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A GENERALIZED CONTINUITY AND PRODUCT SPACES

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A generalization of Kempisty's notion of quasicontinuity is known as somewhat continuity. The present paper shows that a separate somewhat continuity need not imply somewhat continuity, while a function $f(x, y)$ quasicontinuous in one variable and somewhat continuous in the other variable is shown to be somewhat continuous. Kempisty's classical theorem on the quasicontinuity of separately quasicontinuous functions is obtained in a general setting as a corollary.

The notion of a quasicontinuous function $f: X \to Y$ where X, Y are topological spaces was introduced for the case of Euclidean spaces by Kempisty in [2]. In the general case (see e. g. [3]) f is said to be quasicontinuous at $x_0 \in X$ if $f^{-1}(V) \cap U$ contains a nonempty open set for any open sets U, V where $x_0 \in U, f(x_0) \in V$. It is said to be quasicontinuous if it is quasicontinuous at any $x_0 \in X$. The notion of a somewhat continuous function (see [1]) generalizes the notion of quasicontinuity. $f: X \to Y$ is said to be somewhat continuous if for any open $V \subset Y$ such that $f^{-1}(V) \neq \emptyset$ we have $\operatorname{int} f^{-1}(V) \neq \emptyset$.

A theorem concerning quasicontinuity on product spaces, stating that separate quasicontinuity implies quasicontinuity, was given for the case of the function $f(x, y)$ of two real variables by Kempisty in [2]. N. F. G. Martin [3] has given a general version of Kempisty's theorem for the functions $f: X \times Y \to Z$, where X is a Baire space, Y second countable and Z metric.

If f is a function defined on the product space $X \times Y$, we shall call an x — section for a given $x \in X$ the function f_x defined on Y such that $f_x(y) = f(x, y)$. The y — section f_y for a given $y \in Y$ is defined analogouslv.

Theorem 1. *Let X be a Baire space, Y second contable and Z regular. Let $f: X \times Y \to Z$ have all x — sections somewhat continuous and all y — sections quascicontinuous. Then f is somewhat continuous.*

Proof. Let f not be somewhat continuous. There exists $G \neq \emptyset$ open such that $f^{-1}(G) \neq \emptyset$ and $\operatorname{int} f^{-1}(G) = \emptyset$.

Let (x_0, y_0) $f^{-1}(G)$. Choose G_1 open such that $\bar{G}_1 \subset G$, $f(x_0, y_0) \in G_1$. This is possible because of the regularity of Z. Owing to the quasicontinuity and hence somewhat continuity of f_{y_0} at the point x_0 we have $\operatorname{int} f_{y_0}^{-1}(G_1) \neq \emptyset$.

Put $U = \operatorname{int} f_{y_0}^{-1}(G_1)$. For any $x \in U$ form $f_x^{\,1}(G_1)$. Since $f_x(y_0) = f(x, y_0) \in G_1$, we have $f_x^{\,1}(G_1) \neq \emptyset$. The somewhat continuity of f_x gives int $f_x^{\,1}(G_1) \neq \emptyset$ for any $x \in U$.

Let $\{V_n\}$ be a countable basis of the space Y. Define A_n as the set of all $x \in U$ for which $V_n \subset \operatorname{int} f_x^{-1}(G_1)$. Evidently $\bigcup_{n=1}^{\infty} A_n = U$.

Let $S \subset U$ be any nonempty open set. Let us form $S \times V_n$ for given n. Because of the fact int $f^{-1}(G) = \emptyset$ there exists $(x^*, y^*) \in S \times V_n$ such that $f(x^*, y^*) \notin G$.

Choose a neighbourhood G^* of $f(x^*, y^*)$ such that $G^* \cap G_1 = \emptyset$, Using the quasicontinuity of f_{y^*} at x^* we obtain that there exists a nonempty set $S' \subset S$ such that $f(x, y^*) \in G^*$ for any $x \in S'$, hence $f(x, y^*) \notin G_1$.Thus $y^* \notin f_x^{-1}(G_1)$. This implies $V_n \not\subset f_x^{-1}(G_1)$, hence $x \notin A_n$. Thus $S' \cap A_n = \emptyset$. This means that A_n is nowhere dense and the set $U = \bigcup_{n=1} A_n$ is of the first category. This is a contradiction.

Theorem 2. *Let X be a Baire space, Y second countable and Z regular. Then a function $f: X \times Y \to Z$ quasicontinuous in each variable separately is quasicontinuous on $X \times Y$.*

To prove the above Theorem we shall prove first the following.

Lemma. *A function $f: X \to Y$ (X, Y arbitrary topological spaces) is quasicontinuous on X if and only if there exists a basis $\mathscr{B}$ of the space X such that for any element $B \in \mathscr{B}$ the restriction f/B is somewhat continuous.*

Proof. Necessity. Let $B \in \mathscr{B}$. Suppose that $(f/B)^{-1}\,(V) \neq \emptyset$ for some V open. Then there exists $x_0 \in B$ such that $(f/B)\,(x_0) \in V$. From the quasicontinuity of f at x_0 it immediately follows that there exists a nonempty open set $G \subset B$ such that $(f/B)\,(G) \subset V$.

Hence int $(f/B)^{-1}\,(V) \neq \emptyset$.

Sufficiency. Let $x_0 \in X$ be any point, U an open set containing x_0 and V an open set containing $f(x_0)$. Let $B \in \mathscr{B}$ be such that $x_0 \in B \subset U$.

Consider the restriction f/B. We have $(f/B)^{-1}\,(V) \neq \emptyset$, hence int $(f/B)^{-1}\,(V) \neq \neq \emptyset$. Put $G = \operatorname{int}(f/B)^{-1}\,(V)$. Evidently $G \subset U$ and $f(G) \subset V$. The quasi continuity of f at x_0 is proved. Since x_0 was arbitrary, the quasicontinuity of f on X follows.

Proof of Theorem 2. Let $\{V_n\}_{n-1}^{\infty}$ be a basis of Y and $\mathscr{B}$ any basis of X. The collection of $B \times V_n$, where $n = 1, 2, \ldots$ and B runs over $\mathscr{B}$, is a basis of $X \times Y$. Considering the restriction $f/B \times V_n$, we see that it satisfies on each $B \times V_n$ the assumptions of Theorem 1. ($B \times V_n$ is considered with the relative topology). Hence $f/B \times V_n$ is somewhat continuous. Now the result follows from the lemma.

The following example shows that the somewhat continuity of $f(x, y)$ in each variable separately does not imply the somewhat continuity of $f(x, y)$ as a function of two variables.

Example. Define the functions f_1, f_2, f_3, f_4 on

$$\langle 0, \tfrac{1}{2}) \times \langle 0, 1\rangle, \langle \tfrac{1}{2}, 1\rangle \times \langle 0, 1\rangle, \langle 0, 1\rangle \times \langle -\tfrac{1}{2}, 0)$$

$$\langle 0, 1) \times \langle -1, -\tfrac{1}{2}) \text{ respectively.}$$

Put $f_1(x, y) = \begin{cases} 1 & \text{if } y \text{ is rational} \\ 0 & \text{if } y \text{ is irrational} \end{cases}$

$f_2(x, y) = \begin{cases} 0 & \text{if } y \text{ is rational} \\ 1 & \text{if } y \text{ is irrational} \end{cases}$

$f_3(x, y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

$f_4(x, y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

Functions f_5, f_6, f_7, f_8 are defined on

$$\langle -1,0) \times \langle \tfrac{1}{2}, 1\rangle, \langle -1, 0) \times \langle 0, \tfrac{1}{2}), \langle -1, \tfrac{1}{2}) \times \langle -1, 0)$$

$\langle -\tfrac{1}{2}, 0) \times \langle -1, 0)$ respectively, as follows

$f_5(x, y) = f_4(-x, -y)$, $f_6(x, y) = f_3(-x, -y)$

$f_7(x,y) = f_2(-x, -y)$, $f_8(x, y) = f_1(-x, -y)$

Denote the interval $\langle -1, 1\rangle \times \langle -1, 1\rangle$ as I. Put $f(x, y) = f_i(x, y)$, where $1 \leqq i \leqq 8$. f is unambiguously defined on I by means of the functions f_i. It is easy to check that f is not somewhat continuous on I while the sections f_x and f_y are somewhat continuous for every

$x \in \langle -1, 1\rangle$, $y \in \langle -1, 1\rangle$, respectively.

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