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PARITY OF NUMBERS OF CROSSINGS FOR COMPLETE n-PARTITE GRAPHS

HEIKO HARBORTH

Dedicated to Professor Dr. H.—J. Kanold on the occasion of his sixtieth birthday

1. Introduction

For the vertices of a graph G (without loops and multiple edges) we draw distinct points or small circles, called nodes, in the plane. Then we connect every pair of these nodes by a simple Jordan arc if the corresponding vertices of G are adjacent in G. Doing this we further take care that two arcs have at most one point in common, either a node, with which both arcs are incident, or a point of intersection, called a crossing. Crossings of more than two arcs in one point are not allowed. We finally call this mapping of G onto the Euclidean plane a drawing $D(G)$ of G ("good drawing" in [1]).

Two nodes, two crossings, or a node and a crossing are called adjacent in $D(G)$, if they are connected by a part of an arc without any further crossing. Two simple regions of the plane, being bounded by polygons with such parts of arcs as sides, are called adjacent in $D(G)$, if their polygons have sides in common. Then two drawings $D_1(G)$ and $D_2(G)$ will be called isomorphic, if there exists a one-to-one correspondence between their nodes, crossings, arcs, and regions, which preserves the adjacency properties.

Besides the question for planarity of G only a few of the problems concerning nonisomorphic drawings of G have been investigated. Several authors take into account the minimum number of crossings for special classes of graphs (for references see [1]).

In this paper we will consider complete n-partite graphs $G(x_1, x_2, \ldots, x_n) = G(x_{1/n})$, which are graphs with $m = x_1 + x_2 + \ldots + x_n$ vertices ($n \geqq 2$), being the complement of n disjoint complete graphs with x_1, x_2, ..., and x_n vertices, respectively. If we use n different colors for these n classes of vertices, it becomes clear that $G(x_{1/n})$ also may be called a complete n-colorable graph. As introduced in [2], we distinguish three types of crossings: four-, three-, or two-colorable crossings in case the four nodes determining a crossing are of four, three, or two different colors, respectively. From this we have to consider seven different numbers S of crossings for a drawing $D(G(x_{1/n}))$: $S2(x_1, x_2, \ldots, x_n) = S2(x_{1/n}) = S2$, $S3$, $S4$, $S23$, $S24$, $S34$, and $S234 = S$.

The minimum of $S = S(x_{1/n})$, the so-called crossing number $cr(x_{1\ n})$, has been estimated in [2] and [4]. Since by the concept of drawing used here maximum numbers of crossings CR are easily to be found, we will list them in Section 2. In studying all integers occurring as numbers of crossings for all nonisomorphic drawings of $G(x_{1/n})$, we observe, that in some cases only one residue class modulo 2 is possible. Therefore it will be the purpose of this paper to give necessary and sufficient conditions for the numbers of crossings of $G(x_{1/n})$ to be only of one parity. In [3] this parity argument already is used (however, not convincingly proved) for complete bipartite graphs $G(x_1, x_2)$ (only two-colorable crossings), and in [1] a theorem for complete graphs $G(1, \ldots, 1) = K_n$ (only four-colorable crossings) was announced for 1973, but has not yet materialized.

2. Maximum numbers of crossings

As two arcs of a drawing are allowed to have at most one crossing, we get the following results.

Theorem 1. *The maximum numbers of crossings for a complete n-partite graph $G(x_{1/n})$ are*

(1) $$CR2(x_{1/n}) = \sum_{1 \leqq i<j \leqq n} \binom{x_i}{2}\binom{x_j}{2},$$

(2) $$CR3(x_{1/n}) = \sum_{1 \leqq i\ \ j<r \leqq n} \frac{1}{2}\, x_i x_j x_r (x_i + x_j + x_r - 3),$$

(3) $$CR4(x_{1/n}) = \sum_{1 \leqq i<j<r\ \ s \leq n} x_i x_j x_r x_s,$$

(4) $$CR23(x_{1/n}) = CR2(x_{1/n}) + CR3(x_{1/n}),$$

(5) $$CR24(x_{1/n}) = CR2(x_{1/n}) + CR4(x_{1/n}),$$

(6) $$CR34(x_{1/n}) = CR3(x_{1/n}) + CR4(x_{1/n}),$$

(7) $$CR(x_{1/n}) = CR2(x_{1/n}) + CR3(x_{1/n}) + CR4(x_{1/n}) = \binom{m}{4} - \sum_{i=1}^{n} \left\{ \binom{x_i}{4} + (m - x_i)\binom{x_i}{3} \right\}$$

with

(8) $$m = x_1 + x_2 + \ldots + x_n.$$

Proof. ($\leqq$) At most every pair of nodes of one color i together with every pair of another color j, or every pair of nodes of color i together with all pairs of nodes of colors j and r, or every quadruple of nodes with different colors i, j, r, s, determine at most one two-, one three-, or one four-colorable crossing, respectively. Hence $S2 \leqq CR2$, and $S4 \leqq CR4$ follows immediately, and $S3 \leqq CR3$ is seen to be valid by

$$\binom{x_i}{2}x_jx_r + x_i\binom{x_j}{2}x_r + x_ix_j\binom{x_r}{2} = \frac{1}{2}x_ix_jx_r(x_i + x_j + x_r - 3).$$

That "$\leqq$" holds in (4), (5), (6), and in the first relation of (7) is trivial. If we consider all quadruples of the m nodes of $D(G(x_{1/n}))$, then at least every quadruple of nodes of any color i, so as every triple of nodes of color i together with every node being not of this color i, cannot determine a crossing. Thus the second term in (7) also gives an upper bound of $CR(x_{1/n})$.

($\geqq$) We now describe a special drawing of $G(x_{1/n})$ in which the numbers of (1) to (7) will be attained. For nodes we take the point-vertices of a convex m-gon. Then for $i = 1, 2, \ldots, n$ we color x_i consecutive nodes by the color i. We then draw the arcs from all nodes of one color to all nodes of another color in bundles inside the polygon (see Fig. 1). Two-colorable crossings occur

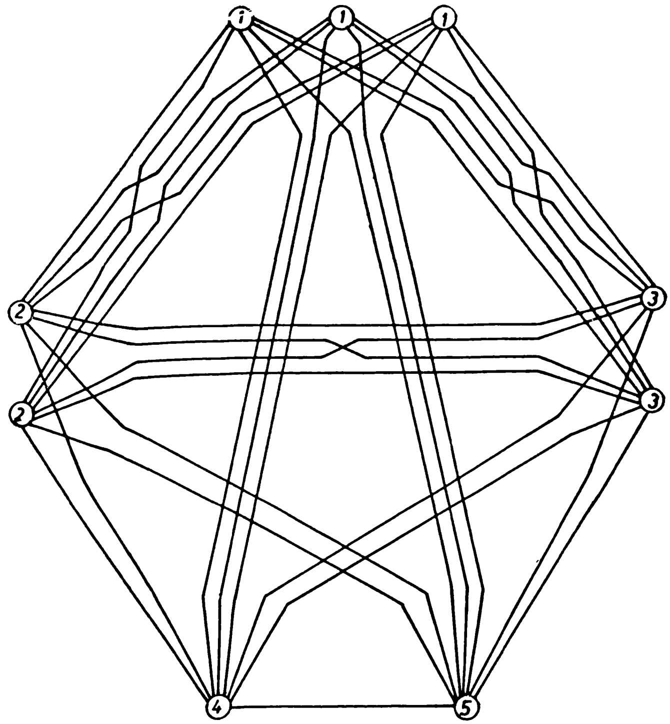

Fig. 1. $D(G(3, 2, 2, 1, 1))$ with maximum numbers of crossings.

inside these bundles. Three-colorable crossings converge near the nodes of that color, two of them have a share in the crossing. Four-colorable crossings are to be found, where bundles intersect. By counting the different crossings the proof is finished.

3. Parity of *S*2

In this section only two-colorable crossings are of interest.

Lemma 1. *Any drawing of* $G(3, 3)$ *has* 1, 3, 5, 7, *or* 9 *crossings.*

Proof. It may be possible to give simpler proofs (see for instance [3]), however, checking all nonisomorphic drawings of the Kuratowski graph $G(3, 3)$ will imply Lemma 1, and to have listed these drawings is of interest in itself. Hence in Fig. 2 we present all drawings of $G(3, 3)$. There are 1, 9, 33, 48, and 11 drawings with 1, 3, 5, 7, and 9 crossings, respectively.

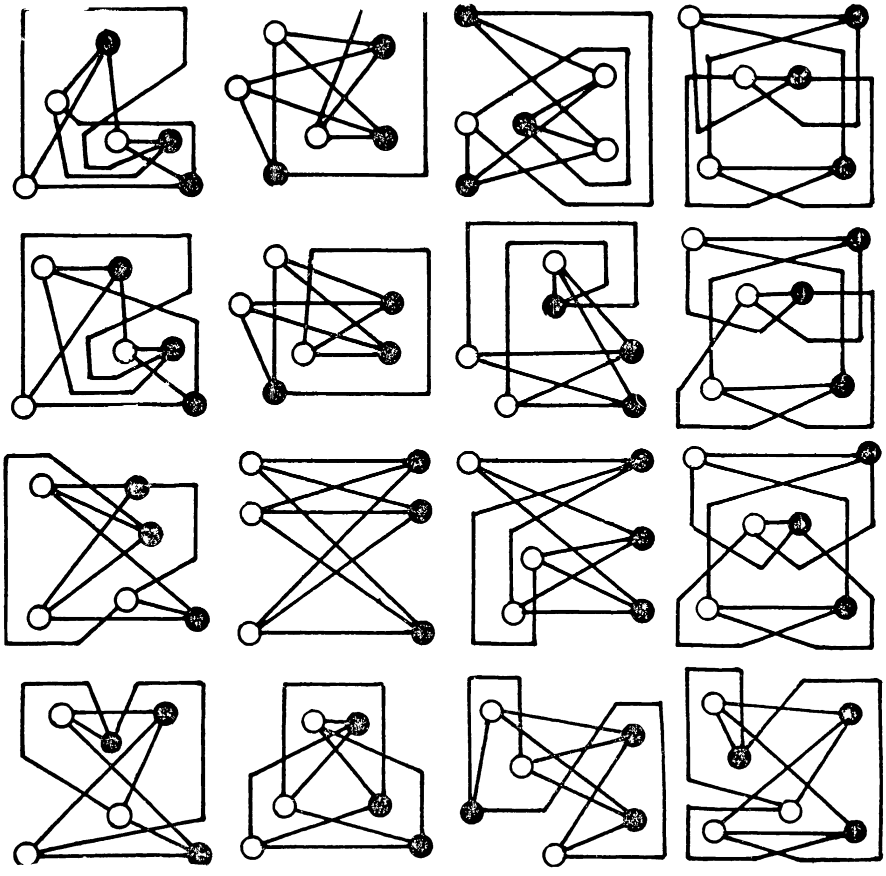

Fig. 2. All 2, 6, and 102 nonisomorphic drawings $D(G(2,2))$, $D(G(3,2))$, and $D(G(3,3))$.

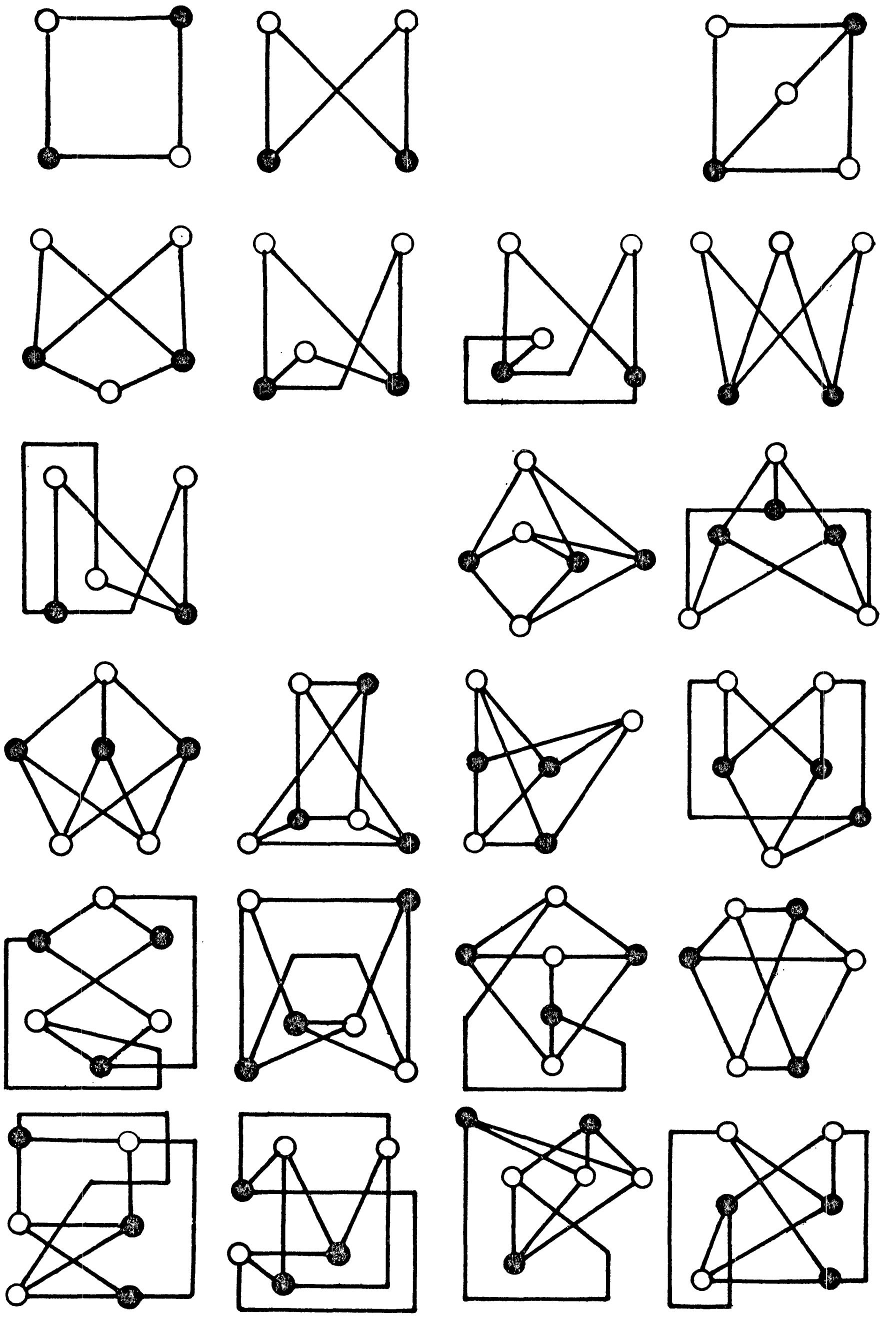

Fig. 2(1)

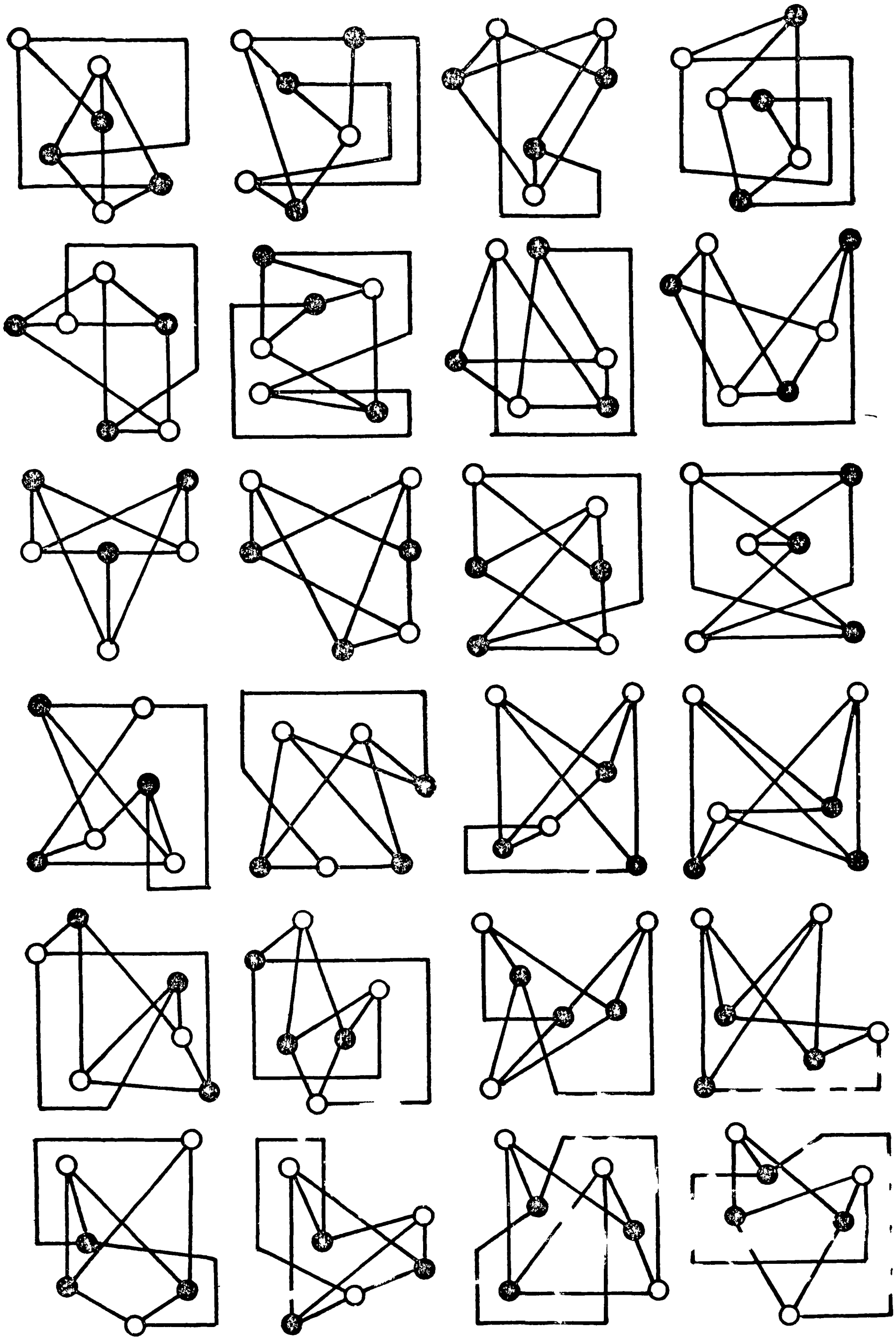

Fig. 2(2)

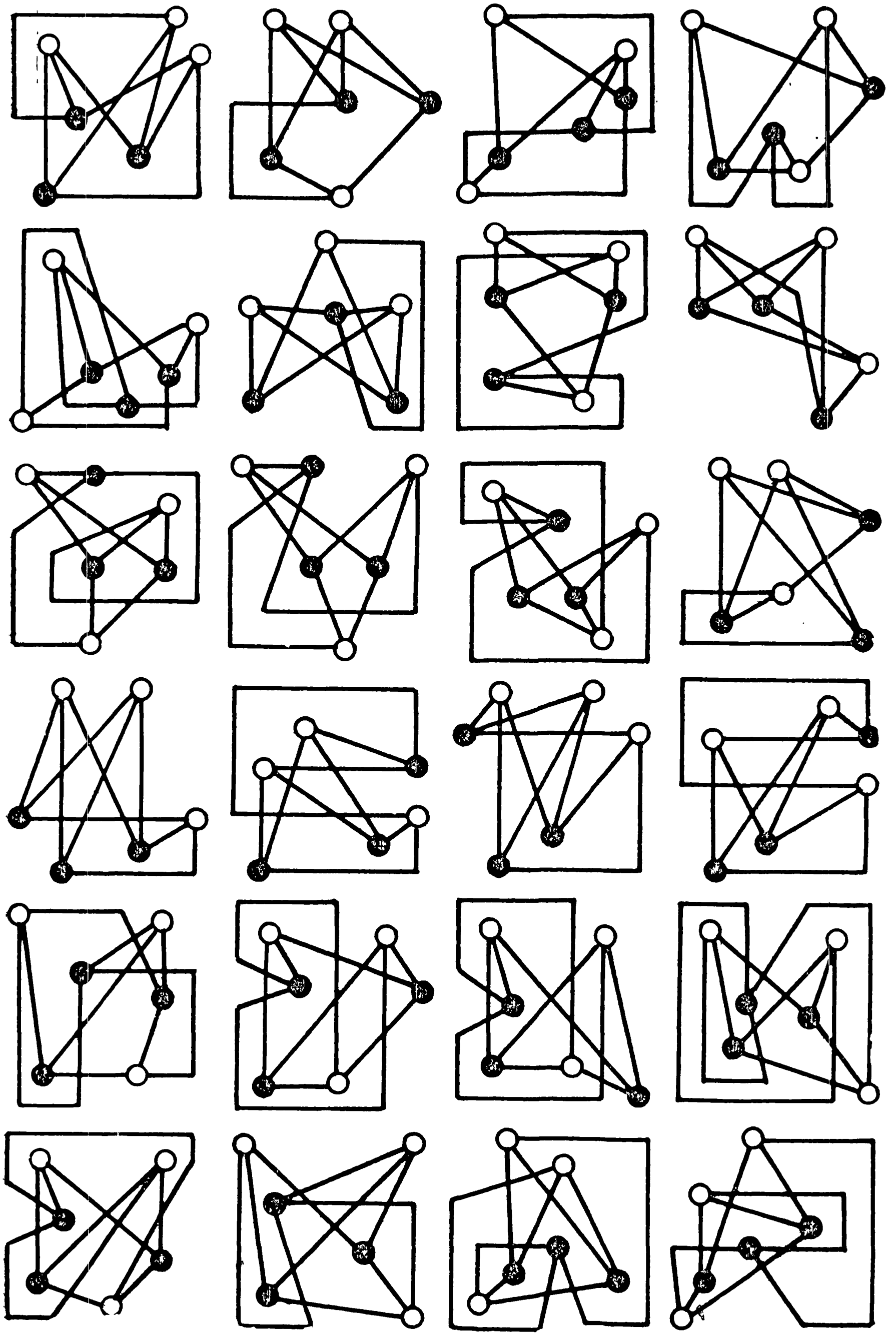

Fig. 2(3)

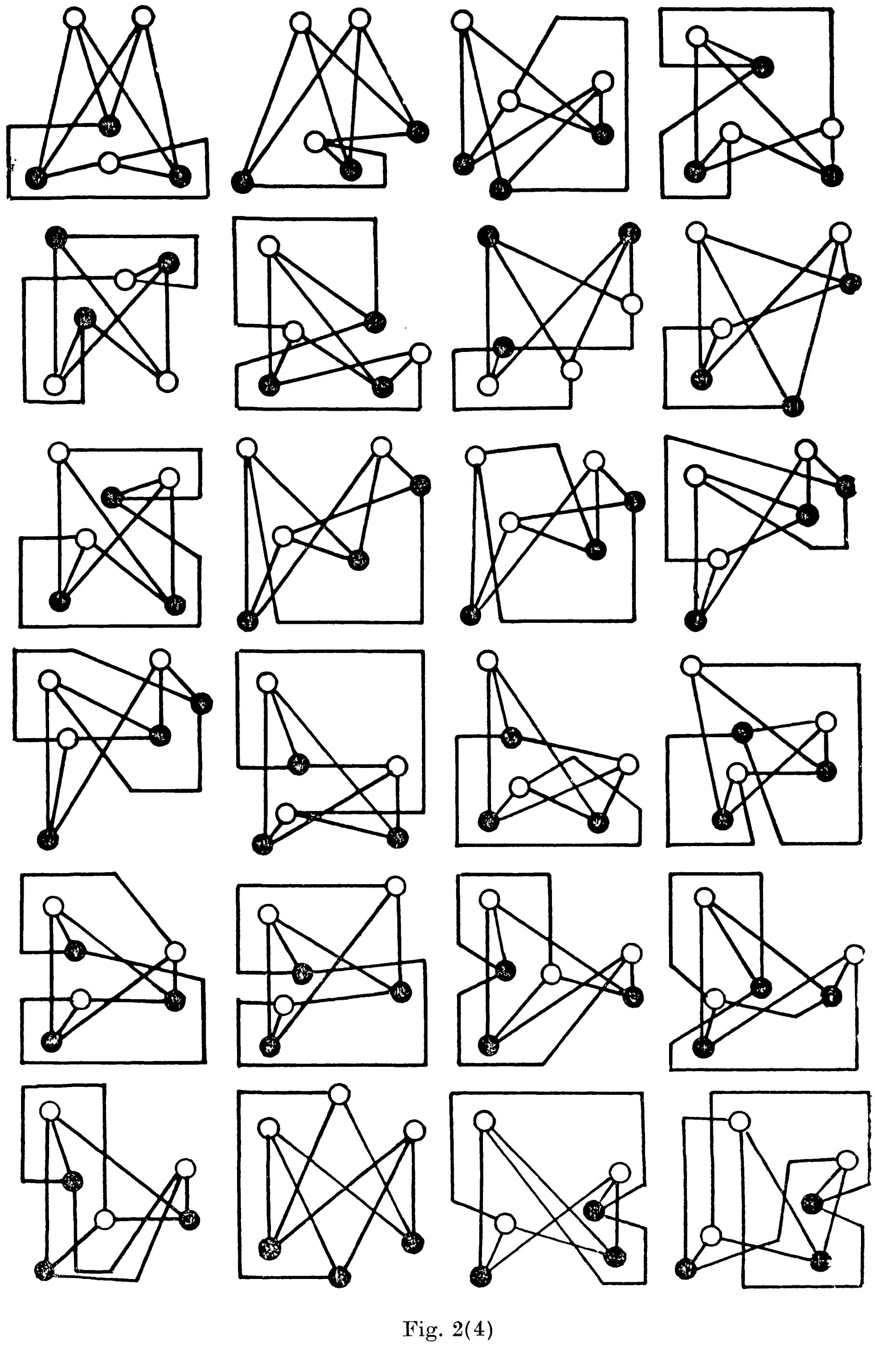

Fig. 2(4)

Let G_2 be a graph having one more vertex P than a graph G_1. Any drawing of G_1 dissects the plane in to simple regions. We put a further node (corresponding to P) successively into each of these regions. Then we draw in all possible ways those arcs the corresponding edges of which are incident with P in G_2. We do this by going from one region to each neighbouring region if the common part of an arc is still allowed to be intersected. Finally we get a finite number of drawings $D(G_2)$. Some of them being isomorphic may be neglected. As, conversely, by omitting from $D(G_2)$ the node corresponding to P so as all arcs being incident with this node, we always get a drawing $D(G_1)$, we are sure to receive all nonisomorphic drawings $D(G_2)$ by this procedure from all such drawings of G_1. There are 2 drawings of $G(2, 2)$, 6 drawings of $G(3, 2)$, and 102 drawings of $G(3, 3)$ (see Fig. 2).

Theorem 2. *Consider $G(x_{1/n})$ with at least two values $x_i \geqq 2$. Then the parity of all two-colorable numbers of crossings of drawings $D(G(x_{1/n}))$ is the same, iff $x_1, x_2, \ldots, x_n$ are all odd. Let l denote the number of these values x_i being $\equiv 3 (\mathrm{mod}\ 4)$, then*

$$S2(x_{1/n}) \equiv \begin{cases} 0 (\mathrm{mod}\ 2) & \text{if } l \equiv 0,\ 1 (\mathrm{mod}\ 4), \\ 1 (\mathrm{mod}\ 2) & \text{if } l \equiv 2,\ 3 (\mathrm{mod}\ 4). \end{cases} \tag{9}$$

Proof. ($\Leftarrow$) We consider two colors i and j for the present. With these colors there are $\binom{x_i}{3}\binom{x_j}{3}$ different subgraphs $G(3, 3)$ of $G(x_i, x_j)$, being a subgraph of $G(x_{1/n})$. If $\alpha_{2r+1}(i, j)$ subgraphs $G(3, 3)$ have drawings with exactly $2r + 1$ crossings of $D(G(x_{1/n}))$ for $r = 0, 1, 2, 3, 4$, then by Lemma 1

$$\binom{x_i}{3}\binom{x_j}{3} = \sum_{r=0}^{4} \alpha_{2r+1}(i, j). \tag{10}$$

Every two-colorable crossing of $D(G(x_i, x_j))$ is counted in $(x_i - 2)(x_j - 2)$ drawings $D(G(3, 3))$, so that

$$(x_i - 2)(x_j - 2) S2(x_i, x_j) = \sum_{r=0}^{4} (2r + 1)\alpha_{2r+1}(i, j). \tag{11}$$

We use

$$S2(x_{1/n}) = \sum_{1 \leqq i < j \leqq n} S2(x_i, x_j), \tag{12}$$

and get by summation of (11) and substitution of (10)

$$S2(x_{1/n}) + \sum_{1 \leqq i < j \leqq n} \{(x_i - 2)(x_j - 2) - 1\} S2(x_i, x_j) = \tag{13}$$

$$= \sum_{1 \leqq i < j \leqq n} \left\{\binom{x_i}{3}\binom{x_j}{3} + 2\sum_{r=0}^{4} r\alpha_{2r+1}(i, j)\right\}.$$

If now all values x_i are odd we get from (13)

$$(14) \qquad S2(x_{1/n}) \equiv \sum_{1 \leqq i < j \leqq n} \binom{x_i}{3}\binom{x_j}{3} \pmod 2,$$

and this congruence is independent of a special drawing.

Every summand in (14) is divisible by two if $x_i \equiv 1 \pmod 4$ or $x_j = 1 \pmod 4$, so that there remain $\binom{l}{2}$ odd summands, that is

$$(15) \qquad S2(x_{1/n}) \equiv \binom{l}{2} \pmod 2.$$

From (15) now (9) follows immediately.

($\Rightarrow$) Let 1 and 2 be colors with $x_1 \equiv 0 \pmod 2$ and $x_2 \geqq 2$. We consider a drawing $D(G(x_{1/n}))$ as described in Section 2. The consecutive nodes of colors 1 and 2 are labelled clockwise by $P_1, P_2, \ldots, P_{x_1}$, and $Q_1, Q_2, \ldots, Q_{x_2}$, respectively, and P_{x_1} has to be followed immediately by Q_1. Then on the arc (P_{x_1}, Q_2) there are exactly $x_1 - 1$ two-colorable crossings induced by (P_1, Q_1), (P_2, Q_1), $\ldots$, $(P_{x_1}\ 1, Q_1)$. If we now connect P_{x_1} and Q_2 by an arc outside the convex m-gon instead of inside, we get another drawing of $G(x_1\ n)$ with $CR2(x_{1/n}) - (x_1 - 1)$ crossings. The numbers $CR2$ and $CR2 - x_1\ \ 1$, however, are modulo 2 incongruent.

4. Parity of $S3$

In studying three-colorable crossings we start with two Lemmas.

Lemma 2. *The three-colorable number of crossings for any drawing of* $G(3, 1, 1, 1)$ *takes one of the values* 1, 3, 5, 7, *or* 9.

Proof. There are only three- and four-colorable crossings in a drawing $D(G(3, 1, 1, 1))$. We consider those three nodes each of which is the single one of a color, and the three arcs connecting them. On these arcs only four -colorable crossings are to be found, and, conversely, every four-colorable crossing of $D(G(3, 1, 1, 1))$ lies on these arcs. Thus, if we omit these three arcs, there remains a drawing $D(G(3, 3))$ with all three-colorable crossings of $D(G(3, 1, 1, 1))$. Lemma 1 then yields Lemma 2.

Lemma 3. *Any drawing* $D(G(2, 2, 2))$ *has an even number of three-colorable crossings.*

Proof. Let the nodes of the first, second, and third color be denoted by P_1 and P_2, P_3 and P_4, and P_5 and P_6, respectively. We distinguish the following four cases

$$i\ \ 1: P_1, P_3, P_5; \quad i = 2: P_1, P_3, P_6;$$

$$i\ \ 3: P_1, P_4, P_5; \quad i = 4: P_1, P_4, P_6.$$

In these cases $i = 1, 2, 3$, and 4 we use a new color for the given nodes, and the occasionally remaining three nodes of $G(2, 2, 2)$ are colored by another new color. We further omit those arcs connecting nodes of the same new color. Thus we receive drawings of subgraphs $G^{(i)}(3, 3)$ of $G(3, 1, 1, 1)$ with the numbers of crossings $S2^{(i)}(3, 3)$. We easily check that every two-colorable crossing of $D(G(2, 2, 2))$ is counted exactly twice in all drawings $D(G^{(i)}(3, 3))$, $i = 1, 2, 3, 4$, whereas every three-colorable one is counted exactly once, that is

$$S3(2, 2, 2) + 2S2(2, 2, 2) = \sum_{i=1}^{4} S2^{(i)}(3, 3). \tag{16}$$

By Lemma 1 the four summands on the right of (16) are odd, and so the value of $S3(2, 2, 2)$ is always even.

We now will prove the following assertion.

Theorem 3. *If* $n \geqq 3$, *and* $x_i \geqq 2$ *for at least one index* i, *then the parity of three-colorable numbers of crossings is the same for all nonisomorphic drawings* $D(G(x_{1/n}))$, *iff* (a) *every* x_i *is odd, and* n *is even, or* (b) *every* x_i *is even* $(1 \leqq i \leqq n)$. *Let* l *values* x_i *be* $\equiv 3 (\mathrm{mod}\ 4)$, *then in case* (a)

$$S3(x_{1/n}) \equiv \begin{cases} 1 (\mathrm{mod}\ 2), & \textit{if } l \equiv 1 (\mathrm{mod}\ 2),\ n \equiv 0 (\mathrm{mod}\ 4), \\ 0 (\mathrm{mod}\ 2) & \textit{otherwise}, \end{cases} \tag{17}$$

and in case (b)

$$S3(x_{1/n}) \equiv 0 (\mathrm{mod}\ 2). \tag{18}$$

Proof. ($\Leftarrow$(a)) The number of three-colorable crossings determined by two nodes of color i, one node of color j, and one of color r, will be denoted by $S3_{i;j,r}$. Next, $\alpha_{2r+1}(i)$, $r = 0, 1, 2, 3, 4$, will be the number of subgraphs $G(3, 1, 1, 1)$ of $G(x_{1/n})$ containing as part of a drawing $D(G(x_{1/n}))$ exactly $2r + 1$ three-colorable crossings, each with two nodes of color i. By Lemma 2 we get for the number of subgraphs $G(3, 1, 1, 1)$ of $G(x_{1/n})$ having three nodes of color i

$$\binom{x_i}{3} \sum_{\substack{1 \leqq j < r < s \leqq n \\ j, r, s \neq i}} x_j x_r x_s = \sum_{r-0}^{4} \alpha_{2r+1}(i). \tag{19}$$

Every three-colorable crossing with its nodes of colors i, i, j, and r may be completed by one of $x_i - 2$ nodes of color i, one of $m - x_i - x_j - x_r$ nodes being not of the colors i, j, or r, so as by the corresponding arcs to drawings $D(G(3, 1, 1, 1))$ with three nodes of color i. Thus

$$(x_i - 2) \sum_{\substack{1 \leqq j < r \leqq n \\ j, r \neq i}} (m - x_i - x_j - x_r) S3_{i;j,r} = \sum_{r=0}^{4} (2r + 1) \alpha_{2r+1}(i). \tag{20}$$

Together with

(21) $$S3(x_{1/n}) = \sum_{i=1}^{n} \sum_{\substack{1 \leqq j < r \leqq n \\ j,\, r \neq i}} S3_{i;j,r}(x_{1/n})$$

we get from (19) and (20)

(22) $$S3(x_{1/n}) + \sum_{i=1}^{n} \sum_{\substack{1 \leqq j < r \leqq n \\ j,r \neq i}} \{(x_i - 2)(m - x_i - x_j - x_r) - 1\} S3_{i;j\,r} =$$
$$= \sum_{i=1}^{n} \binom{x_i}{3} \sum_{\substack{1 \leqq j < r < s \leqq n \\ j,r,s \neq i}} x_j x_r x_s + 2 \sum_{i=1}^{n} \sum_{r=0}^{4} r\alpha_{2r+1}(i).$$

Now in case (a) the congruences

(23) $$x_i - 2 \equiv 1 (\mathrm{mod}\ 2) \text{ and } m - x_i - x_j - x_r \equiv 1 (\mathrm{mod}\ 2)$$

are fulfilled for all summands in the first sum of (22), and we conclude from this

(24) $$S3(x_{1/n}) \equiv \sum_{i=1}^{n} \binom{x_i}{3} \sum_{\substack{1 \leqq j < r < s \leqq n \\ j,r,s \neq i}} x_j x_r x_s \ (\mathrm{mod}\ 2).$$

The inner sums of (24) consist of $\binom{n-1}{3}$ odd terms, and $\binom{x_i}{3}$ is odd only if $x_i \equiv 3 (\mathrm{mod}\ 4)$, so that (24) yields

(25) $$S3(x_{1/n}) \equiv \binom{n-1}{3} \sum_{i=1}^{n} (x_i 3) \equiv l \binom{n-1}{3} \quad (\mathrm{mod}\ 2).$$

From (25) we get (17) at once.

Let us remark that the preceding part of the proof ($\Leftarrow$(a)) may be obtained also by using

(26) $$S3(x_{1/n}) = \sum_{i=1}^{n} S2(x_i, m - x_i) - 2S2(x_{1/n}),$$

and by discussing in all possible combinations the residue classes of l and n modulo 4. The validity of (26) is realized straight away.

($\Leftarrow$(b)) By $S3_i$ we denote the number of three-colorable crossings with two determining nodes of color i. For a drawing $D(G(x_i, x_j, x_r))$ we add up the numbers of three-colorable crossings for the drawings of all subgraphs $G(2, 2, 2)$ of $G(x_i, x_j, x_r)$. Then because of Lemma 3 this sum is even. On the other hand every three-colorable crossing with two nodes of color i is counted in $(x_j - 1)(x_r - 1)$ different subgraphs $G(2, 2, 2)$. Thus

(27) $$(x_j - 1)(x_r - 1) S3_i(x_i, x_j, x_r) + (x_i - 1)(x_r - 1) S3_j(x_i, x_j, x_r)$$
$$+ (x_i - 1)(x_j - 1) S3_r(x_i, x_j, x_r) \equiv 0 (\mathrm{mod}\ 2).$$

Then by using

$$(28) \qquad S3(x_i, x_j, x_r) = S3_i(x_i, x_j, x_r) + S3_j(x_i, x_j, x_r) + S3_r(x_i, x_j, x_r)$$

we conclude from (27)

$$(29) \quad \begin{aligned} &\{(x_j - 1)(x_r - 1) + (x_i - 1)(x_r - 1) + (x_i - 1)(x_j - 1)\}S3(x_i, x_j, x_r) \\ &\quad - \{(x_i - 1)(x_r - 1) + (x_i - 1)(x_j - 1)\}S3i(x_i, x_j, x_r) \\ &\quad - \{(x_j - 1)(x_r - 1) + (x_j - 1)(x_i - 1)\}S3_j(x_i, x_j, x_r) \\ &\quad - \{(x_r - 1)(x_j - 1) + (x_r - 1)(x_i - 1)\}S3_r(x_i, x_j, x_r) \\ &\quad \equiv 0(\text{mod } 2). \end{aligned}$$

In case (b) all x_i are even. Therefore the coefficients of $S3_i$, $S3_j$, and $S3_r$ in (29) are even. Furthermore the coefficient of $S3(x_i, x_j, x_r)$ is odd, so that we can divide by it in (29). Thus $S3(x_i, x_j, x_r)$ is even, and together with

$$(30) \qquad S3(x_{1/n}) = \sum_{1 \leqq i < j < r \leqq n} S3(x_i, x_j, x_r) \equiv 0(\text{mod } 2)$$

we have obtained (18).

($\Rightarrow$) Again we consider a drawing $D(G(x_{1/n}))$ with maximum numbers of crossings, as described in Section 2. The nodes of colors 1 and 2 are clockwise consecutive points $P_1, P_2, \ldots, P_{x_1}, Q_1, Q_2, \ldots, Q_{x_2}$ on the m-gon. The numbers of crossings are not changed if the colors 1 and 2 are arbitrarily chosen. On the arc (P_{x_1}, Q_2) there are exactly $m - x_1 - x_2$ three-colorable crossings.

If $m \equiv 0(\text{mod } 2)$, we choose $x_1 \equiv 0(\text{mod } 2)$, and $x_2 \equiv 1(\text{mod } 2)$, which is always possible. Namely, because of (b) there will be at least one odd x_i, and all x_i odd, together with m even would be equivalent to (a). If $m \equiv 1(\text{mod } 2)$, we may choose either $x_1 \equiv x_2 \equiv 0(\text{mod } 2)$ or $x_1 \equiv x_2 \equiv 1(\text{mod } 2)$, as $n \geqq 3$. In any case $m - x_1 - x_2$ will be odd. Now we omit (P_{x_1}, Q_2), and we draw a new arc outside the m-gon. We then have two drawings of $G(x_{1/n})$ with $CR3$ and $CR3 - m + x_1 + x_2$ three-colorable crossings, where both numbers are of different residue classes modulo 2.

5. Parity of $S23$

Theorem 4. *If $n \geqq 3$, and $x_i \geqq 2$ for at least one of the values x_i, then the numbers $S23(x_{1/n})$ of not four-colorable crossings in all nonisomorphic drawings $D(G(x_{1/n}))$ are of the same parity, iff all x_i are odd and n is even* ($1 \leqq i \leqq n$). *Let l times $x_i \equiv 3(\text{mod } 4)$ hold, then*

$$(31) \qquad S23(x_{1/n}) \equiv \begin{cases} 0(\text{mod } 2), \text{ if } n \equiv 0(\text{mod } 4),\ l \equiv 0, 3(\text{mod } 4), \\ \qquad\qquad \text{or if } n \equiv 2(\text{mod } 4),\ l \equiv 0, 1(\text{mod } 4), \\ 1(\text{mod } 2), \text{ if } n \equiv 0(\text{mod } 4),\ l \equiv 1, 2(\text{mod } 4), \\ \qquad\qquad \text{or if } n \equiv 2(\text{mod } 4),\ l \equiv 2, 3(\text{mod } 4). \end{cases}$$

Proof. ($\Rightarrow$) The nodes $P_1, P_2, \ldots, P_{x_2}, Q_1, Q_2, \ldots, Q_{x_1}$ are consecutive in a drawing with maximum numbers of crossings (Section 2). Then the arc (P_{x_2}, Q_2) has exactly $m - x_1 - x_2$ three-colorable and $x_2 - 1$ two-colorable crossings, that is together $m - x_1 - 1$. If $m - x_1 \equiv 0 (\mathrm{mod}\ 2)$, then by drawing (P_{x_2}, Q_2) inside or outside the m-gon we have two drawings of $G(x_{1/n})$ with modulo 2 different numbers $S23$ of crossings. We now are able to choose color 1 with $x_1 \equiv m (\mathrm{mod}\ 2)$ in all cases besides m even and all x_i odd, which means, however, that n is even.

($\Leftarrow$) This part of the proof follows immediately from Theorems 3 and 4 together with

$$S23(x_{1/n}) = S2(x_{1/n}) + S3(x_{1/n}). \tag{32}$$

If only one $x_i \geqq 2$, then $S2(x_{1/n}) = 0$, trivially. Equations (9) and (17) yield (31).

6. Parity of $S4$

We now will be engaged in four-colorable crossings.

Lemma 4. *Any drawing of the complete graph* $G(1, 1, 1, 1, 1) = K_5$ *has* 1, 3, *or* 5 *crossings.*

Proof. Using the same procedure as described in Section 3 we get five nonisomorphic drawings of the Kuratowski graph K_5. There are 1, 2, and 2 drawings with 1, 3, and 5 crossings, respectively, shown in Fig. 3.

Lemma 5. *For* $G(2, 2, 2, 2)$ *the numbers* $S4(2, 2, 2, 2)$ *of four-colorable crossings are always even.*

Proof. Let P_1 and P_2 be vertices of the same color in $G(2, 2, 2, 2)$. We add a new edge (P_1, P_2), and obtain a graph G'. There are 8 different subgraphs of the type K_5 in G', having the vertices P_1, P_2, and one vertex of each of the remaining three colors. The corresponding numbers of crossings of $D^{(i)}(K_5)$ as part of $D(G')$ may be denoted by $S4^{(i)}(2, 2, 2, 2)$, $i = 1, 2, \ldots, 8$. There are no two-colorable crossings of $D(G(2, 2, 2, 2))$ in any $D^{(i)}(K_5)$. Every four--colorable crossing of $D(G(2, 2, 2, 2))$ occurs in exactly one $D^{(i)}(K_5)$. Those crossings for which both P_1 and P_2 are determining nodes are counted in two different drawings $D^{(i)}(K_5)$. Let S' be the number of such crossings in $D(G')$, then

$$S4(2, 2, 2, 2) + 2S' = \sum_{i=1}^{8} S4^{(i)}(2, 2, 2, 2). \tag{33}$$

As by Lemma 4 the values $S4^{(i)}$ are odd, it follows from (33), that $S4(2, 2, 2, 2)$ is even.

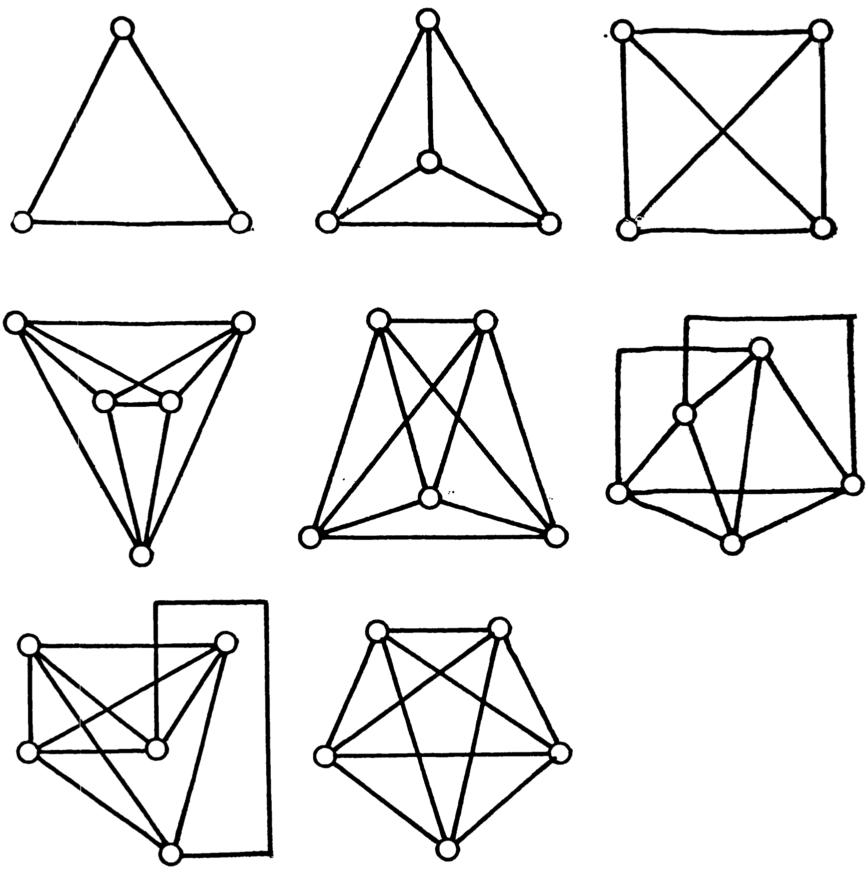

Fig. 3. All 1, 2, and 5 nonisomorphic drawings $D(G(1, 1, 1))$, $D(G(1, 1, 1, 1))$, and $D(G(1, 1, 1, 1, 1))$.

Theorem 5. *If $n \geqq 4$, the parity of $S4(x_{1/n})$ is the same for any drawing of $G(x_{1/n})$, iff* (a) *all values x_i are odd and n is odd, or* (b) *all values x_i are even* $(1 \leqq i \leqq n)$. *There holds in case* (a)

$$(34) \qquad S4(x_{1/n}) \equiv \begin{cases} 0(\text{mod } 2), \textit{ if } n \equiv 1, 3(\text{mod } 8), \\ 1(\text{mod } 2), \textit{ if } n \equiv 5, 7(\text{mod } 8), \end{cases}$$

and in case (b)

$$(35) \qquad S4(x_{1/n}) \equiv 0(\text{mod } 2).$$

Proof. ($\Leftarrow$(a)) We may assume $n \geqq 5$. As parts of $D(G(x_{1/n}))$ there are drawings $D(K_5)$ of all subgraphs K_5 of $G(x_{1/n})$. Let α_1, α_3, and α_5 be the numbers of such drawings $D(K_5)$, in which there occur 1, 3, and 5 crossings, respectively. With Lemma 4 we conclude

$$(36)\qquad \sum_{1 \leqq i<j<r<s \;\; t \leqq n} x_i x_j x_r x_s x_t = \alpha_1 + \alpha_3 + \alpha_5 .$$

Every four-colorable crossing of $D(G(x_{1/n}))$ is counted in $m - x_i - x_j - x_r - x_s$ different subgraphs K_5. That is

$$(37)\qquad \sum_{1 \leqq i<j<r<s \leqq n} (m - x_i - x_j - x_r - x_s) S4(x_i, x_j, x_r, x_s) =$$

$$= \alpha_1 + 3\alpha_3 + 5\alpha_5 .$$

We use

$$(38)\qquad S4(x_{1/n}) = \sum_{1 \leqq i<j<r \;\; s \leqq n} S4(x_i, x_j, x_r, x_s)$$

to get from (36) and (37)

$$(39)\quad S4(x_{1/n}) + \sum_{1 \leqq i<j \;\; r<s \leqq n} (m - x_i - x_j - x_r - x_s - 1) S4(x_i, x_j, x_r, x_s) -$$

$$= \sum_{1 \leqq i<j<r<s<t \leqq n} x_i x_j x_r x_s x_t + 2\alpha_3 + 4\alpha_5 .$$

If now x_i is odd for all i, and n is odd, then m is odd, too, and the coefficients of $S4(x_i, x_j, x_r, x_s)$ in (39) are even, so that

$$(40)\qquad S4(x_{1/n}) \equiv \sum_{1 \leqq i<j<r<s<t \leqq n} x_i x_j x_r x_s x_t \equiv \binom{n}{5} \pmod 2 .$$

From (40), independent of a special drawing, we infer (34) at once.

($\Leftarrow$(b)) We consider subgraphs $G(2, 2, 2, 2)$ of $G(x_{1/n})$ with colors i, j, r, s. Their numbers of four-colorable crossings in $D(G(x_{1/n}))$ are always even (Lemma 5). Every four-colorable crossing is counted in $(x_i - 1)(x_j - 1)(x_r - - 1)(x_s - 1)$ subgraphs $G(2, 2, 2, 2)$, that is

$$(41)\qquad (x_i - 1)(x_j - 1)(x_r - 1)(x_s - 1) S4(x_i, x_j, x_r, x_s) \equiv 0 \pmod 2 .$$

As all x_i are even in case (b), we are allowed to divide (44) by the coefficient of $S4$. Together with (38) we then get (35).

($\Rightarrow$) We take into account a special drawing $D'(G(x_{1/n}))$ with $S4'(x_{1\ n})$ four-colorable crossings. The nodes are distributed on a circular line, in such a way that there are three consecutive nodes P, Q, R of different colors, which still have to be chosen suitably. There are x_1, x_2, x_3 nodes with colors like P, Q, R, respectively. The arcs of $D'(G(x_{1/n}))$ are to be drawn inside the circle. On the arc (P, R) we find $m - x_1 - x_2 - x_3$ four-colorable crossings.

If $m \equiv 0$ or $1 \pmod 2$, we may choose $x_1 \equiv 1$ or $0 \pmod 2$, as otherwise (b) or (a) would hold, respectively. Because of $n \geqq 4$, there remain at least three colors, that is x_2 and x_3 may be chosen either both even or both odd. In any case $m - x_1 - x_2 - x_3$ becomes odd. Thus, in drawing (P, R) outside the circle instead of inside, we get a drawing with $S4'(x_{1/n}) - m + x_1 + x_2 + x_3$ four-colorable crossings. But this number differs from $S4'(x_{1/n})$ by an odd number.

7. Parity of $S24$

Theorem 6. *For $n \geqq 3$, and $G(x_{1/n}) \neq G(x, 1, 1)$ the parity of the numbers $S24(x_{1/n})$ of not three-colorable crossings is the same, iff all values x_i as well as n are odd* $(1 \leqq i \leqq n)$. *In detail, with l values $x_i \equiv 3 \pmod 4$ the following congruences are valid.*

$$S24(x_{1/n}) \equiv \begin{cases} 0 \pmod 2, \text{ if } n \equiv 1, 3 \pmod 8,\ l \equiv 0, 1 \pmod 4, \\ \qquad \text{or if } n \equiv 5, 7 \pmod 8,\ l \equiv 2, 3 \pmod 4, \\ 1 \pmod 2, \text{ if } n \equiv 1, 3 \pmod 8,\ l \equiv 2, 3 \pmod 4, \\ \qquad \text{or if } n \equiv 5, 7 \pmod 8,\ l \equiv 0, 1 \pmod 4. \end{cases} \tag{42}$$

Proof. $(\Rightarrow)$ A drawing, corresponding to $D'(G(x_{1/n}))$ of Section 6, where $P_1, P_2, \ldots, P_{x_1}, Q, R_1, R_2, \ldots, R_{x_3}$ are consecutive nodes on the circular line, has on (P_{x_1}, R_2) exactly $m - x_1 - x_2 - x_3$ four-colorable and $x_1 - 1$ two-colorable, that is together $m - x_2 - x_3 - 1$ not three-colorable crossings. If this number is odd, then (P_{x_1}, R_2) may be drawn outside or inside the circle to get two drawings with an even and an odd number $S24$.

If $m \equiv 0 \pmod 2$, we may choose x_2 and x_3 either both even or both odd $(n \geqq 3)$. In case of $m \equiv 1 \pmod 2$ it is possible to choose x_2 odd and x_3 even, as all x_i even would contradict m odd, and all x_i odd would yield n odd, which is just the condition of the Theorem.

$(\Leftarrow)$ This and (42) follow directly from

$$S24(x_{1/n}) = S2(x_{1/n}) + S4(x_{1/n}), \tag{43}$$

as well as from Theorems 2 and 5 in case $n \geqq 4$ and two values $x_i \geqq 2$. If $n = 3$, then $S4 = 0$ in (43), and we apply Theorem 2. For $G(x, 1, 1, \ldots, 1)$ there holds $S2 = 0$ in (43), and then Theorem 5 finishes the proof $(n \geqq 4)$.

8. Parity of $S34$

Theorem 7. *If $n \geqq 3$, and at least one value $x_i \geqq 2$, then for the number $S34(x_{1/n})$ of not two-colorable crossings the parity is the same, iff all x_i are even* $(1 \leqq i \leqq n)$. *In this case there is always*

$$S34(x_{1/n}) \equiv 0 \pmod 2. \tag{44}$$

Proof. (⇒) In the drawing of Section 7 there are now on the arc (P_{x_1}, R_1) exactly $m - x_1 - x_2 - x_3$ four-colorable and $x_1 - 1 + x_3 - 1$ three-colorable crossings, which are together $m - x_2$ not two-colorable crossings. If $m - x_2$ is odd, the proof follows as before.

In case of $m \equiv 0 (\bmod 2)$, we choose x_2 odd, for otherwise all x_i would be even. If $m \equiv 1 (\bmod 2)$, then either at least one x_i is even, say x_2, or all x_i are odd. In the latter case also n is odd. Then $S4$ always is of the same parity (Theorem 5), and $S3$ takes both residue classes modulo 2 (Theorem 3), so that with

$$S34(x_{1/n}) = S3(x_{1/n}) + S4(x_{1/n}) \tag{45}$$

the numbers $S34$ may be odd as well as even.

(⇐) Theorems 3 and 5 complete the proof for $n \geqq 4$. If $n = 3$, then $S4 = 0$ is trivial, and in (45) Theorem 3 is to be used. Theorems 3 and 5 together with (45) also yield (44).

9. Parity of S

Finally we combine the results of Sections 3, 4, and 6 to get statements for the parity of

$$S(x_{1/n}) = S234(x_{1/n}) = S2(x_{1/n}) + S3(x_{1/n}) + S4(x_{1/n}). \tag{46}$$

Theorem 8. If $n \geqq 3$, *and at least one value* $x_i \geqq 2$, *then the parity of the numbers* $S(x_{1/n})$ *of the crossings for all nonisomorphic drawings* $D(G(x_{1/n}))$ *is never the same.*

Proof. We take into account a drawing as in Section 7. On (P_{x_1}, R_2) there are $m - x_1 - x_2 - x_3$ four-colorable, $x_1 - 1 + x_3 - 2 + m - x_1 - x_3 - 1$ three-colorable, and $x_1 - 1$ two-colorable crossings, which are together $2m - x_2 - x_3 - 5$ crossings. This number is odd, if $x_2 \not\equiv x_3 (\bmod 2)$, and in these cases the proof is accomplished.

If $x_i \equiv 0 (\bmod 2)$ for all i, then (46) and Theorems 3 and 5 yield $S \equiv S2 (\bmod 2)$. But Theorem 2 shows that $S2$ is not of only one parity.

If $x_i \equiv 1 (\bmod 2)$ for all i, we distinguish two cases. First let n be even. $S2$ is of the same parity (Theorem 2, and $S2 = 0$, if only one $x_i \geqq 2$). Further $S3$ takes only one residue class modulo 2 (Theorem 3). Thus it follows from (46) and Theorem 5 that S and $S4$ are of odd as well as even values. Secondly, let n be odd. Then again $S2$ is of the same parity. Also $S4$ is of only one parity (Theorem 5, and $S4 = 0$ for $n = 3$). As $S3$ takes both residue classes modulo 2 (Theorem 3), by (46) this is right also for S.

Theorem 9. *Besides the trivial case* $S(x, 1) = 0$, *the parity of the numbers* $S(x_{1/n})$ *of the crossings for all nonisomorphic drawings* $D(G(x_{1/n}))$ *is the same only for*

$$S(x_1, x_2) = S2(x_1, x_2), \text{ if } x_1 \equiv x_2 \equiv 1 (\text{mod } 2),$$

and for

$$S(1, 1, \ldots, 1) = S4(1, 1, \ldots, 1), \text{ if } n \equiv 1 (\text{mod } 2),$$

that is, for complete bipartite and for complete graphs.

Proof. Theorem 8 gives the proof for $n \geqq 3$ and at least one $x_i \geqq 2$. If $n = 2$, then we have either $G(x, 1)$ (trivial) or $G(x_1, x_2)$ with $x_1, x_2 \geqq 2$, so that Theorem 2 may be used. If there is always $x_i = 1$, then we have the complete graph K_n. For $n = 3$ there holds $S = S4 = 0$, and for $n \geqq 4$ we apply Theorem 5.

REFERENCES

[1] GUY, R. K.: Crossing numbers of graphs. In: Graph theory and applications, Springer Lecture Notes in Mathematics 303, 1972, 111—124.

[2] HARBORTH, H.: Über die Kreuzungszahl vollständiger, n-geteilter Graphen. Math. Nachr. 48, 1971, 179—188.

[3] KLEITMAN, D. J.: The crossing number of $K_{5,n}$. J. Combin. Theory 9, 1970, 315 to 323.

[4] KOMAN, M.: On the crossing numbers of graphs. Acta Univ. Carolinae, Math. Phys. 10, 1969, 9—46.

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