
&w_3(t)>0, \quad w_3'(t)>0, \quad w_3''(t)>0,\\
&w_3'''(t)>0 \quad \textit{for} \quad t\in(t_0, \infty),\\
&w_3(t_0)=w_3'(t_0)=0
\end{aligned}
$$

and

$$L[w_1]\leqq 0, \quad L[w_2]\geqq 0, \quad L[w_3]\leqq 0 \quad \textit{for} \quad t\in(t_0, \infty).$$

Then equation (L) *is disconjugate on* $[t_0, \infty)$.

Proof. Conditions (9) imply $W(w_1, w_2; t)>0$, $W(w_1, w_3; t)>0$ on $[t_0, \infty)$. We will show that $W(w_2, w_3; t)>0$ and $W(w_1, w_2, w_3; t)>0$ on (t_0, ∞). Let

$$\alpha(t)=w_2'(t)w_3''(t)-w_2''(t)w_3'(t) \quad \text{for} \quad t\geqq t_0.$$

Then

$$\alpha(t_0)=w_2'(t_0)w_3''(t_0)\geqq 0$$

and

$$\alpha'(t)=w_2'(t)w_3'''(t)-w_2'''(t)w_3'(t)>0 \quad \text{on} \quad (t_0, \infty).$$

It follows from this that $\alpha(t)>0$ on (t_0, ∞). Since

$$
\begin{aligned}
W(w_2, w_3; t_0)&=w_2(t_0)w_3'(t_0)-w_2'(t_0)w_3(t_0)=0\\
W'(w_2, w_3; t_0)&=w_2(t_0)w_3''(t_0)-w_2''(t_0)w_3(t_0)\geqq 0
\end{aligned}
$$

and

$$
\begin{aligned}
W''(w_2, w_3; t)&=w_2'(t)w_3''(t)-w_2''(t)w_3'(t)+\\
&+w_2(t)w_3'''(t)-w_2'''(t)w_3(t)=\alpha(t)+w_2(t)w_3'''(t)-\\
&-w_2'''(t)w_3(t)>0 \quad \text{on} \quad (t_0, \infty),
\end{aligned}
$$

then

$$W'(w_2, w_3; t)>0 \quad \text{and} \quad W(w_2, w_3; t)>0 \quad \text{on} \quad (t_0, \infty).$$

Hence we again obtain from (9) that

$$
\begin{aligned}
W(w_1, w_2, w_3; t)&=w_1(t)[w_2'(t)w_3''(t)-w_2''(t)w_3'(t)]-\\
&-w_1'(t)[w_2(t)w_3''(t)-w_2''(t)w_3(t)]+w_1''(t)[w_2(t)w_3'(t)-\\
&-w_2'(t)w_3(t)]=w_1(t)\alpha(t)-w_1'(t)W'(w_2, w_3; t)+\\
&+w_1''(t)W(w_2, w_3; t)>0
\end{aligned}
$$

on (t_0, ∞). It follows from Lemma 6 that eqation (L) is disconjugate on $[t_0, \infty)$ and the proof is complete.

The following consequences follow from Theorems 4 and 5.

Corollary 1. *Let* (L) *have solutions* y_1, y_2 and y_3 *with*

$$\begin{aligned}
&y_1(t)>0, \quad y_1'(t)<0, \quad y_1''(t)>0 \quad \text{on} \quad [t_0, \infty),\\
&y_2(t)>0, \quad y_2'(t)>0, \quad y_2''(t)\leqq 0 \quad \text{on} \quad [t_0, \infty),\\
&y_3(t)>0, \quad y_3'(t)>0, \quad y_3''(t)>0 \quad \text{on} \quad (t_0, \infty),\\
&y_3(t_0)=0.
\end{aligned}$$

Then (L) *is disconjugate on* $[t_0, \infty)$.

Corollary 2. *Let* (L) *have solutions* y_1, y_2 *and* y_3 *with*

$$\begin{aligned}
&y_1(t)>0, \quad y_1'(t)<0, \quad y_1''(t)>0, y_1'''(t)<0 \quad \text{on} \quad [t_0, \infty),\\
&y_2(t)>0, \quad y_2'(t)>0, \quad y_2''(t)>0, y_2'''(t)\leqq 0 \quad \text{on} \quad [t_0, \infty),\\
&y_3(t)>0, \quad y_3'(t)>0, \quad y_3''(t)>0, y_3'''(t)>0 \quad \text{on} \quad (t_0, \infty),\\
&y_3(t_0)=y_3'(t_0)=0.
\end{aligned}$$

Then (L) *is disconjugate on* $[t_0, \infty)$.

The following sufficient conditions for (L) to be disconjugate are simple consequences of Theorems 1, 2, 4 and 5.

Corollary 3. *Suppose that* (A) *holds and let there be function* $w \in C^4[t_0, \infty)$, $t_0 \in I$ *such that* $w>0$, $w'>0$, $w''\leqq 0$, $L[w]\geqq 0$ *on* (t_0, ∞). *Then* (L) *is disconjugate on* $[t_0, \infty)$.

Corollary 4. *Suppose that* (A) *holds and let there be function* $w \in C^4[t_0, \infty)$, $t_0 \in I$ *such that* $w>0$, $w'>0$, $w''>0$, $w'''\leqq 0$ *and* $L[w]\geqq 0$ *on* (t_0, ∞). *Then* (L) *is disconjugate on* $[t_0, \infty)$.

5. Necessary and sufficient conditions for oscillatory and nonoscillatory equations

Theorem 6. *Suppose that* (A) *holds. Then equation* (L) *is oscillatory if and only if for every nonoscillatory solution* $y(t)$ *of* (L) *there holds either*

(10) $$y(t)y'(t)>0, \quad y(t)y''(t)>0, \quad y(t)y'''(t)>0$$

on $[t_0, \infty)$ *for some* $t_0 \in I$, *or*

(10′) $$y(t)y'(t)<0, \quad y(t)y''(t)>0, \quad y(t)y'''(t)<0 \quad \text{on} \quad I.$$

Proof. Assume that (L) is oscillatory and let $y(t)$ be a nonoscillatory solution of (L). Then by Lemma 2 there exists a number $t_1 \in I$ such that either

$$y(t)y'(t)>0, \quad y(t)y''(t)>0,$$

or

$$y(t)y'(t)<0, \quad y(t)y''(t)>0,$$

or

$$y(t)y'(t)>0, \quad y(t)y''(t)<0$$

for all $t \geqq t_1$. There is no loss of generality in assuming that $y(t)>0$ for all $t \in [t_1, \infty)$. We note that if $y''(t)>0$, it then follows from (L) that $y^{(4)}(t) \geqq 0$ (not identically zero in any subinterval). Hence these cases are possible:

a) $\quad y(t)>0, \quad y'(t)>0, \quad y''(t)>0, \quad y'''(t)>0,$

b) $\quad y(t)>0, \quad y'(t)>0, \quad y''(t)>0, \quad y'''(t)<0,$

c) $\quad y(t)>0, \quad y'(t)<0, \quad y''(t)>0, \quad y'''(t)>0,$

d) $\quad y(t)>0, \quad y'(t)<0, \quad y''(t)>0, \quad y'''(t)<0,$

e) $\quad y(t)>0, \quad y'(t)>0, \quad y''(t)<0$

for $t \geqq t_0$, where t_0 is some number greater than or equal to t_1. In the case c) this is easily seen to be impossible. Only the cases a), b), d), e) may occur. Suppose that $y(t)$ does not satisfy the conditions (10), (10′). Then either b) or e) holds. If a solution satisfying condition b) or e) existed, equation (L) then would be nonoscillatory, by Corollaries 1 and 2 of Theorems 4 and 5, contrary to the hypothesis. This completes the proof of the first half of Theorem 6.

If $y(t)$ is an arbitrary nonoscillatory solution of (L), which satisfies condition (10) or (10′), we could then construct oscillatory solutions u and v of (L) given by

$$u = b_0 z_0(t) + b_3 z_3(t)$$
$$v = c_2 z_2(t) + c_3 z_3(t),$$

where $b_0^2 + b_3^2 = c_2^2 + c_3^2 = 1$. The proof of this part of the Theorem is similar to that of Theorem 3 ([6], p. 293) and will be omitted.

Remark 1. An argument, similar to the one given to show that u and v are oscillatory, can be given to show that any nontrivial linear combination of u and v is oscillatory.

Further, we note that u and v are linearly independent since, otherwise, we would have $u = cz_3$, $c \neq 0$ and this would contradict the fact that u is oscillatory.

Remark 2. If (L) is oscillatory, then it has three linearly independent oscillatory solutions.

The proof of this is virtually the same as that of Theorem 4 ([6], p. 294).

Remark 3. We note that in view of Theorem 6 and Remark 2, the conditions (10), (10′) are equivalent to the existence of three linearly independent oscillatory solutions.

Theorem 7. *Suppose that* (A) *holds. Then equation* (L) *is nonoscillatory on I if and only if there exists a number $t_0 \in I$ and a solution $y(t)$ of* (L) *such that either*

$$y(t)>0, \quad y'(t)>0, \quad y''(t)<0,$$

or

$$y(t)>0, \quad y'(t)>0, \quad y''(t)>0, \quad y'''(t)<0$$

for all $t \geqq t_0$.

Proof. The sufficient condition follows from Corollaries 3 and 4.

It is easy to show that the existence of such a solution is also necessary. Indeed, if (L) is nonoscillatory there must exist a nonoscillatory solution $y(t)$ which does not satisfy the conditions (10), (10'). Then by Lemma 2 and from the proof of Theorem 6 it follows that there exists a number $t_0 \in I$ such that either

$$y(t)>0, \quad y'(t)>0, \quad y''(t)<0,$$

or

$$y(t)>0, \quad y'(t)>0, \quad y''(t)>0, \quad y'''(t)<0$$

for all $t \geqq t_0$.

Theorem 8. *Suppose that* (A) *holds. Then equation* (L) *is nonoscillatory on* I *if and only if there exists a function* $w(t) \in C^4\,[t_0, \infty)$, $t_0 \in I$, *such that either*

$$w(t)>0, \quad w'(t)>0, \quad w''(t)<0, \quad L[w] \geqq 0$$

or

$$w(t)>0, \quad w'(t)>0, \quad w''(t)>0, \quad w'''(t)<0, \quad L[w] \geqq 0.$$

Proof. Suppose that (L) is nonoscillatory on I. It follows from Theorem 7 that there exists a number $t_0 \in I$ and a function $w(t)$ such that either

$$w(t)>0, \quad w'(t)>0, \quad w''(t)<0, \quad L[w]=0$$

or

$$w(t)>0, \quad w'(t)>0, \quad w''(t)>0, \quad w'''(t)<0, \quad L[w]=0$$

for all $t \geqq t_0$.

The proof of the second part of the Theorem follows from Corollaries 3 and 4.

Theorem 9. *Suppose that* (A) *holds. Then equation* (L) *is nonoscillatory on* I *if and only if there exists a number* $t_0 \in I$ *such that* (L) *is disconjugate on* $[t_0, \infty)$.

Proof. The necessity of the condition follows from Theorem 8 and Corollaries 3 and 4. The proof of the sufficiency is based on the fact that the solution has only a finite number of zeros on the compact interval $[a, t_0]$. Hence, if (L) is disconjugate on $[t_0, \infty)$, then it is nonoscillatory on $[a, \infty)$.

6. The properties of the zeros of solutions of oscillatory differential equations

We will now show when the zeros of two linearly independent oscillatory solutions separate. First we state the following theorem.

Theorem 10. *Suppose that* (A) *holds and equation* (L) *is oscillatory. Let* u *and* v *be the solutions as constructed in the proof of Theorem 6. If* Y_1 *and* Y_2 *are two* *independent linear combinations of u and v, then there exists a number $t_0 \in I$ such that for all $t > t_0 Y_1^{(j)}$ and $Y_2^{(j)}$ cannot have any common zeros, $j = 0, 1, 2, 3$.*

Proof. To prove the Theorem it is sufficient to show that there exists at most one point $s \in I$ such that $Y_1^{(j)}(s) = Y_2^{(j)}(s) = 0$ for some $j = 0, 1, 2$. Suppose that $Y_1^{(j)}(\tau) = Y_2^{(j)}(\tau) = 0$ for some $j = 0, 1, 2$ and some other point $\tau > s$. Then there exist constants c_1 and c_2, $c_1^2 + c_2^2 \neq 0$, satisfying

$$c_1 Y_1^{(j)}(\tau) + c_2 Y_2^{(j)}(\tau) = 0, \tag{11}$$

$$c_1 Y_1^{(j+1)}(\tau) + c_2 Y_2^{(j+1)}(\tau) = 0. \tag{12}$$

Let $Y = c_1 Y_1 + c_2 Y_2$. Then it follows, by Lemmas 4 and 5, that either

$$\operatorname{sgn} Y(t) = \operatorname{sgn} Y^{(j)}(t), \quad j = 1, 2, 3, \quad \text{for} \quad t > \tau \tag{13}$$

or

$$\operatorname{sgn} Y^{(j)}(t) \neq \operatorname{sgn} Y^{(j+1)}(t), \quad j = 0, 1, 2, \quad \text{for} \quad t \in [a, \tau]. \tag{14}$$

The case (13) contradicts the fact that every linear combination of u and v is oscillatory (Remark 1). It follows from (14) that $Y^{(j)}(t) \neq 0$ for $t \in [a, \tau]$, $j = 0, 1, 2$. This contradicts the assumption $Y^{(j)}(s) = 0$ for some $j = 0, 1, 2$. If $j = 3$, the proof is similar; we replace (12) by

$$c_1 Y_1^{(j-1)}(\tau) + c_2 Y_2^{(j-1)}(\tau) = 0.$$

Hence there exists a number $t_0 \geqq s \geqq a$ such that the assertion of the Theorem holds.

Theorem 11. *Suppose that* (A) *holds and equation* (L) *is oscillatory. Let u and v be the solutions as constructed in the proof of Theorem 6. Then there exists a number $t_0 \in I$ such that the zeros of any two independent linear combinations of u and v separate on (t_0, ∞).*

Proof. Let Y_1 and Y_2 be any two independent linear combinations of u and v. According to Theorem 10 we can choose $t_0 \in I$ such that $Y_1^{(j)}$ and $Y_2^{(j)}$, $j = 0, 1, 2$, have no common zeros in (t_0, ∞). Let t_1 and t_2 $(t_0 < t_1 < t_2)$ be any two consecutive zeros of Y_1. Suppose that Y_2 has no zero between t_1 and t_2. Then by Theorem 10 $Y_2(t_1) Y_2(t_2) \neq 0$ and hence Y_2 does not wanish in the interval $[t_1, t_2]$. Thus, by Rolle's Theorem there exists a point $\tau \in (t_1, t_2)$ such that

$$\left(\frac{Y_1}{Y_2}\right)'_{t=\tau} = 0,$$

and hence $Y_2 Y_1' - Y_2' Y_1$, wanishes at a point τ.

Therefore, there exist the constants c_1 and c_2, $c_1^2 + c_2^2 \neq 0$ satisfying

$$c_1 Y_1(\tau) + c_2 Y_2(\tau) = 0$$
$$c_1 Y_1'(\tau) + c_2 Y_2'(\tau) = 0.$$

The solution $y = c_1 Y_1 + c_2 Y_2$ by Remark 1 is oscillatory and hence necessarily $y''(\tau)y'''(\tau) < 0$. Since $Y_1'(a) = Y_2'(a) = 0$ $(a < \tau)$, then $y'(a) = 0$. By Lemma 5 $y(t)y'(t) < 0$ for $t < \tau$, which contradicts $y'(a) = 0$. The proof is complete.

The following theorem gives a condition for the uniqueness of the solution $z(t)$ of Theorem 2.

Theorem 12. *Suppose that* (A) *holds and let equation* (L) *be nonoscillatory. Then there exists at most one solution (with the exception of constant multiples) of* (L) *such that*

$$\operatorname{sgn} y^{(j)}(t) \neq \operatorname{sgn} y^{(j+1)}(t), \quad j = 0, 1, 2$$

for $t \in I$ *and*

$$\lim_{t \to \infty} y(t) = 0.$$

Proof. Suppose that there exists some other solution $z(t)$ linearly independent of $y(t)$, having the same property. Then there exists a constant c such that $z(\tau) + cy(\tau) = 0$, $\tau \in I$. Let $Y(t) = z(t) + cy(t)$. Since $Y(t)$ is nonoscillatory solution of (L) and $Y(\tau) = 0$, by Lemma 2 and 5 there exists a number $t_0 \geqq \tau$ such that either

$$YY' > 0, \quad YY'' > 0$$

or

$$YY' > 0, \quad YY'' < 0$$

for all $t \geqq t_0$. But this contradicts the fact that $z(t)$ and $y(t)$ are both bounded and $\lim_{t \to \infty} Y(t) = 0$. This contradiction proves the Theorem.

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ОСЦИЛЛЯЦИОННЫЕ И НЕОСЦИЛЛЯЦИОННЫЕ СВОЙСТВА РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

Ян Регенда

Резюме

В работе приведены необходимые и достаточные условия для осцилляции и неосцилляции решений уравнения. Кроме того, рассматривается вопрос о том, чередуются-ли нули двух линейно независимых решений.