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ON CONVERGENCES OF SIGNED STATES

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In the paper the notion of the uniform and weak convergences of signed states on a logic will be studied. Some theorems about convergences will be proved.

1. Uniform convergence

Let L be a σ-lattice with the first and the last elements 0 and 1, respectively, and an orthocomplementation $\perp : a \mapsto a^\perp$, a, $a^\perp \in L$ such that

$$\begin{aligned} &\text{(i)}\ (a^\perp)^\perp = a \text{ for all } a \in L; \\ &\text{(ii)}\ \text{if } a < b, \text{ then } b^\perp < a^\perp; \\ &\text{(iii)}\ a \vee a^\perp = 1 \text{ for all } a \in L. \end{aligned} \tag{1}$$

An orthocomplemented σ-lattice L satisfying the condition if $a < b$, then $b = a \vee (b \wedge a^\perp)$ is called a logic.

Two elements a, b of a logic L are orthogonal and we write $a \perp b$ if $a < b^\perp$. A nonzero element a in a logic L is called an atom if for any element $b < a$ either $b = a$ or $b = 0$. An observable is a map x from the Borel sets $B(R_1)$ of R_1 into a logic L such that (i) $x(R_1) = 1$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$; (iii) $x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$ if $\{E_i\}$ is a sequence of disjoint elements of $B(R_1)$. We denote by $\sigma(x)$ the smallest closed set $C \subset R_1$ such that $x(C) = 1$. If there is a compact set $K \subset R_1$ such that $x(K) = 1$, x will be called bounded. For a bounded observable x we denote $\|x\| = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

A signed state on a logic L is a map m from L into $R_1 \cup \{-\infty\} \cup \{+\infty\}$ such that

(i) $m(0) = 0$;

(ii) $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$, $a_i \perp a_j$, $i \neq j$, $\{a_i\} \subset L$;

and from the values $\pm\infty$ it may obtain at most one value. A signed state m on L such that $m : L \to \langle 0, 1 \rangle$, $m(1) = 1$ is called a state. We denote by $S(L)$ the set of all states on a logic L. It may be empty ([4]). A logic is quite full if the statement $m(b) = 1$ whenever $m(a) = 1$ implies $a < b$, where m is a state.

Let $M(L)$ be the set of all bounded signed states on L. $M(L)$ is a real Banach space with respect to the norm $\|m\| = \sup_{a \in L} |m(a)|$, the usual addition and the multiplication by real scalars of signed states. The convergence with respect to this norm is called uniform (in symbols $m_n \xrightarrow{u} m$).

If x is a bounded observable and m be a signed state on L, then the function

$$m_x(E) = m(x(E)), \quad E \in B(R_1), \tag{3}$$

is a signed measure on $B(R_1)$. Therefore it may be written as a difference of two measures, that is $m_x = m_x^+ - m_x^-$. If the sets $A, B \in B(R_1)$ form the Hahn decomposition of a signed measure m_x, then $m_x^+(E) = m_x(E \cap A) = m(x(E \cap A))$, $m_x^-(E) = -m(x(E \cap B))$ for all $E \in B(R_1)$. Hence for the norm of m_x^+ we have $\|m_x^+\| = \sup\{m_x^+(E): E \in B(R_1)\} = \sup\{m(x(E \cap A)): E \in B(R_1)\} \leq \leq \sup_{a \in L} |m(a)| \leq \|m\|$. Likewise $\|m_x^-\| \leq \|m\|$.

Lemma 1.1. *Let L be a logic and x be a bounded observable, then the function* $\bar{x}$

$$\bar{x}(m) = \int \lambda \, dm_x = \int \lambda \, dm_x^+ - \int \lambda \, dm_x^-, \qquad m \in M(L), \tag{4}$$

is a bounded real linear functional on $M(L)$ *and*

$$\|\bar{x}\| \leq 2\|x\|. \tag{5}$$

If L is quite full, then

$$\|x\| \leq \|\bar{x}\| \leq 2\|x\|.$$

Proof. The function $\bar{x}$ is well defined. It is homogeneous and linear as it follows from the equality

$$(m+n)_x = (m+n)_x^+ - (m+n)_x^- = m_x^+ + n_x^+ - (m_x^- + n_x^-).$$

For an estimate of $\bar{x}$ we have $\bar{x}(m) = \int \lambda \, dm_x = \int \lambda^+ \, dm_x^+ - \int \lambda^- \, dm_x^+ - \int \lambda^+ \, dm_x^- + + \int \lambda^- \, dm_x^-$, where λ^+, λ^- are the positive and the negative parts of $f(\lambda) = \lambda$. The first and fourth member is positive, the others are negative. Therefore $\alpha = \int \lambda^+ \, dm_x^+ + \int \lambda^- \, dm_x^- \leq \|x\| \, (m_x^+(R_1) + m_x^-(R_1)) \leq 2 \, \|x\| \, \|m\|$, likewise $\beta = \int \lambda^+ \, dm_x^- + \int \lambda^- \, dm_x^+ \leq 2 \, \|x\| \, \|m\|$. But $|\bar{x}(m)| \leq \max\{\alpha, \beta\} \leq 2 \, \|x\| \, \|m\|$, hence $\|\bar{x}\| \leq 2 \, \|x\|$.

Let now L be a quite full logic. In [4, Theorem 6.1] it is shown that $\|x\| = \sup\{|\int \lambda \, dm_x| : m \in S(L)\}$. This equality implies $\|x\| \leq \|\bar{x}\|$. q.e.d.

The logic $L(H)$ of all closed subspaces of a separable Hilbert space H (real or complex) is one of the most important examples of a logic. A signed state m of the form

$$m(M) = \mathrm{tr}\,(TP^M), \quad M \in L(H)\,, \tag{6}$$

where P^M is the projector of M and T is the Hermitean operator of the trace class, is called a regular signed state. Theorem 3.4([2]) asserts that every bounded signed state on a logic $L(H)$, where H is a separable Hilbert space of a dimension at least 3, is regular.

Because of a one-to-one correspondence between the set of all bounded observables x on $L(H)$ and the set of all Hermitean operators A on H, and by using the theorems about operators of the trace class, it may be shown that every bounded real linear functional on $M(H) = M(L(H))$, $\dim H \geq 3$, is given by the formula (4) and therefore $\bar{x}(m) = \mathrm{tr}(TA)$, where x corresponds to A, and m to T, by (6).

In [6] there is given the characterization of the uniform convergence of regular signed states on $L(H)$: A sequence of regular signed states $m_n(M) = \mathrm{tr}(T_n P^M)$, $n = 1, 2, \ldots$ converges uniformly to zero iff the following condition is satisfied

$$\lim_n \mathrm{tr}\,|T_n| = 0\,. \tag{7}$$

2. Weak convergence

A system of seminorms $\{\|m\|_a = |m(a)| : a \in L\}$ defines the weak topology on $M(L)$. We obtain a locally convex Hausdorff topologic linear space. The weak convergence in this topology of a net $\{m_\alpha\}$ to a signed state m (in symbols $m_\alpha \xrightarrow{w} m$) is given by $m_\alpha(a) \to m(a)$ for all $a \in L$. If $m_n \xrightarrow{u} m$, then $m_n \xrightarrow{w} m$. The converse implication does not hold in general. For example, let $\Omega = \{1, 2, \ldots\}$, $L = \{\emptyset, \Omega, \{1, k\}, \{1, k\}^c, k \geq 2\}$. Let us define a sequence $\{m_n\}$ of states by $m_n(a) = X_a(n)$, $a \in L$ for $n = 1, 2, \ldots$. Then $m_n \xrightarrow{w} m$, where $m(\{1, k\}^c) = m(\Omega) = 1$, $m(\emptyset) = m(\{1, k\}) = 0$, $k = 2, 3, \ldots$, but $\|m_n - m\| = 1$ for every n.

Lemma 2.1. *Let there be a constant $K > 0$ for a net $\{m_\alpha\}$ of $M(L)$ such that $\|m_\alpha\| < K$ for all α. Then $m_\alpha \xrightarrow{w} m$ iff $\bar{x}(m_\alpha) \to \bar{x}(m)$ for each bounded observable x on L.*

Proof. The sufficiency is trivial because of defining an observable q_a (i.e. such an observable that $q_a(\{0\}) = a^\perp$, $q_a(\{1\}) = a$) for all $a \in L$.

The necessity. Let x be a bounded observable, then there is a sequence of simple observables $\{x_n\}$ such that $x_n = \sum_{i=1}^{k_n} \lambda_i^n q_{a_i^n}$, where $a_1^n, a_2^n, \ldots, a_{k_n}^n$ are orthogonal elements for every $n=1, 2, \dots$ and $\|x_n - x\| \to 0$ [5, Lemma 7.1.] (If x, y are simultaneous observables, then $x-y$ has the sense and $\bar{x}(m) - \bar{y}(m) = \overline{(x-y)}(m)$ [5]). Hence

$$|\bar{x}(m_\alpha) - \bar{x}(m)| \leq |\bar{x}(m_\alpha) - \bar{x}_n(m_\alpha)| + |\bar{x}_n(m_\alpha) - \bar{x}_n(m)| +$$

$$+ |\bar{x}_n(m) - \bar{x}(m)| \leq |\overline{(x-x_n)}\,(m_\alpha)| + \left|\sum_{i=1}^{k_n} \lambda_i^n (m_\alpha(a_i^n) - m(a_i^n))\right| +$$

$$+ |\overline{(x_n - x)}\,(m)| \leq 2\,\|m_\alpha\|\,\|x - x_n\| + \|x\| \sum_{i=1}^{k_n} |m_\alpha(a_i^n) - m(a_i^n)| +$$

$$+ 2\,\|m\|\,\|x_n - x\| \leq 2\,(K + \|m\|)\,\|x_n - x\| + \|x\| \sum_{i=1}^{k_n} |m_\alpha(a_i^n) - m(a_i^n)|\,,$$

and Lemma 2.1 is proved. q.e.d.

Theorem 2.2. *Let $\{m_n\}$ be a sequence of finite signed states on L. If there is a finite limit $m(a) = \lim_n m_n(a)$ for all $a \in L$, then m is a finite signed state on L and the σ-additivity of a sequence $\{m_n\}$ is uniform with respect to n.*

Proof. It is evident that m is a finite finitely additive real valued function on L, and $m(0)=0$. We can show that m is a σ-additive function, that is, if $\{a_i\}_{i=1}^{\infty}$ is a sequence of mutually orthogonal elements from L with a lattice sum $a = \bigvee_{i=1}^{\infty} a_i$, then $m(a) = \sum_{i=1}^{\infty} m(a_i)$. Without the loss of a generality we may assume that $a_i \neq$ for all i.

Let us denote by $\mathscr{A}$ the Boolean σ-algebra composed from elements of the form $\bigvee_{i \in D} a_i$, where D is an arbitrary subset of $\Omega = \{1, 2, \dots\}$ (if $D = \emptyset$, then $\bigvee_{i \in \emptyset} a_i = 0$). The measurable space $(\Omega, 2^\Omega)$ is isomorphic to $\mathscr{A}$. The prescription $\psi(\{i\}) = a_i$ defines uniquely an isomorphism and hence $\psi(\Omega) = a$. A sequence of signed measures $\mu_n(A) = m_n(\psi(A))$ has a finite limit $\mu(A) = \lim_n \mu_n(A)$ for all $A \subset \Omega$. By Nikodym's theorem ([1]), μ is a finite signed measure on $(\Omega, 2^\Omega)$. Therefore $m(a) = \lim_n m_n(a) = \lim_n \mu_n(\Omega) = \mu(\Omega) = \sum_{i=1}^{\infty} \mu(\{i\}) = \sum_{i=1}^{\infty} m(a_i)$.

The σ-additivity of $\{m_n\}$ is uniform with respect to n because of the uniform σ-additivity with respect to n of $\{\mu_n\}$ on $(\Omega, 2^\Omega)$ (Nikodym's theorem [1]). q.e.d.

Corollary 2.2.1. *The cone of all positive negative bounded signed states is sequentially weakly complete.*

Proof. Let $\{m_n\}$ be a Cauchy sequence of the elements of the cone. Then there exists $\lim_n m_n(a)$ for all $a \in L$ and therefore m is an element of the given cone, by Theorem 2.2.

q.e.d.

Corollary 2.2.2. *The set $S(L)$ is sequentially weakly complete.*

Theorem 2.3. *Suppose that L is such a logic that for any element $b \neq 0$ there exists a countable system of orthogonal atoms $\{a_i\}$ from L such that $b = \bigvee_i a_i$. Then the sufficient and necessary conditions for a sequence $\{m_n\}$ on finite signed states to converge weakly to a signed state m are the following*

(i) *the sequence $\{m_n(a)\}$ has a finite limit for any atom $a \in L$;*

(ii) *for every orthogonal sequence $\{a_k\}$ of the atoms of L the series $\sum_k m_n(a_k)$ converge uniformly with respect to n.*

Then the limit $m(b) = \lim_n m_n(b)$ exists and it is finite signed state on L.

Proof. The necessity. The condition (i) is evident and (ii) follows from Theorem 2.2.

The sufficiency. A sequence $\{m_n(b)\}$ is a Cauchy one for any $b \in L$ because if $\bigvee_k a_k = b$, where $\{a_k\}$ are orthogonal atoms for b, then

$$|m_n(b) - m_m(b)| \leqslant \left| m_n(b) - m_n\left(\bigvee_{k=1}^{i} a_k\right)\right| + \\ + \left| m_n\left(\bigvee_{k=1}^{i} a_k\right) - m_m\left(\bigvee_{k=1}^{i} a_k\right)\right| + \left| m_m\left(\bigvee_{k=1}^{i} a_k\right) - m_m(b)\right| .$$

Theorem 2.2 ensures that $\lim_n m_n(b) = m(b)$ is a finite signed state on L. q.e.d.

Theorem 1 of [6] follows from the above theorem, as it can easy be seen.

Theorem 2.4. *Let a logic L satisfy the finite chain condition (f.c.c.), that is, if $\{a_n\} \subset L$ with $a_1 > a_2 > \ldots$ implies that there exists an integer N such that $a_n = a_N$ for $n > N$. Then the unit sphere $\mathcal{G} = \{m \in M(L) : \|m\| \leqslant 1\}$ of $M(L)$ is weakly compact.*

Proof. Let us assign an interval $D_x = \{t \in R_1 : |t| \leqslant 2\, \|x\|\}$ to every bounded observable x. The cartesian product $D = \prod_{x \in O(L)} D_x$, where $O(L)$ is the set of all bounded observables on L, is a compact space in the product topology.

A map $\delta : \mathscr{G} \to D$, defined by $\delta(m)(x) = \bar{x}(m)$, $x \in O(L)$ is a homeomorphism between the sphere $\mathscr{G}$ (in the weak topology) and its image $\delta(\mathscr{G})$ (in the product topology). Indeed, δ is one-to-one, because if $\delta(m_1) = \delta(m_2)$, then $\bar{x}(m_1) = \bar{x}(m_2)$ for any $x \in O(L)$. Especially, there holds $\bar{q}_a(m_1) = m_1(a) = m_2(a)$ for any observably q_a, $a \in L$. δ and δ^{-1} are continuous maps because of $m_\alpha \xrightarrow{w} m$ iff $\delta(m_\alpha)(x) \to \delta(m)(x)$, by Lemma 2.1. To prove that $\mathscr{G}$ is a compact set, it is sufficient to show that the image of $\mathscr{G}$ by δ is closed, because then $\delta(\mathscr{G})$ will be a compact set.

Let us $\delta(m_\alpha) \to \xi \in D$ in the product topology. We define $m(a) = \lim_\alpha m_\alpha(a)$, $a \in L$, that means $\xi(q_a) = m(a)$. Then m is a signed state in the norm $\|m\| \leq 1$. We shall show that $\delta(m) = \xi$. We may assume that $x \in O(L)$ is of the form $x = \sum_{i=1}^{k} \lambda_i X_{\{\lambda_i\}} \circ x$ because L satisfies f.c.c.. Then $\delta(m)(x) = \bar{x}(m) = \sum_{i=1}^{k} \lambda_i \, m(a_i) = \lim_\alpha \sum_{i=1}^{k} \lambda_i m_\alpha(a_i) = \lim_\alpha \bar{x}(m_\alpha) = \xi(x)$, where $a_i = x(\{\lambda_i\})$. q.e.d.

If H is a real separable Hilbert space, then the cone of all positive (negative) regular signed states is weakly metrizable ([6]). For the space $M(H)$, dim $H \geq 3$, the problem of metrizability seems to be open.

Theorem 2.5. *Let $R(H)$ be a space of all regular signed states on $L(H)$. Then $R(H)$ is sequentially weakly complete.*

Proof. Let $\{m_n\}$ be a weakly fundamental sequence of regular signed states on $L(H)$ and $\{T_n\}$ be a corresponding sequence of operators of the trace class. There is a finite limit $m(M) = \lim_n m_n(M)$ for every $M \in L(H)$. m is a finite signed state on $L(H)$, by Theorem 2.2.

We shall show that m is regular. Denote by $\hat{f}$ the onedimensional subspace of H generated by a unit vector f. Then $m(\hat{f}) = \lim_n m_n(\hat{f}) = \lim_n (T_n f, f)$. Hence there is a Hermitean operator T such that $T = w - \lim_n T_n$. T is an operator of the trace class because if $\{f_i\}$ is an orthonormal base, then $\sum_i (Tf_i, f_i) = \sum_i m(\hat{f}_i) = m(H)$. The series $\sum_i (Tf_i, f_i)$ converges absolutely because of the absolute convergence of $\sum_i m(\hat{f}_i)$. We have thus $m(M) = \operatorname{tr}(TP^M)$, $M \in L(H)$. q.e.d.

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О СХОДИМОСТИ ОБОБШЕННЫХ СОСТОЯНИИ

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Резюме

В работе исследуется понятие равномерной и слабой сходимости обобшенных состоянии на логике. Теорема Никодыма и другие теоремы о сходимости здесь доказаны.