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OSCILLATORINESS OF SOLUTIONS OF A NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION

PAVEL ŠOLTÉS

Consider a differential equation

$$x'' + a(t)x'' + b(t)f(x)h(x') = 0 \tag{1}$$

where $a(t) \in C_0\langle t_0, \infty)$, $b(t) \in C_1\langle t_0, \infty)$, $f(x) \in C_1(-\infty, \infty)$, $h(y) \in C_0(-\infty, \infty)$, $xf(x) > 0$ for $x \neq 0$, $h(y) > 0$ for all $y \in (-\infty, \infty)$, with $t_0 \in (-\infty, \infty)$.

Put

$$F(x) = \int_0^x f(s)\,\mathrm{d}s, \qquad H(y) = \int_0^y \frac{s}{h(s)}\,\mathrm{d}s\,.$$

We have then the following

Theorem 1. (*Theorem 4 of* [2]): *Suppose that* $a \in C_1\langle t_0, \infty)$ *and that the following conditions hold for all* $t \in \langle t_0, \infty)$ *and* $x \in (-\infty, \infty)$:

1. $a(t) \geqslant 0, \quad a'(t) \leqslant 0, \quad b(t) \geqslant 0, \quad b'(t) \leqslant 0, \quad f'(x) \geqslant \varepsilon > 0;$

2. $\displaystyle\int_{t_0}^{\infty} a(s)\,\mathrm{d}s \leqslant A < \infty\,, \qquad \int_{t_0}^{\infty} b(s)\,\mathrm{d}s = +\infty\,.$

If $\lim_{|y| \to \infty} H(y) = H \leqslant +\infty$, *then any solution* $x(t)$ *of* (1) *such that*

$$K_0 = H(x'(t_0)) + b(t_0)F(x(t_0)) < H$$

is either oscillatory, or $\lim_{t \to \infty} x(t) = 0$.

A similar statement can be proved also under weaker assumptions. Note that it is the consequence of the hypotheses of Theorem 1 that $b(t) > 0$ for all $t \in \langle t_0, \infty)$.

Throughout this paper we shall suppose that, for every $t \geqslant t_0$,

$$a(t) \geqslant 0, \quad b(t) > 0, \quad \int_{t_0}^{\infty} \frac{\{b'(s)\}+}{b(s)}\,\mathrm{d}s = K < \infty\,,$$

where $\{b'(t)\}_+ = \max\{b'(t), 0\}$.

We have

Theorem 2. *Suppose that*

1. $$\lim_{t\to\infty} a(t)-0, \quad f'(x)\geq\varepsilon>0 \quad \text{for all} \quad x\in(- \quad ,\infty)$$

2. $$\int_{t_0} b(s)\,\mathrm{d}s=+\infty\,, \qquad \int_{t_{(}} a(s)\,\mathrm{d}s\leq A<\infty$$

If $\lim_{|y|\to\infty} H(y)=H\leq+\infty$, *then ny olution* $x(t)$ *of* (1) *such that*

$$[H(x'(t_0))+b(t_0)Fx(\ )\ ]\exp K<H \tag{2}$$

is either oscillatory or $\mathrm{l}\,\mathrm{m}_{t\to} x(t)-0$.

Proof. From the equation (1) we ha e

$$H(x'(t))+b(t)F(\ (t))<H(\ '(t \ ))+b(t_0)F(x(t_,))+$$
$$+\int_{,}^{t}\{b'(s)\}_+F(x(s))\,\mathrm{d}s$$

and hence

$$H(x'(t))+b(t)F(x(t))\leq K \ +\int_{t}^{t}\frac{\{b'(s)\}_+}{b(s)}[H(x'(s))+b(s)F(x(s))]\,\mathrm{d}s\,,$$

an using this in conjuction with Bellman s l mma, we get

$$H(x \ (t))< \quad \mathrm{xp}\int\frac{\{b \ (\ )\}_+}{b(s)}\,\mathrm{d}s\,,$$

where $K_0=H(x'(t_0))+b(t_0)F(\ (\ )$.

Suppose that the solution $x(t)$ ex st on $\langle t\,,t)$ Using (2) and the last derived relation, we see that $x'(t)$ i bound d on $\langle t\,,t)$. Now if $t<+\infty$, then $x(t)$ is also bounded on $\langle t_0, t)$ and therefore $x(t)$ exi t on $\langle t\,,\infty)$.

Suppose that $x(t)$ is not o cill t , i e th t there exists $t_1\geq t_0$ uch that $x(t)\neq 0$ for all $t\geq t_1$. Suppose e.g that $x(t)>0$ (th proof is quite analogous for $x(t)<0$) By methods similar to tho e u d in [ ] it i po ible t show th t there ex ts $t_2\geq t_1$ such that for $t\geq t_2$

$$\frac{x'(t)}{f(\ (t))}<K_1 \quad h(\alpha)\int b(s)\,\mathrm{d}s\,,$$

therefore

$$\frac{x'(t)}{f(x(t)} \quad - \quad \mathrm{r} \tag{3}$$

so that $x(t)$ is a decreasing function. We shall now prove that $\lim_{t\to\infty} x(t)=0$. It is a consequence of (3) that for any $k>0$ there exists $t_3 \geqslant t_2$ such that for any $t \geqslant t_3$ we have

$$\frac{x'(t)}{f(x(t))} < -k .$$

Integrating this from t_3 to $t \geqslant t_3$, we have

$$\int_{t_3}^{t} \frac{x'(s)}{f(x(s))} \mathrm{d}s = \int_{x(t_3)}^{x(t)} \frac{\mathrm{d}\tau}{f(\tau)} < -k(t-t_3) , \tag{4}$$

and therefore $\lim_{t\to\infty} x(t)=0$, since $f(x)$ is continuous.

Obviously, we also have

Theorem 3. *Suppose, in addition to the assumption of Theorem 2, that for* $x>0$

$$\lim_{t\to 0+} \int_{t}^{x} \frac{\mathrm{d}s}{f(s)} < \infty , \qquad \lim_{t\to 0-} \int_{t}^{-x} \frac{\mathrm{d}s}{f(s)} < \infty . \tag{5}$$

Then any solution $x(t)$ *of* (1) *satisfying* (2) *is oscillatory.*

Proof. It is necessary to prove the impossibility of $\lim_{t\to\infty} x(t)=0$. This is a direct consequence of (4). In fact, if (5) holds, then the left part of (4) is bounded, yielding a contradiction.

Remark 1. If $a(t)\equiv 0$, it is sufficient to replace the assumption of Theorem 2 that $f'(x)\geqslant \varepsilon>0$ by the weaker assumption that $f'(x)\geqslant 0$.

Theorem 4. *Suppose that the following assumptions hold:*

1. $$f'(x)\geqslant \varepsilon>0 \quad \text{for all} \quad x\in(-\infty, \infty)$$

2. $$\int_{t_0}^{\infty} s a(s) \mathrm{d}s \leqslant A < \infty , \qquad \int_{t_0}^{\infty} s b(s) \mathrm{d}s = +\infty ,$$

and that, for every $x>0$,

$$\int_{x}^{\infty} \frac{\mathrm{d}s}{f(s)} < \infty , \qquad \int_{-x}^{-\infty} \frac{\mathrm{d}s}{f(s)} < \infty . \tag{6}$$

Then any solution $x(t)$ *of* (1) *satisfying* (2) *is either oscillatory or* $\lim_{t\to\infty} x(t)=0$.

If, in addition to this, (5) *holds, then any solution satisfying* (2) *is oscillatory.*

Proof. By methods similar to those of [2] we show that a solution $x(t)$ of (1) which satisfies (2) exists on $\langle t_0, \infty)$ and that $x'(t)$ is bounded. Suppose that it is not oscillatory and that $t_1 \geq t_0$ is a number such that $x(t) \neq 0$ for all $t \geq t_1$. We shall assume that $x(t) > 0$, since the method of proof is similar if $x(t) < 0$.

Then

$$\frac{tx''(t)}{f(x(t))} + \frac{ta(t)x'(t)}{f(x(t))} = -tb(t)h(x'(t)) .$$

Integrating this from t_1 to $t \geq t_1$, we get

$$\frac{tx'(t)}{f(x(t))} - \int_{t_1}^{t} \frac{x'(s)}{f(x(s))}\,\mathrm{d}s + \int_{t_1}^{t} \frac{sf'(x(s))x'^2(s)}{f^2(x(s))}\,\mathrm{d}s + \int_{t_1}^{t} \frac{sa(s)x'(s)}{f(x(s))}\,\mathrm{d}s =$$

$$= \frac{t_1 x'(t_1)}{f(x(t_1))} - \int_{t_1}^{t} sb(s)h(x'(s))\,\mathrm{d}s ,$$

so that

$$\frac{tx'(t)}{f(x(t))} + \int_{t_1}^{t} \frac{sx'^2(s)}{f^2(x(s))}\left[f'(x(s)) - \tfrac{1}{2}\,a(s)\right]\mathrm{d}s \leq \frac{t_1 x'(t_1)}{f(x(t_1))} +$$

$$+ \int_{x(t_1)}^{x(t)} \frac{\mathrm{d}\tau}{f(\tau)} + \tfrac{1}{2}\int_{t_1}^{t} sa(s)\,\mathrm{d}s - \int_{t_1}^{t} sb(s)h(x'(s))\,\mathrm{d}s .$$

Since $|x'(t)| \leq M < \infty$ and $h(y)$ is continuous, there exists α such that, for all $t \geq t_1$, $h(x'(t)) \geq h(\alpha)$. A further consequence of the assumptions of the Theorem is that $\lim_{t\to\infty} a(t) = 0$; hence there exists $t_2 \geq t_1$ such that, for all $t \geq t_2$,

$$f'(x(t)) - \tfrac{1}{2}\,a(t) \geq 0 .$$

Because of (7), we have, for all $t \geq t_2$,

$$\frac{tx'(t)}{f(x(t))} \leq K_1 - h(\alpha)\int_{t_2}^{t} sb(s)\,\mathrm{d}s ,$$

where K_1 is a constant, and therefore

$$\frac{tx'(t)}{f(x(t))} \to -\infty \quad \text{for} \quad t \to \infty . \tag{8}$$

Now we note the following two consequences of (8): first, $x(t)$ is a monotonic decreasing function, so that $\lim_{t\to\infty} x(t)$ exists; second, for any $k > 0$ there exists $t_3 \geq t_2$ such that, for all $t \geq t_3$,

$$\frac{tx'(t)}{f(x(t))} < -k ,$$

so that

$$\int_{t_3}^{t} \frac{x'(s)}{f(x(s))}\, ds = \int_{x(t_3)}^{x(t)} \frac{d\tau}{f(\tau)} < ln\left[\left(\frac{t_3}{t}\right)^k\right]. \tag{9}$$

Hence $\lim_{t\to\infty} x(t) = 0$.

If (5) holds as well as (9), then evidently $x(t)$ is oscillatory. This completes the proof.

Remark 2. It is evident from the proof of Theorem 4 that the conclusion of the Theorem will also hold if assumption 2 is replaced by the assumption that

$$a(t)\to 0, \qquad \tfrac{1}{2}\int_{t_0}^{t} sa(s)\,ds - c\int_{t_0}^{t} sb(s)\,ds \to -\infty \quad \text{for} \quad t\to\infty$$

with $c>0$ an arbitrary constant.

A theorem analogous to that concerning the oscillatoriness of solutions of the equation (1) is also true for the equation

$$x'' + a(t)g(x, x') + b(t)f(x)h(x') = 0 \tag{10}$$

where $a(t)$, $b(t)$, $f(x)$ and $h(y)$ are the same functions as in (1) and $g(x, y)$ is continuous for all $(x, y)\in(-\infty, \infty)x, x(-\infty, \infty)$. Namely we have

Theorem 5. *Suppose that the following assumptions hold:*

1. $$g(x, y)y \geq 0, \quad y^2 f'(x) \geq g^2(x, y) \quad \textit{for all} \quad (x, y)\in(-\infty, \infty)\times(-\infty, \infty)$$

2. $$\tfrac{1}{2}\int_{t_0}^{t} sa^2(s)\,ds - c\int_{t_0}^{t} sb(s)\,ds \to -\infty \quad \textit{for} \quad t\to\infty,$$

with $c>0$ an arbitrary constant.

If (6) *holds, then any solution $x(t)$ of* (10) *satisfying* (2) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

If, in addition to this, (5) *holds, then any solution $x(t)$ satisfying* (2) *is* oscillatory.

Proof. The existence of a solution $x(t)$ on $\langle t_0, \infty)$ and the boundedness of $x'(t)$ are proved in a way similar to that of Theorem 2. Let the solution $x(t)$ be oscillatory and suppose e.g. that $x(t)>0$ for all $t\geq t_1\geq t_0$, $t_1>0$. The proof is similar for the case $x(t)<0$. Equation (10) yields

$$\frac{tx'(t)}{f(x(t))} - \int_{t_1}^{t} \frac{x'(s)}{f(x(s))}\, ds + \int_{t_1}^{t} \frac{sx'^2(s)f'(x(s))}{f^2(x(s))}\, ds +$$
$$+ \int_{t_1}^{t} \frac{sa(s)g(x(s), x'(s))}{f(x(s))}\, ds = \frac{t_1 x'(t_1)}{f(x(t_1))} - \int_{t_1}^{t} sb(s)h(x'(s))\, ds$$

hence

$$\frac{tx'(t)}{f(x(t))}+\int_{t_1}^{t}\frac{s}{f^2(x(s))}\left[x'^2(s)f'(x(s))-\tfrac{1}{2}\,g^2(x(s),x'(s))\right]\mathrm{d}s\leqslant$$

$$\leqslant K_2+\tfrac{1}{2}\int_{t_1}^{t} sa^2(s)\,\mathrm{d}s-h(\alpha)\int_{t_1}^{t} sb(s)\,\mathrm{d}s$$

where K_2 is a constant.

The rest of the proof is similar to that of Theorem 4.

Remark 3. Results similar to those stated as Theorem 4 and 5 are stated in Theorems 1 and 2 of [1] which concern solutions of the equation (1) for $a(t)\equiv 0$ and $h(y)\equiv 1$.

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КОЛЕБАТЕЛЬНЫЕ СВОЙСТВА РЕШЕНИЙ НЕЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА

Павел Шолтес

В работе решаются вопросы колебательности решений нелинейного дифференциального уравнения второго порядка

$$x''+a(t)x'+b(t)f(x)h(x')=0\,. \tag{1}$$

На основании свойств функций $a(t)$, $b(t)$, $f(x)$, $h(x')$ приведены достаточные условия, при выполнении которых или решение $x(t)$ уравнения (1) колеблется или $\lim_{t\to\infty} x(t)=0$.

Тоже приведены достаточные условия, при которых все решения, которые выполнЯит условие (2), колеблются.