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THE HADWIGER NUMBER OF AN INFINITE GRAPH

BOHDAN ZELINKA

In the papers [3] and [4] the Hadwiger numbers of finite graphs were studied. Here we would like to remark on the Hadwiger numbers of infinite graphs.

Let G and G' be two undirected graphs. We say [2] that G can be contracted onto G', if and only if G' can be obtained from G by a finite number of the following operations:
(a) deleting an edge;
(b) deleting an isolated vertex;
(c) contracting an edge.

To contract an edge e of G with the end vertices u and v means to delete the vertices u and v and the edge e from G and to add a new vertex w to G and to join w with all vertices of G different from u and v which were adjacent to u or v.

For finite graphs the following definition of the Hadwiger number was used [2].

The Hadwiger number $\eta(G)$ of an undirected graph G is the maximal number of vertices of a complete graph onto which G can be contracted.

In [4] it was shown that for finite connected graphs this definition is equivalent to the following one.

The Hadwiger number $\eta(G)$ of a connected undirected graph G is the maximal number of the classes of a partition $\mathscr{P}$ of the vertex set of G with the property that each class of $\mathscr{P}$ induces a connected subgraph of G and to any two of these classes there exists an edge of G joining a vertex of one of them with a vertex of the other.

For infinite graphs these definitions are not equivalent. If for some connected graph there exists the Hadwiger number according to the first definition, there exists also the Hadwiger number of it according to the second definition and it is equal to the Hadwiger number according to the first definition. If G is contracted onto some complete graph K, then there exists a connected homomorphism [1] of G onto K; the partition $\mathscr{P}$ is obtained so that two vertices belong to the same class of $\mathscr{P}$ if and only if their images in this homomorphism are equal. On the other hand, if we have such a partition $\mathscr{P}$ and if a class of $\mathscr{P}$

is finite, then the subgraph induced by it can be contracted onto a graph consisting of one vertex by a finite number of operations (*a*), (*b*), (*c*). If it is infinite, this is not possible. Therefore there exist infinite connected graphs, for which there exists the Hadwiger number according to the second definition, but not according to the first definition. An example is a graph consisting of one (one-way or two-way) infinite path. The Hadwiger number of this graph according to the first definition is undefined, according to the second definition it is equal to 2.

Thus if G is a connected undirected graph, we shall consider three numbers $\eta_1(G)$, $\eta_2(G)$, $\eta(G)$ for it. The number $\eta_1(G)$ is the maximal number of vertices of a complete graph onto which G can be contracted. The number $\eta_2(G)$ is the maximal number of classes of a partition $\mathscr{P}$ of the vertex set of G with the property that each class of P induces a connected subgraph of G and to any two of these classes there exists an edge of G joining a vertex of one of them with a vertex of the other. Using the concept of connected homomorphism, we can express the definition of $\eta_2(G)$ in other words. The number $\eta_2(G)$ is the maximal cardinality of the vertex set of a complete graph which is an image of G in a connected homomorphism. Finally, $\eta(G)$ will denote the supremum of cardinalities of vertex sets of complete graphs onto which the graph G can be mapped by connected homomorphisms. This number $\eta(G)$ will be called the Hadwiger number of the graph G.

It is easy to see that $\eta(G)$ exists for every connected undirected graph G. If G is finite, it is equal to the Hadwiger number of G, as it was defined for finite graphs. If G is infinite, we have seen that $\eta_1(G)$ need not exist. We shall show that $\eta_2(G)$ need not exist, either.

Theorem 1. *There exists a connected graph G for which $\eta_2(G)$ is undefined.*

Proof. Let $K_1, K_2, \ldots$ be an infinite (of the ordinal number ω_0) sequence of pairwise vertex-disjoint complete graphs; let the number of vertices of K_n be n for each positive integer n. In each K_n we choose a vertex u_n. Each u_n will be joined with u_{n+1} by an edge. The graph thus obtained will be denoted by G. Suppose that $\eta_2(G)$ exists. If it is finite, then denote $\eta_2(G) = m$, where m is some positive integer. Define $\mathscr{P}$ so that one class of $\mathscr{P}$ consists of u_n and of all vertices of G not belonging to K_{m+1} and each other class of $\mathscr{P}$ consists of only one vertex which belongs to K_{n+1} and is different from u_{m+1}. The partition $\mathscr{P}$ satisfies the conditions from the definition and has $m+1$ classes thus we have a contradiction. Now suppose that $\eta_2(G)$ is infinite. Let $\mathscr{P}$ be the corresponding partition. For any two edges joining vertices of different classes of $\mathscr{P}$ there exists a circuit containing both of them; thus they lie in the same block of G. Therefore all such edges lie in the same block of G and in this block there is at least one vertex from each class of $\mathscr{P}$. But the classes of $\mathscr{P}$ are pairwise disjoint and they are infinitely many, which is a contradiction. The number $\eta_2(G)$ can be neither finite, nor infinite, therefore it does not exist.

Thus we see that the numbers $\eta_1(G)$ and $\eta_2(G)$ need not exist, but evidently $\eta(G)$ exists for every graph G. Further, if $\eta_1(G)$ or $\eta_2(G)$ exists, it is equal to $\eta(G)$. For the graph G from the proof of Theorem 1 we have $\eta(G) = \aleph_0$.

Theorem 2. *Let $\mathfrak{n}$ be a regular transfinite cardinal number. Let G be a connected undirected graph with $\eta_2(G) = \mathfrak{n}$. Then G contains at least $\mathfrak{n}$ vertices of an at least $\mathfrak{n}$ degree.*

Proof. The vertex set V of G can be partitioned into $\mathfrak{n}$ pairwise disjoint subsets which satisfy the condition from the definition. Let this partition be denoted by $\mathscr{P}$. If $C_0 \in \mathscr{P}$, then there exist at least $\mathfrak{n}$ edges joining a vertex of C_0 with a vertex of another class of $\mathscr{P}$. Suppose that each vertex C_0 has a degree less than $\mathfrak{n}$. As $\mathfrak{n}$ is regular, there exists a transfinite cardinal number $\mathfrak{m} < \mathfrak{n}$ such that the degree of any vertex of C_0 is at most $\mathfrak{m}$. Let G_0 be the subgraph of G induced by C_0. The degree of any vertex of C_0 is at most $\mathfrak{m}$ also in G_0. Let $u \in C_0$. As G_0 is connected, each vertex of C_0 is connected with u by some path and therefore its distance from u in G_0 is equal to some non--negative integer. The number of vertices which have the distance d from u, where d is some non-negative integer, is at most $\mathfrak{m}^d$. But $\mathfrak{m}^d = \mathfrak{m}$, because $\mathfrak{m}$ is a transfinite cardinal number. The number of all vertices of C_0 is at most $\mathfrak{m} \cdot \aleph_0 = \mathfrak{m}$. From any of the vertices of C_0 at most $\mathfrak{m}$ vertices go into other classes of $\mathscr{P}$, therefore the total number of vertices which do not belong to C_0 and are adjacent to vertices of C_0 is at most $\mathfrak{m}^2 = \mathfrak{m}$. As the number of classes of $\mathscr{P}$ is $\mathfrak{n} > \mathfrak{m}$, there are classes of $\mathscr{P}$, none of whose vertices is adjacent to a vertex of C_0, which is a contradiction. Thus C_0 contains at least one vertex of degree at least $\mathfrak{n}$. As C_0 was chosen arbitrarily, each class of $\mathscr{P}$ contains such a vertex and the number of such vertices is at least $\mathfrak{n}$.

Theorem 3. *Let $\mathfrak{n}$ be a singular transfinite cardinal number different from $\aleph_0$. Let G be a connected undirected graph with $\eta_2(G) = \mathfrak{n}$. If $\mathfrak{m} < \mathfrak{n}$, then there exist at least $\mathfrak{n}$ vertices of G of a degree greater than $\mathfrak{m}$.*

Proof. Let $\mathscr{P}$ be the same as in the proof of Theorem 2, let $C_0 \in \mathscr{P}$. Suppose that $\mathfrak{m}$ is transfinite. If we suppose that the degrees of the vertices of C_0 do not exceed $\mathfrak{m}$, we obtain the same contradiction as in the proof of Theorem 2. Thus each $C_0 \in \mathscr{P}$ contains a vertex of a degree greater than $\mathfrak{m}$ and the number of such vertices is at least $\mathfrak{n}$. We have proved the assertion for each transfinite $\mathfrak{m}$, therefore it holds evidently also for each finite cardinal number $\mathfrak{m}$.

Analogously we can prove a theorem concerning $\eta(G)$.

Theorem 4. *Let $\mathfrak{n}$ be a singular transfinite cardinal number different from $\aleph_0$.*

Let G be a connected undirected graph such that $\eta(G) = \mathfrak{n}$. If $\mathfrak{m} < \mathfrak{n}$, then G contains more than $\mathfrak{m}$ vertices of a degree greater than $\mathfrak{m}$.

Now we shall prove a theorem which differs substantially from a similar theorem for finite graphs [3].

Theorem 5. *For each transfinite cardinal number $\mathfrak{n}$ there exists a connected undirected graph in which all vertices have the degree $\mathfrak{n}$ and whose Hadwiger number is 2.*

Proof. Let M be a set of the cardinality $\mathfrak{n}$. The vertex set of the graph G is the set of all finite sequences (including the empty sequence) of elements of M. Two vertices are joined by an edge, if and only if one of them is obtained from the other by adding one term after its last term. The graph G is a tree therefore its Hadwiger number is equal to 2.

Theorem 6. *There exists a graph in which the degree of any vertex does not exceed 3 and whose Hadwiger number is $\aleph_0$.*

Proof. Let us have an infinite sequence (of the ordinal number ω_0) of vertex-disjoint simple one-way infinite paths $P_0, P_1, P_2, \ldots$. The vertices of the path P_n, where n is some non-negative integer, are $u_n^0, u_n^1, u_n^2, \ldots$, the edges are $u_n^i u_n^{i+1}$ for $i = 0, 1, 2, \ldots$. Now we shall join each vertex u_n^m, where $m \neq n$, with the vertex u_m^n. Thus we obtain a graph G. A vertex u_n^m, where $m \neq n$, $m \neq 0$, has degree 3, because it is adjacent only to the vertices u_n^{m-1}, u_n^{m+1}, u_m^n. A vertex u_n^0, where $n \neq 0$, is adjacent only to the vertices u_n^1, u_0^n, therefore it has degree 2. A vertex u_n^n, where $n \neq 0$, is adjacent to the vertices a_n^{n-1}, a_n^{n+1}; it has degree 2. The vertex u_0^0 is adjacent only to u_0^1 and has degree 1. Each path P_n is a connected graph. To any m and n, $m \neq n$, there exists an edge $u_n^m u_m^n$ joining a vertex of P_m with a vertex of P_n. Thus $\eta_2(G) = \eta(G) = \aleph_0$.

Theorem 7. *Let G be a connected undirected graph, let $\eta(G) = n$, where n is a finite number. Then there exists a finite subgraph G_0 of G such that $\eta(G_0) = n$.*

Proof. If $\eta(G)$ is finite, then evidently $\eta_2(G)$ exists and $\eta_2(G) = n$. Consider again the partition $\mathscr{P}$. It has n classes. If C_1, C_2 are two different classes of $\mathscr{P}$, we delete all edges joining vertices of C_1 with vertices of C_2, except for one. If we do this for any two classes of $\mathscr{P}$, we obtain a graph G' in which for any two classes of $\mathscr{P}$ there exists only one edge joining a vertex of one with a vertex of another. Now let $C \in \mathscr{P}$. There are at most $n - 1$ vertices of C which are adjacent to vertices of other classes. For any two of these vertices choose a path connecting them in the subgraph of G induced by C (this subgraph is connected) and delete all vertices of C which do not belong to any of these paths. If we do this for each $C \in \mathscr{P}$, we obtain a graph G_0. Let $\mathscr{P}_0$ be the partition of the vertex set of G_0 such that two vertices of G_0 belong to the same class of $\mathscr{P}_0$, if and only if they belong to the same class of $\mathscr{P}$. Each class of $\mathscr{P}_0$ is a finite set and induces a connected subgraph of G_0 and to any two of these classes an edge exists which joins a vertex of one of them with a vertex of the other. Thus G_0 is finite and $\eta(G_0) \geqq n$. As G_0 is a subgraph of G, its Hadwiger number cannot exceed the Hadwiger number of G and we have $\eta(G) = n$.

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