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# ON QUASI-HOMOGENEOUS COPULAS

GASPAR MAYOR, RADKO MESIAR AND JOAN TORRENS

Quasi-homogeneity of copulas is introduced and studied. Quasi-homogeneous copulas are characterized by the convexity and strict monotonicity of their diagonal sections. As a by-product, a new construction method for copulas when only their diagonal section is known is given.

*Keywords:* copula, diagonal section, quasi-homogeneity

*AMS Subject Classification:* 62H05, 62E10

## 1. INTRODUCTION

Homogeneity of order  $k$  of real functions reflects their regularity with respect to the inputs with the same ratio in the form

$$F(\lambda x_1, \dots, \lambda x_n) = \lambda^k F(x_1, \dots, x_n). \quad (1)$$

In several classes of special functions, such as *triangular norms* or *copulas*, the homogeneity is a rather restrictive property. A generalized homogeneity should reflect the multiplicative constant  $\lambda$  as well as the original value  $F(x_1, \dots, x_n)$ , and thus it should be expressed on the form

$$F(\lambda x_1, \dots, \lambda x_n) = G(\lambda, F(x_1, \dots, x_n)) \quad (2)$$

where  $G$  is a binary function. In [9], the concept of quasi-homogeneity was introduced by considering  $G(a, b) = \varphi^{-1}(f(a)\varphi(b))$ , with  $\varphi$  an injective function and  $f$  an arbitrary function. Hence a function  $F$  is called quasi-homogeneous if

$$F(\lambda x_1, \dots, \lambda x_n) = \varphi^{-1}(f(\lambda)\varphi(F(x_1, \dots, x_n))). \quad (3)$$

The aim of this paper is to discuss the class of *quasi-homogeneous* copulas. In the next section, we recall some preliminary notions and results on homogeneity of t-norms and copulas, and on quasi-homogeneity of t-norms. In Section 3, we represent quasi-homogeneous copulas by means of their diagonal sections, while in Section 4 we characterize all diagonal sections of quasi-homogeneous copulas. As a consequence, a new construction method for copulas when only their diagonal section is known, is obtained. Finally several concluding remarks are included.

## 2. PRELIMINARIES

We will suppose the reader to be familiar with some basic concepts and results on copulas, that can be found in [15]. Recall that a binary function  $C : [0, 1]^2 \rightarrow [0, 1]$  is said to be a *copula* if it satisfies the following properties:

$$C1) \quad C(x, 0) = C(0, x) = 0 \text{ for all } x \in [0, 1],$$

$$C2) \quad C(x, 1) = C(1, x) = x \text{ for all } x \in [0, 1],$$

$$C3) \quad \text{for all } x, x', y, y' \text{ in } [0, 1] \text{ with } x \leq x' \text{ and } y \leq y',$$

$$C(x', y') - C(x, y') - C(x', y) + C(x, y) \geq 0.$$

The weakest copula is the Łukasiewicz copula whereas the strongest one is the minimum. They are respectively given by

$$W(x, y) = \max\{0, x + y - 1\} \quad \text{and} \quad M(x, y) = \min\{x, y\}$$

for all  $x, y \in [0, 1]$ .

Similarly, basic notions on *t-norms* are also assumed and they can be found in [14]. Recall that a binary function  $T : [0, 1]^2 \rightarrow [0, 1]$  is said to be a *t-norm* if it is associative, commutative, non-decreasing in each variable and has neutral element 1, that is  $T(x, 1) = T(1, x) = x$  for all  $x \in [0, 1]$ . Thus, we only recall here some definitions and results that will be used in the paper.

**Definition 1.** A function  $F : [0, 1]^2 \rightarrow [0, 1]$  is said to be *homogeneous* of degree  $k > 0$  if it satisfies

$$F(\lambda x, \lambda y) = \lambda^k F(x, y) \quad \text{for all } x, y, \lambda \in [0, 1]. \quad (4)$$

The homogeneity condition has been characterized for t-norms as well as for copulas and the results are as follows.

**Theorem 1.** (Alsina et al. [2], Theorem 3.4.1) A t-norm  $T$  is homogeneous if and only if either  $k = 1$  and  $T$  is the minimum t-norm, or  $k = 2$  and  $T$  is the product t-norm.

**Theorem 2.** (Nelsen [15], Theorem 3.4.2) A copula  $C$  is homogeneous if and only if  $1 \leq k \leq 2$  and  $C$  is the member  $C_\theta$  of the Cuadras–Augé family with  $\theta = 2 - k$ .

Recall that the Cuadras–Augé family is the parametric family of copulas given by

$$C_\theta(x, y) = (\min\{x, y\})^\theta (xy)^{1-\theta} \quad \text{for all } x, y \in [0, 1]$$

with  $\theta \in [0, 1]$ .

Some generalizations of the homogeneity condition have been studied, specially in the framework of t-norms (see [2]). One of these generalizations is introduced by substituting  $\lambda^k$  by any arbitrary function  $f : [0, 1] \rightarrow [0, 1]$ , but this leads to no new solutions for t-norms (see [2], Corollary 3.4.2). The widest generalization of homogeneity introduces the so-called quasi-homogeneity in the following terms.

**Definition 2.** A function  $F : [0, 1]^2 \rightarrow [0, 1]$  is said to be *quasi-homogeneous* if there exists a continuous, strictly monotonic function  $\varphi : [0, 1] \rightarrow \mathbb{R}$  and a function  $f : [0, 1] \rightarrow [0, 1]$  such that

$$F(\lambda x, \lambda y) = \varphi^{-1}(f(\lambda)\varphi(F(x, y))) \quad \text{for all } x, y, \lambda \in [0, 1]. \tag{5}$$

In this case it will be said that  $F$  is  $(\varphi, f)$ -quasi-homogeneous.

Quasi-homogeneous t-norms have been also characterized allowing for new solutions. The result is due to Ebanks in 1998 (see [9]), see also [2] for the current version.

**Theorem 3.** (Alsina et al. [2], Theorem 3.4.3) A t-norm  $T$  is quasi-homogeneous if and only if it is a member of the family  $T_\alpha$  with  $0 \leq \alpha \leq +\infty$ , where

$$T_\alpha(x, y) = \begin{cases} (x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha} & \text{if } \min\{x, y\} > 0 \\ 0 & \text{otherwise} \end{cases}$$

for  $\alpha$  such that  $0 < \alpha < +\infty$ , and  $T_0 = T_{\mathbf{P}}$  is the product t-norm and  $T_{+\infty} = T_{\mathbf{M}}$  is the minimum t-norm.

Here,  $f_\alpha(\lambda) = \lambda^c$  with arbitrary  $c > 0$  for all  $\alpha \in [0, +\infty]$ , and the  $\varphi_\alpha$  are given by

$$\varphi_\alpha(x) = k(1 + x^\alpha)^{-c/\alpha}, \quad \text{for } 0 < \alpha < +\infty$$

and

$$\varphi_0(x) = kx^{c/2} \quad \text{and} \quad \varphi_{+\infty}(x) = kx^c.$$

### 3. QUASI-HOMOGENEOUS COPULAS

In this section we want to characterize quasi-homogeneous copulas, that is, those copulas that satisfy Definition 2. Firstly, let us deal with the easier generalization of homogeneity that consists in substituting  $\lambda^k$  by an arbitrary function  $f$ .

**Proposition 1.** Let  $f : [0, 1] \rightarrow [0, 1]$  be an arbitrary function and let  $C$  be a copula such that

$$C(\lambda x, \lambda y) = f(\lambda)C(x, y) \quad \text{for all } x, y, \lambda \in [0, 1].$$

Then  $f(\lambda) = \lambda^k$  with  $1 \leq k \leq 2$  and  $C$  is a member of the Cuadras–Augé family.

**Proof.** Taking  $x = y = 1$  we have  $C(\lambda, \lambda) = f(\lambda)$  for all  $\lambda \in [0, 1]$ , that is,  $f$  has to be the diagonal section of  $C$  and, in particular,  $f$  must be continuous with  $f(0) = 0$  and  $f(1) = 1$ . On the other hand,

$$f(\lambda x) = C(\lambda x, \lambda x) = f(\lambda)f(x)$$

for all  $\lambda, x \in [0, 1]$ . Consequently,  $f$  must be of the form  $f(\lambda) = \lambda^k$  for some  $k > 0$  (see for instance [1]). That is  $C$  must be homogeneous of degree  $k$  and hence the result. □

Thus, as for the case of t-norms, no new solutions appear for copulas with this generalization. On the contrary, for the quasi-homogeneity condition, we will see that there are a lot of copulas satisfying Equation (5). First let us characterize the structure of such copulas.

**Theorem 4.** A copula  $C$  is quasi-homogeneous if and only if its diagonal section is strictly increasing and  $C$  is given by

$$C(x, y) = \begin{cases} \delta \left( (x \vee y) \delta^{-1} \left( \frac{x \wedge y}{x \vee y} \right) \right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \tag{6}$$

In this case,  $C$  is  $(\varphi, f)$ -quasi-homogeneous with  $f(\lambda) = \lambda^c$  and  $\varphi(x) = (\delta^{-1}(x))^c$  for arbitrary  $c > 0$ .

*Proof.* Let us consider a copula  $C$  that verifies equation (5) for a continuous strictly monotonic function  $\varphi : [0, 1] \rightarrow \mathbb{R}$  and an arbitrary function  $f : [0, 1] \rightarrow [0, 1]$ . We can write

$$\varphi(C(x, x)) = f(x)\varphi(C(1, 1)) = f(x)\varphi(1)$$

for all  $x \in [0, 1]$ . It is clear that if  $C$  satisfies equation (5) for functions  $\varphi, f$  it also satisfies it for functions  $k\varphi, f$  with  $k \neq 0$ , and consequently we can suppose  $\varphi(1) = 1$ . Thus we have  $f(x) = \varphi(C(x, x)) = \varphi(\delta(x))$  where  $\delta$  is the diagonal section of  $C$ , and

$$f(xy) = \varphi(C(xy, xy)) = f(x)\varphi(C(y, y)) = f(x)f(y).$$

That is,  $f$  satisfies the multiplicative Cauchy equation and since  $\varphi$  is strictly monotonic this implies that also  $f$  is monotonic, and, hence that  $f(\lambda) = \lambda^c$  for all  $\lambda \in [0, 1]$  with  $c > 0$  (see again [1]). Thus  $C$  satisfies

$$\varphi(C(\lambda x, \lambda y)) = \lambda^c \varphi(C(x, y)) \quad \text{for all } x, y, \lambda \in [0, 1] \tag{7}$$

with  $c > 0$ . Now, taking  $x = y = 1$  we obtain  $\varphi(\delta(\lambda)) = \lambda^c$  which implies that  $\delta$  must be strictly increasing and that  $\varphi(x) = (\delta^{-1}(x))^c$  with  $c > 0$ .

Finally, equation (7) can be written now as

$$(\delta^{-1}(C(\lambda x, \lambda y)))^c = \lambda^c (\delta^{-1}(C(x, y)))^c$$

or equivalently

$$\delta^{-1}(C(\lambda x, \lambda y)) = \lambda \delta^{-1}(C(x, y))$$

for all  $x, y, \lambda \in [0, 1]$ . If we consider the function  $F : [0, 1]^2 \rightarrow [0, 1]$  defined by  $F(x, y) = \delta^{-1}(C(x, y))$  we obtain that  $F$  is homogeneous of degree 1. Moreover, whereas  $\max\{x, y\} > 0$  we can write

- if  $x \leq y$ 

$$F(x, y) = F\left(y \frac{x}{y}, y\right) = yF\left(\frac{x}{y}, 1\right) = y\delta^{-1}\left(C\left(\frac{x}{y}, 1\right)\right) = y\delta^{-1}\left(\frac{x}{y}\right)$$
- if  $y \leq x$ 

$$F(x, y) = F\left(x, x \frac{y}{x}\right) = xF\left(1, \frac{y}{x}\right) = x\delta^{-1}\left(C\left(1, \frac{y}{x}\right)\right) = x\delta^{-1}\left(\frac{y}{x}\right).$$

That is,  $F$  is given by

$$F(x, y) = \begin{cases} (x \vee y)\delta^{-1}\left(\frac{x \wedge y}{x \vee y}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

where  $\vee$  stands for maximum and  $\wedge$  for minimum. Thus  $C$  must be given by equation (6).

Reciprocally, if  $C$  is a copula given by equation (6) with diagonal section  $\delta$  strictly increasing then clearly  $C$  is quasi-homogeneous with functions  $f(\lambda) = \lambda^c$  and  $\varphi(x) = (\delta^{-1}(x))^c$  with  $c > 0$ . □

**Remark 1.** i) Due to special properties of copulas, it is possible to relax the requirements of continuity and strict monotonicity of the function  $\varphi$  in Definition 2 into the requirements that  $Rang(\varphi)$  contains at least three elements.

ii) Observe that a function  $S : [0, 1]^2 \rightarrow [0, 1]$  is called a semi-copula whenever it is non-decreasing in both coordinates and  $S(1, x) = S(x, 1) = x$  for all  $x \in [0, 1]$  (see [8]). A semi-copula  $Q : [0, 1]^2 \rightarrow [0, 1]$  is called a quasi-copula (see [8, 12, 13] or [15]) if it is 1-Lipschitz, i. e.,

$$|Q(x_1, y_1) - Q(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2| \quad \text{for all } x_1, x_2, y_1, y_2 \in [0, 1].$$

Note that Proposition 1 as well as Theorem 4 can be applied to continuous semi-copulas and quasi-copulas without any modification.

iii) Note also that all quasi-homogeneous copulas are symmetric in view of the representation (6).

From the previous theorem, it is clear that for finally characterizing quasi-homogeneous copulas we only need to find those admissible diagonals of copulas, that are strictly increasing and for which equation (6) effectively gives a copula. This will be done in next section, but in the more general case where the function  $\delta$  needs not to be strict.

#### 4. DIAGONAL SECTIONS OF QUASI-HOMOGENEOUS COPULAS

Given a copula  $C$  it is well known that its diagonal section is a function  $\delta : [0, 1] \rightarrow [0, 1]$  that satisfies:

- d1)  $\delta(x) \leq x$  for all  $x \in [0, 1]$  with  $\delta(0) = 0$  and  $\delta(1) = 1$ ,
- d2)  $\delta$  is non-decreasing,
- d3)  $\delta$  is 2-Lipschitz, i. e.  $|\delta(x) - \delta(y)| \leq 2|x - y|$  for all  $x, y \in [0, 1]$ .

Let us denote by  $\mathbf{D}$  the set of all functions  $\delta : [0, 1] \rightarrow [0, 1]$  that can be the diagonal section of a copula, that is, satisfying conditions from d1) to d3). In general, there are a lot of copulas with the same diagonal section  $\delta \in \mathbf{D}$ . In this sense many authors have studied, fixing a function  $\delta \in \mathbf{D}$ , how to construct a copula

$C$  with diagonal section  $\delta$ . This has been done in different manners and contexts (see [7, 10, 11] obtaining respectively Bertino copulas, diagonal copulas, MT-copulas and so on (see [5]).

Our interest now is to study in what cases a copula  $C$  can be obtained from its diagonal through equation (6). In fact, note that such equation can be generalized for diagonals  $\delta \in \mathbf{D}$  in general, not necessarily strictly increasing, by using the pseudo-inverse function  $\delta^{(-1)}$ . Specifically, given a non-decreasing function  $\delta : [0, 1] \rightarrow [0, 1]$  its pseudo-inverse  $\delta^{(-1)} : [0, 1] \rightarrow [0, 1]$  is given by (see [14])

$$\delta^{(-1)}(x) = \sup\{t \in [0, 1] \mid \delta(t) \leq x\} \quad \text{for all } x \in [0, 1]. \tag{8}$$

Now we can study when a copula  $C$  can be constructed from its diagonal through the expression

$$C_{(\delta)}(x, y) = \delta \left( (x \vee y) \delta^{(-1)} \left( \frac{x \wedge y}{x \vee y} \right) \right) \quad \text{for all } (x, y) \neq (0, 0) \tag{9}$$

and  $C(0, 0) = 0$ . This generalization is important because it will allow us to obtain many more representative examples. For instance the following one.

**Example 1.** The weakest copula  $W(x, y) = \max\{x + y - 1, 0\}$  has diagonal section  $\delta_W$  given by  $\delta_W(x) = \max\{2x - 1, 0\}$  and it can be constructed from  $\delta$  through equation (9). Note that however  $W$  is not quasi-homogeneous since its diagonal is not strictly increasing.

**Theorem 5.** Let  $\delta \in \mathbf{D}$ . If  $\delta$  is convex then the binary operation  $C_{(\delta)}$  given by equation (9) is a (commutative) copula with diagonal section  $\delta$ .

*Proof.* Evidently,  $C_{(\delta)}(x, 1) = C_{(\delta)}(1, x) = x, C_{(\delta)}(x, 0) = C_{(\delta)}(0, x) = 0$  and  $C_{(\delta)}(x, y) = C_{(\delta)}(y, x)$  for all  $x, y \in [0, 1]$ . Thus, the only thing to show  $C_{(\delta)}$  is a copula is its 2-increasingness. Denote

$$a = \sup\{x \in [0, 1] \mid \delta(x) = 0\},$$

since  $\delta$  is convex it must be strictly increasing on  $[a, 1]$ . We denote by  $d^{-1}$  the inverse of  $\delta : [a, 1] \rightarrow [0, 1]$ , then  $d^{-1} : [0, 1] \rightarrow [a, 1]$  is given by

$$d^{-1}(x) = \delta^{(-1)}(x) = \sup\{z \in [0, 1] \mid \delta(z) \leq x\},$$

where  $\delta^{(-1)}$  is the pseudo-inverse of  $\delta$  (see (8)) and  $C_{(\delta)}(x, y)$  can be written as

$$C_{(\delta)}(x, y) = \delta \left( (x \vee y) d^{-1} \left( \frac{x \wedge y}{x \vee y} \right) \right)$$

for all  $(x, y) \neq (0, 0)$ . It is easy to see that for  $y \leq x$ , it is

$$C_{(\delta)}(x, y) = 0 \quad \iff \quad y \leq x\delta \left( \frac{a}{x} \right) \quad \text{or} \quad x \leq a.$$

Moreover, for  $y < x$ ,  $C_{(\delta)}$  is non-decreasing in  $y$ , and this fact together with the symmetry of  $C_{(\delta)}$  reduces the cases for 2-increasingness to be checked for two cases:

- i) 2-increasingness on squares whose diagonal is on the main diagonal of  $[0, 1]^2$ ,
- ii) 2-increasingness on rectangles contained in the region where  $C_{(\delta)}$  is positive for  $y < x$ .

In the first case it should be shown that for  $0 \leq u < v \leq 1$  it holds

$$\delta(u) + \delta(v) - 2\delta\left(vd^{-1}\left(\frac{u}{v}\right)\right) \geq 0. \tag{10}$$

Since  $\delta \geq \delta_W$  we have

$$\delta\left(\frac{u+v}{2v}\right) \geq 2\frac{u+v}{2v} - 1 = \frac{u}{v}, \quad \text{that is,} \quad \frac{u+v}{2} \geq vd^{-1}\left(\frac{u}{v}\right),$$

and thus

$$v - vd^{-1}\left(\frac{u}{v}\right) \geq vd^{-1}\left(\frac{u}{v}\right) - u.$$

The convexity of  $\delta$  then ensures

$$\delta(v) - \delta\left(vd^{-1}\left(\frac{u}{v}\right)\right) \geq \delta\left(vd^{-1}\left(\frac{u}{v}\right)\right) - \delta(u)$$

which is equivalent to (10).

To prove case ii), one should show that for  $[u, u'] \times [v, v']$  in positive area of  $C_{(\delta)}$  and such that  $v' \leq u$ , it holds

$$\delta\left(ud^{-1}\left(\frac{v}{u}\right)\right) + \delta\left(u'd^{-1}\left(\frac{v'}{u'}\right)\right) - \delta\left(ud^{-1}\left(\frac{v'}{u}\right)\right) - \delta\left(u'd^{-1}\left(\frac{v}{u'}\right)\right) \geq 0 \tag{11}$$

Due to concavity of  $d^{-1}$ , the function  $h(x) = d^{-1}(x)/x$  is decreasing and it holds

$$\frac{d^{-1}\left(\frac{v'}{u'}\right) - d^{-1}\left(\frac{v}{u'}\right)}{\frac{v'-v}{u'}} \geq \frac{d^{-1}\left(\frac{v'}{u}\right) - d^{-1}\left(\frac{v}{u}\right)}{\frac{v'-v}{u}},$$

that is,

$$u' \left( d^{-1}\left(\frac{v'}{u'}\right) - d^{-1}\left(\frac{v}{u'}\right) \right) \geq u \left( d^{-1}\left(\frac{v'}{u}\right) - d^{-1}\left(\frac{v}{u}\right) \right).$$

Now, due to convexity of  $\delta$  and decreasingness of  $h$ , the last inequality reads

$$\delta\left(u'd^{-1}\left(\frac{v'}{u'}\right)\right) - \delta\left(u'd^{-1}\left(\frac{v}{u'}\right)\right) \geq \delta\left(ud^{-1}\left(\frac{v'}{u}\right)\right) - \delta\left(ud^{-1}\left(\frac{v}{u}\right)\right)$$

which is equivalent to (11). □

**Example 2.** Fix  $\alpha \in [0, 1/2]$  and let  $\delta_\alpha : [0, 1] \rightarrow [0, 1]$  be the convex function in  $D$  given by

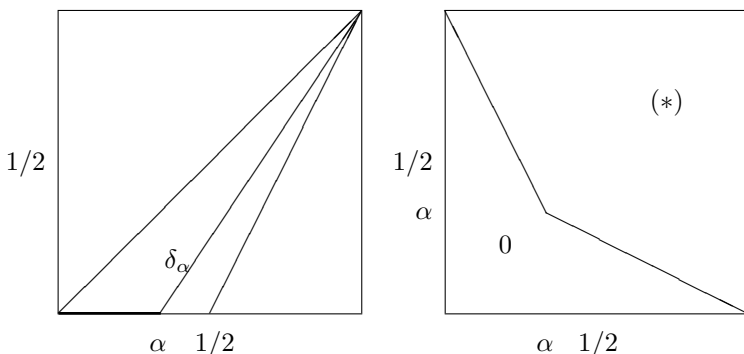
$$\delta_\alpha(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ \frac{x-\alpha}{1-\alpha} & \text{otherwise.} \end{cases}$$



By applying the previous theorem to these functions  $\delta_\alpha$ , we obtain a family of parametric copulas  $C_{(\delta_\alpha)}$  given by

$$C_{(\delta_\alpha)}(x, y) = \max \left\{ 0, \frac{\alpha(x \vee y) + (1 - \alpha)(x \wedge y) - \alpha}{1 - \alpha} \right\},$$

with boundary members  $C_{(\delta_0)} = M$  and  $C_{(\delta_{1/2})} = W$ . In Figure 1 we can see the parametric family of diagonals  $(\delta_\alpha)$  with  $\alpha \in [0, 1/2]$ , and the corresponding copulas  $C_{(\delta_\alpha)}$ .



**Fig 1.** Parametric family of diagonals  $(\delta_\alpha)$  (left) and copulas  $C_{(\delta_\alpha)}$  (right) of Example 2, where  $(*)$  stands for  $\frac{\alpha(x \vee y) + (1 - \alpha)(x \wedge y) - \alpha}{1 - \alpha}$ .

**Theorem 6.** Let  $\delta \in \mathbf{D}$  be strictly increasing. Then the binary operation  $C_{(\delta)}$  given by equation (6) is a (commutative) copula with diagonal section  $\delta$  if and only if  $\delta$  is convex.

*Proof.* From the previous theorem we only need to prove that when  $C_{(\delta)}$  is a copula then  $\delta$  must be convex. But if  $C_{(\delta)}$  is a copula (quasi-copula is enough) it is 1-Lipschitz, i. e.,

$$C_{(\delta)}(x, y_2) - C_{(\delta)}(x, y_1) \leq y_2 - y_1 \quad \text{for all } x, y_1, y_2 \text{ with } y_1 \leq y_2.$$

For  $z \in ]0, 1[$  and  $\varepsilon \in ]0, 1 - z[$  put

$$x = \frac{z}{z + \varepsilon}, \quad y_1 = \frac{z\delta(z)}{z + \varepsilon}, \quad y_2 = \frac{z\delta(z + \varepsilon)}{z + \varepsilon}.$$

Then, since  $\delta$  is strictly increasing we have  $\delta^{(-1)} = \delta^{-1}$ , and thus

$$C_{(\delta)}(x, y_2) - C_{(\delta)}(x, y_1) = \delta(z) - \delta \left( \frac{z^2}{z + \varepsilon} \right) \leq y_2 - y_1 = \frac{\delta(z + \varepsilon) - \delta(z)}{\frac{z + \varepsilon}{z}}.$$

Note that  $\frac{z^2}{z+\varepsilon} = z - \varepsilon \frac{z}{z+\varepsilon}$  and thus the equation above can be written as

$$\frac{\delta(z) - \delta\left(z - \varepsilon \frac{z}{z+\varepsilon}\right)}{\varepsilon \frac{z}{z+\varepsilon}} \leq \frac{\delta(z + \varepsilon) - \delta(z)}{\varepsilon}. \tag{12}$$

Finally, since  $\delta$  is 2-Lipschitz, it has continuous derivative on a union of open subintervals of  $[0, 1]$  of the form  $\cup_{k \in K} a_k, b_k[$  with  $\sum_{k \in K} (b_k - a_k) = 1$ . This fact together with equation (12) implies the convexity of  $\delta$ .  $\square$

**Corollary 1.** Let  $C : [0, 1]^2 \rightarrow [0, 1]$  be a binary operation with continuous diagonal  $\delta(x) = C(x, x)$ . Then  $C$  is a quasi-homogeneous copula if and only if  $\delta$  is a strictly increasing convex function and  $C$  is given by equation (6).

**Example 3.** Fix  $k \in [0, 1[$  and  $\alpha$  such that  $\max\{0, 2k - 1\} \leq \alpha \leq k$ . Let  $\delta_{k,\alpha} : [0, 1] \rightarrow [0, 1]$  be the convex strictly increasing function in  $\mathbf{D}$  given by

$$\delta_{k,\alpha}(x) = \begin{cases} \frac{\alpha x}{k} & \text{if } x \leq k \\ \frac{\alpha-1}{k-1}x + \frac{k-\alpha}{k-1} & \text{otherwise.} \end{cases}$$

By applying Theorem 6 to these functions  $\delta_{k,\alpha}$ , we obtain a family of two-parametric quasi-homogeneous copulas  $C_{(\delta_{k,\alpha})}$  given by  $C_{(\delta_{k,\alpha})}(x, y) =$

$$\begin{cases} x \wedge y & \text{if } (x \vee y) \geq \alpha(x \wedge y) \\ x \wedge y + \frac{\alpha-k}{k-1}(x \vee y) + \frac{k-\alpha}{k-1} & \text{if } \frac{(k-1)(x \wedge y) + (\alpha-k)(x \vee y)}{\alpha-1} \leq k \\ \frac{\alpha}{k(\alpha-1)}((k-1)(x \wedge y) + (\alpha-k)(x \vee y)) & \text{otherwise} \end{cases}$$

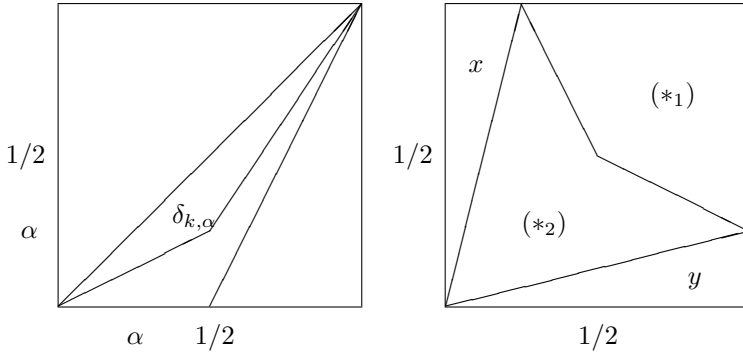
with boundary member  $C_{(\delta_{0,0})} = M$  and whose limit when  $k \rightarrow 1$  is given by the weakest copula  $W$ . This parametric family for the case  $k = 1/2$  and  $0 \leq \alpha \leq 1/2$  can be viewed in Figure 2.

**Remark 2.** In view of the fact that the class of all diagonal sections of copulas coincides with the class of all diagonal sections of quasi-copulas, it can be shown that there are no proper quasi-homogeneous quasi-copulas, i. e., each quasi-homogeneous quasi-copula is necessarily a copula. On the other hand, a continuous semi-copula  $S$  is quasi-homogeneous and given by (6) if and only if its diagonal section  $\delta : [0, 1] \rightarrow [0, 1]$  given by  $\delta(x) = S(x, x)$  is an automorphism of  $[0, 1]$  such that the function  $h : ]0, 1] \rightarrow [0, 1]$  given by  $h(x) = \delta(x)/x$  is non-decreasing. Put, for example,  $\delta(x) = x^c$  with  $c > 0$ . Then  $h(x) = x^{c-1}$  is non-decreasing for  $c \geq 1$ , and the corresponding semi-copula  $S$  is given by

$$S(x, y) = (x \wedge y)(x \vee y)^{c-1} \quad \text{for all } x, y \in [0, 1].$$

Recall that  $S$  is a copula (quasi-copula) only if  $\delta$  is 2-Lipschitz, i. e., if  $c \in [1, 2]$  (and then it belongs to Cuadras–Augé family).

Similarly,  $\delta(x) = \max\{x/3, 3x - 2\}$  is an automorphism of  $[0, 1]$  such that  $h(x) = \max\{1/3, 3 - 2/x\}$  is increasing. The corresponding semi-copula  $S$  is given on the triangle determined by points  $(1, 1/4)$ ,  $(1, 1)$  and  $(3/4, 3/4)$  by  $S(x, y) = y + 2x - 2$  and thus it is not a copula (quasi-copula).



**Fig. 2.** Parametric family of diagonals  $(\delta_{k,\alpha})$  (left) and copulas  $C_{(\delta_{k,\alpha})}$  (right) of Example 3 with  $k = 1/2$ , where  $(*_1)$  stands for  $x \wedge y + (1 - 2\alpha)(x \vee y) + 2\alpha - 1$  and  $(*_2)$  stands for  $\frac{\alpha}{1-\alpha}((x \wedge y) + (1 - 2\alpha)(x \vee y))$ .

5. CONCLUDING REMARKS

We have completely solved the problem of representing quasi-homogeneous copulas by means of their diagonal sections. Moreover, a new method of constructing copulas from convex diagonal sections was introduced. Recall that there are several methods of constructing a copula when a diagonal section  $\delta : [0, 1] \rightarrow [0, 1]$  (non-decreasing, 2-Lipschitz, bounded from above by the identity function and  $\delta(1) = 1$ ) is given. The weakest copula  $C_{[\delta]} : [0, 1]^2 \rightarrow [0, 1]$  such that  $C_{[\delta]}(x, x) = \delta(x)$  is the so-called Bertino copula given by

$$C_{[\delta]}(x, y) = (x \wedge y) - \min\{t - \delta(t) \mid t \in [x \wedge y, x \vee y]\}$$

see [3, 11], or [13]. On the other hand, the diagonal copula  $C_\delta : [0, 1]^2 \rightarrow [0, 1]$  introduced in [10] and given by

$$C_\delta(x, y) = \min \left\{ x, y, \frac{\delta(x) + \delta(y)}{2} \right\}$$

is the strongest symmetric copula satisfying  $C_\delta(x, x) = \delta(x)$  (but not necessarily the strongest copula with diagonal section  $\delta$ ).

Other methods known from the literature, see, e. g., [4, 5] or [7], are restricted to special classes of diagonal sections. Observe that in the case of our construction method (restricted to convex diagonal sections), the only diagonal copula  $C_\delta$  coinciding with  $C_{(\delta)}$  is the strongest copula  $M$  ( $\delta = id$  is the only diagonal section related to the unique copula  $C = M$ ). On the other hand, the only Bertino copulas which can be obtained by our construction are related to a parametric class  $(\delta_a)_{a \in [0, 1/2]}$  of diagonal sections given by

$$\delta_a(x) = \max \left( 0, \frac{x - a}{1 - a} \right)$$

and the corresponding copulas  $C_{[\delta_a]} = C_{(\delta_a)} : [0, 1]^2 \rightarrow [0, 1]$  are given by

$$C_{(\delta_a)}(x, y) = \max \left\{ 0, (x \wedge y) + \frac{a}{1-a}((x \vee y) - 1) \right\}$$

with boundary members  $C_{(\delta_0)} = M$  and  $C_{(\delta_{1/2})} = W$ .

For any copula  $C : [0, 1]^2 \rightarrow [0, 1]$  and a diagonal section  $\delta \in \mathbf{D}$ , the function  $C^{(\delta)} : [0, 1]^2 \rightarrow [0, 1]$  given by

$$C^{(\delta)}(x, y) = C \left( \delta(x \vee y), \frac{x \wedge y}{x \vee y} \right) \quad \text{for all } x, y \in [0, 1] \tag{13}$$

(with the convention  $0/0 = 1$ ) is a function fulfilling the boundary properties of copulas, and  $C^{(\delta)}(x, x) = \delta(x)$ . Note that the same is satisfied if we alternatively take the function

$$C^{(\delta)'}(x, y) = C \left( \frac{x \wedge y}{x \vee y}, \delta(x \vee y) \right) \quad \text{for all } x, y \in [0, 1].$$

It is an interesting open problem for which  $C$  and  $\delta$  also  $C^{(\delta)}$  (or  $C^{(\delta)'}$ ) is a copula. For the product copula  $\Pi$ , (13) can be written as

$$\Pi^{(\delta)}(x, y) = \delta(x \vee y) \frac{x \wedge y}{x \vee y} \quad \text{for all } x, y \in [0, 1] \tag{14}$$

and the complete characterization of all copulas having the form  $\Pi^{(\delta)}$  can be found in [6], where these copulas are called semilinear. Our representation of quasi-homogeneous copulas also contributes to the above mentioned open problem. Indeed, let  $\delta \in \mathbf{D}$  be a convex strictly increasing diagonal section. Then  $\delta^{-1}$  is concave and it is a multiplicative generator of a copula  $D_\delta : [0, 1]^2 \rightarrow [0, 1]$ ,  $D_\delta(x, y) = \delta(\delta^{-1}(x)\delta^{-1}(y))$ . However, then

$$D_\delta^{(\delta)}(x, y) = D_\delta \left( \delta(x \vee y), \frac{x \wedge y}{x \vee y} \right) = \delta \left( (x \vee y)\delta^{-1} \left( \frac{x \wedge y}{x \vee y} \right) \right) = C^{(\delta)}(x, y)$$

is a quasi-homogeneous copula (see Theorem 6).

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*Gaspar Mayor, Department of Mathematics and Computer Science, University of the Balearic Islands, 07122 Palma de Mallorca. Spain.  
e-mail: gmayor@uib.es*

*Radko Mesiar, Department of Mathematics and Descriptive Geometry, Slovak University of Technology, SK-813 68 Bratislava, Slovak Republic and Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8, Czech Republic.  
e-mail: mesiar@math.sk*

*Joan Torrens, Department of Mathematics and Computer Science, University of the Balearic Islands, 07122 Palma de Mallorca. Spain.  
e-mail: dmijts0@uib.es*