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NEW OPERATIONS ON PARTIAL ABELIAN MONOIDS DEFINED BY PREIDEALS

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We consider partial abelian monoids, in particular generalized effect algebras. From the given structures, we construct new ones by introducing a new operation $\oplus$, which is given by restriction of the original partial operation $+$ with respect to a special subset called *preideal*. We bring some derived properties and characterizations of these new built structures, supporting the results by illustrative examples.

Keywords: partial abelian monoid, generalized effect algebra, preideal, Riesz decomposition property, central element

AMS Subject Classification: 81P10, 08A55

1. INTRODUCTION

Effect algebras were introduced in [2] as a system which is an algebraic abstraction of inner properties of the set $\mathcal{E}(H)$ of Hilbert space effects, i. e., self-adjoint operators on a Hilbert space lying between the zero and the identity operator (with respect to usual ordering of self-adjoint operators), which play an important role in quantum measurement theory. Analogously as in the theory of rings or lattices, also the “unbounded versions” (which means the structure does not have the top element) of effect algebras were considered and called *generalized effect algebras*. Generalized effect algebras are equivalent also to positive cancellative *partial abelian monoids* (PAMs). Their importance is given by the fact that every generalized effect algebra can be embedded into an effect algebra (again in analogy with ring theory) as its ideal with half cardinality in unique way (see Remark 1.4). Relations between generalized effect algebras and these effect algebras, concerning Riesz ideals, are studied e. g. in [4], along with questions of preserving some properties within quotient algebras. These ideas follow up, in many lines, those from [5].

In this paper we consider PAMs, but in particular generalized effect algebras and construct new structures based on them by defining a new operation. The operation is induced by a special subset of the PAM. This method was first used in [4, Example 6.3], where we described a generalized effect algebra based on the set $\mathbb{N}_0$ of natural numbers with zero, where the new partial operation $\oplus$ is defined by restriction of the usual addition $+$ to pairs of summands at least one of which is even (Example 4.1). A straightforward generalization is obtained by restricting $+$ to pairs of summands at least one of which is a multiple of $n, n \in \mathbb{N}$. In this way, we obtained new interesting examples of generalized effect algebras which are even lattice ordered, and their Riesz ideals. In what follows, we will generalize these examples.

Definition 1.1. Partial Abelian Monoid is an algebraic structure $(E, +, 0)$ where E is a nonempty set, $0 \in E$ and $+$ is a partial operation such that

(P1) if $a + b$ exists, then $b + a$ exists and $a + b = b + a$,

(P2) if $a+b$ and $(a+b)+c$ exist, then $b+c$, $a+(b+c)$ exist and $a+(b+c) = (a+b)+c$,

(P3) for all $a \in E$, $a + 0$ exists and $a + 0 = a$.

Definition 1.2. Generalized Effect Algebra (GEA) is a PAM which is moreover cancellative and positive, that is

(P4) $a + c = b + c$ implies $a = b$,

(P5) $a + b = 0$ implies $a = b = 0$.

We note that on a GEA, a partial order can be introduced by defining $a \le b$ if there is a c such that $a + c = b$. We may further introduce another partial operation $\ominus$: $a \ominus b$ is defined iff $b \le a$ and $a \ominus b = c$ iff $b \oplus c = a$.

Definition 1.3. An Effect Algebra (EA) $(E, +, 0, 1)$ is a GEA with the greatest element, equivalently a GEA where the following condition holds

(P6) to every $a \in E$ there is $b \in E$ with $a + b = 1$.

Let us have a PAM E and its subset I. We will define a new operation $\oplus$ as follows

(P) $a, b \in E : a \oplus b$ exists $\Leftrightarrow (a \in I \text{ } or \text{ } b \in I)$ and $(a + b$ exists)

in which case, as expected, $a \oplus b := a + b$. Under this definition, we get a new partial structure $(E, \oplus, 0)$ and in the next section we find conditions on I under which E is again a PAM or a GEA.

Remark 1.4. The pattern that was used in the definition of $\oplus$ is similar to that used in one of the well known constructions. As follows from [3], every generalized effect algebra E can be embedded as an ideal into a unique effect algebra E' which is then called *unitization*. An effect algebra E' is unitization of a GEA E iff E is an ideal in E' such that for all $a \in E'$, either $a \in E$ or $a' \in E$. Then $a + b$ is defined in E' only if at least one of a, b is in E. If we put $I = E$, and define $\oplus$ as in (P), we obtain $\oplus = +$.

2. GEA CONDITIONS

In what follows, unless stated otherwise, E (or $(E,+,0)$) will stay for a general GEA. Now let I be a nonempty subset of E without any other requirements. Let us recall that a subset J of E is an *ideal* if (a) $a \in J$, $b \leq a$ implies $b \in J$, and (b) $a, b \in J$ and $a + b$ is defined implies $a + b \in J$.

Proposition 2.1. When $(E,+,0)$ is a GEA, I is a subset of E with $\oplus$ defined by (P), such that

(i) $0 \in I$,

(ii) $a, b \in I$ and $a + b$ exists imply $a \oplus b \in I$,

(iii) $a, a \oplus b \in I$ implies $b \in I$;

then $(E, \oplus, 0)$ is again a GEA.

Proof. The proof is easily done by verifying the axioms (P1) – (P5) of GEA. We check here only (P2), which is the most interesting:

If $a \oplus b$ and $(a \oplus b) \oplus c$ exist, then at least one of a, b is from I and either $a \oplus b$ is in I or c is in I. But then by (iii), either both of a, b are from I or c and one of a, b are in I. Then (ii) implies existence of $b \oplus c$ and $a \oplus (b \oplus c)$. The equality $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ is then clear. □

Remark 2.2. a) It can be easily seen that I with the properties (i) – (iii) is an ideal in $(E, \oplus, 0)$, but in $(E,+,0)$ it is just a special set, which we will call next in the article *preideal*. It is also clear that the symbol $\oplus$ may be replaced by $+$ in the definition of preideal.

b) Although the statement of the last preposition has the form of implication, it holds also in the opposite direction, if the operation $+$ is total, that is, $a + b$ does exist for all $a, b \in E$ (the GEA is then a monoid). If $(E, \oplus, 0)$ is a GEA, then $0 \in I$, because of (P3). Further, $a, b \in I$ implies that for every $c \in E$ $a \oplus (b \oplus c)$ exists, so that by (P2) also $(a \oplus b) \oplus c$ does exist and while c was arbitrary, $a \oplus b$ must be from I. Similarly $a, a \oplus b \in I \Rightarrow \forall c \in E : (a \oplus b) \oplus c$ exists. Therefore $a \oplus (b \oplus c)$ exists, so $b \oplus c$ exists. Thus $b \in I$ and I is a preideal of $(E,+,0)$.

Notice that the requirements of (i) – (iii) are necessary and sufficient for $(E, \oplus, 0)$ be a PAM. So when E is a PAM that is not a GEA, any subset I that satisfies these conditions determines a PAM $(E, \oplus, 0)$ that need not be a GEA. But we may be further interested when it will satisfy the remaining conditions of GEAs.

Proposition 2.3. Let E be a PAM which need not be positive or cancellative and I be its preideal. Then the PAM $(E, \oplus, 0)$ is:

a) positive iff E is positive on I (that is: $a, b \in I$ and $a + b = 0$ imply $a = b = 0$),

b) cancellative iff $[(a \in I \text{ or } b, c \in I) \text{ and } (a + b = a + c)] \Rightarrow \ b = c$.

P r o o f . The introduced conditions are obviously necessary and sufficient for (P4) and (P5) to hold. □

3. SOME OTHER PROPERTIES

Now we are going to deal with some properties that are usually studied when working with GEAs. Namely, we investigate RDP conditions, atoms, central elements and compatibility. Some of these properties are appropriate to find whether they are preserved from $(E, +, 0)$ into $(E, \oplus, 0)$, the others are to show under which circumstances they comply with other conditions.

One of the interesting properties that are investigated in GEAs is the Riesz decomposition property (RDP). This notion was originally utilized for partially ordered abelian groups and then adapted for (generalized) effect algebras ([1, Def. 1.7.4]).

Definition 3.1. A generalized effect algebra E is said to have the Riesz decomposition property when

$$\forall\, a, b, c \in E; a \leq b + c \;\Rightarrow\; \exists\, b_1, c_1 \in E : b_1 \leq b, c_1 \leq c,\; a = b_1 + c_1.$$

Remark 3.2. At this place it will be useful to establish this notation: if we write $a \leq_1 b$, it will mean that there exists an element a_1 such that $a + a_1 = b$. On the other hand, $a \leq_2 b$ will mean the order determined by $\oplus$ (thus $a \leq_2 b$ iff there exists a_2 such that $a \oplus a_2 = b$).
In a general and symmetrical case, if we assume that $a \oplus b$ exists (hence at least one of these elements is from a preideal I), we will assume that the element which is certainly from I, is a; unless said otherwise.

At first, it is quite obvious that in the case the subset I is an ideal of $(E, +, 0)$, RDP is preserved in $(E, \oplus, 0)$. To confirm this, consider $a \leq_2 b \oplus c$. Then $a \leq_1 b + c$, thus $a = b_1 + c_1$ for some $b_1 \leq_1 b$, $c_1 \leq_1 c$. But while b is from I, $b_1 \in I$ and therefore $a = b_1 \oplus c_1$.

Now we show, that I need not be an ideal and $(E, +, 0)$ need not satisfy RDP and still $(E, \oplus, 0)$ will have RDP when I is linearly ordered.

Theorem 3.3. Let $(E, +, 0)$ be a GEA and I be its linearly ordered preideal. Then $(E, \oplus, 0)$ is a GEA with RDP.

P r o o f . Let $a \leq_2 b \oplus c$, so that $b \in I$. Then there is an a_2 such that $a \oplus a_2 = b \oplus c$ and there are two possibilities – either a or a_2 are from I.

a) When $a \in I$ then we have, due to linear ordering in I, $a \leq_1 b$ or $b \leq_1 a$. Then of course we also have $a \leq_2 b$ or $b \leq_2 a$. But if $a \leq_2 b$ then we may write a as $a \oplus 0$ and if $b \leq_2 a$ then we have some b_2 with $b \oplus b_2 = a$ and $b_2 \leq_2 c$ $(b \oplus b_2 \oplus a_2 = b \oplus c)$ so RDP is satisfied in both cases.

b) When $a_2 \in I$, then again by linearity of I, $a_2 \leq_2 b$ or $b \leq_2 a_2$. From the first inequality we get an element a_3 for which $a_2 \oplus a_3 = b$, therefore $a \oplus a_2 = a_2 \oplus a_3 \oplus c$ and by cancellativity $a = a_3 \oplus c$ (and $a_3 \leq_2 b$). From the second inequality, stepwise $b \oplus b_2 = a_2$ for some b_2, $a \oplus b \oplus b_2 = b \oplus c$ and $a \oplus b_2 = c$. Thus $a = (c \ominus b_2) \oplus 0$. Since $c \ominus b_2 \leq_2 c$ and $0 \leq_2 b$, the proof is finished. □

Let us just briefly recall that an atom in a lattice (in a GEA, PAM etc.) is an element a which covers zero, that is $a > 0$ and $0 \leq b \leq a \Rightarrow b = 0$ or $b = a$.

Theorem 3.4. Atoms in $(E, \oplus, 0)$ are exactly those elements $a \in E$, for which at least one of the following conditions holds:

a) a is an atom in $(E, +, 0)$,

b) $\forall b \in E,\ b \neq 0 : b <_1 a \Rightarrow b \notin I$.

In other words, all atoms are preserved and moreover, all elements under which there are no nonzero preideal elements add up to atoms in $(E, \oplus, 0)$ and no others do.

Proof. At first, it is obvious that if a is an atom in $(E, +, 0)$, there can be no x, which is under a ($\leq_2$), therefore a is an atom in $(E, \oplus, 0)$, too. Now if there is an x such that $0 <_1 x <_1 a$, a is atom in $(E, \oplus, 0)$ if and only if for all such elements x, $x \not\leq_2 a$. This holds if and only if for every such x, $x \notin I$. □

Definition 3.5. An element a in a GEA $(E, +, 0)$ is called a central element, if it satisfies the following conditions:

(i) $\forall b \in E\ \exists x, y \in E : b = x + y, x \leq a, y \perp a$,

(ii) $\forall x, y \in E : x \leq a, y \leq a, x \perp y \Rightarrow x + y \leq a$,

(iii) $\forall x, y \in E : x \perp a, y \perp a, x \perp y \Rightarrow \exists x + y + a$.

Proposition 3.6. If a is a central element in $(E, +, 0)$, $a \in I$, and all $x \leq_1 a$ satisfy $x \in I$, then a is central also in $(E, \oplus, 0)$.

Proof. If a is central in E, we have $b = x + y$ and $x \leq_1 a$, $y \perp_1 a$ for every $b \in E$. But then $x \leq_2 a$ because $x \in I$ and $y \perp_2 a$ because $a \in I$. Thus $b = x \oplus y$ and (i) is fulfilled. If $x, y \leq_2 a$ and $x \perp_2 y$, then of course $x, y \leq_1 a$ and $x \perp_1 y$, and we already have $x \oplus y \leq_2 a$, as x, y must be from I. On the other hand, when a and at least one of x, y are from I and they are orthogonal in $(E, +, 0)$, the element $x \oplus y \oplus a$ exists, what is the end of proof. □

Unfortunately, there does not seem to be any other general condition for preserving or not preserving central elements from one GEA to another. But we may check that some new central elements can arise. At first, if we want an element a to be central in $(E, \oplus, 0)$, we see that (i) condition must be satisfied also in $(E, +, 0)$.

Indeed, if $\forall\, b \in E : b = x \oplus y$ and $x \leq_2 a, y \perp_2 a$, then it all remains also when 1 replaces 2. But when (ii) holds for 2, it need not hold for 1 and also the third condition (iii) may fail for expressions with index 1, holding for 2. Examples of these are in the next section.

The last attribute that we are going to analyze is compatibility.

Definition 3.7. We say that elements a, b are (Mackey) compatible ($a \leftrightarrow b$) in $(E, +, 0)$, if $\exists\, x, y, z \in E : a = x + y, b = x + z$ and $x + y + z$ exists.

Remark 3.8. In a GEA with RDP it is easy to prove that any two elements with a common upper bound are compatible, or in other words – in every upwards directed GEA, RDP implies that all elements are compatible ([1, p. 63]).

We will continue with the notation $a \leftrightarrow_1 b$ and $a \leftrightarrow_2 b$ when a, b are compatible in $(E, +, 0)$ or in $(E, \oplus, 0)$, respectively. For example, if I is an ideal, then we have this statement: if at least one of a, b is from I, then $a \leftrightarrow_1 b \Rightarrow a \leftrightarrow_2 b$. It is because in this case $x, y \in I$ or $x, z \in I$, so that $x \oplus y \oplus z$ does exist. If neither a nor b are in I and $a \leftrightarrow_1 b$, then $a \leftrightarrow_2 b$ if and only if $y, z \in I$.

The next proposition deals with compatibility in the case the operation $+$ is total. We say that the subset $A \subset E$ is a *compatible center* of E if $\forall\, a \in A,\ \forall\, b \in E : a \leftrightarrow b$.

Proposition 3.9. Let $(E, +, 0)$ be a GEA with total $+$ operation and with linearly ordered preideal I. Then the compatible center of $(E, \oplus, 0)$ contains I.

Proof. Let $a \in I$. Then for every $b \in E$ we have that $0, a, b$ is a suitable triplet for the decomposition because $a = 0 \oplus a$, $b = 0 \oplus b$ and $0 \oplus a \oplus b$ exists (as $0, a \in I$). □

On the other hand, let us take $a \notin I$ and b which is neither from I and moreover such that a, b are not comparable in $(E, \oplus, 0)$. Then a and b are not compatible ($a \not\leftrightarrow_2 b$). Indeed, if $a \leftrightarrow_2 b$, there had to exist x, y, z, where $a = x \oplus y$, $b = x \oplus z$ and $x \notin I$ and $y, z \in I$. But then a and b were comparable, because y, z are comparable.

Example 3.10. Let us consider the set $E = \mathbb{Z} \times \mathbb{Z}$. Then $(E, +, (0, 0))$ is a GEA with total operation defined coordinatewise, such that E is not linearly ordered. We set $I = 0 \times \mathbb{Z}$, so that I is a linearly ordered preideal of E. It can be seen how the compatibility works here. An element $(0, a)$ is compatible to every element (x, y) from E while for every element (b, a) where $b \neq 0$ and an element (x, y), $x \neq 0$, $x \neq b$ there is no possible decomposition in terms of $\oplus$.

The next section is devoted only to other examples which illuminate the notions from the previous text and demonstrate the concrete validity of the proved theorems.

4. EXAMPLES

Example 4.1. Let us consider the set of natural numbers $\mathbb{N}_0$ and its subset $I = 2k$ of even numbers. A partial operation $\oplus$ is defined by (P). We see that I is a linearly ordered preideal (in the new GEA it is even a so called Riesz ideal, see [4]), and therefore by the Theorem 3.3, $(\mathbb{N}_0, \oplus, 0)$ is a GEA with RDP. At first, in Figure 1 there is a structure of this GEA. It can be seen that it is a lattice. It is evident there are two atoms, 1 and 2, 1 inherited from $(\mathbb{N}_0, +, 0)$ and 2 arises only in the new GEA (according to b) in Theorem 3.4). As for central elements, this is a good example for showing that 1 is central in $(\mathbb{N}_0, \oplus, 0)$ although it is not in $(\mathbb{N}_0, +, 0)$. Indeed, let us set $x = 1, y = 1$ in Definition 3.5, (ii). Then we see that in the chain of $\mathbb{N}_0$ evidently $1 + 1 \not\leq_1 1$, while in the GEA $1 \not\leq_2 1$ and therefore the condition is satisfied. The remaining part of the proof can also be done easily. Since this GEA is upwards directed, by the Remark 3.8 every two elements are compatible.

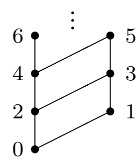

Fig. 1.

Example 4.2. Now we may modify the previous example just by taking another I, $I = nk$ with $n \in \{3, 4 \ldots\}$. The schematic picture for $n = 3$ is in Figure 2. In this case, I is again a preideal and $(\mathbb{N}_0, \oplus, 0)$ is a GEA with RDP, which is a lower semilattice but not a lattice (e. g., there is no common upper bound for 1 and 2). There are no central elements here except zero. Indeed, if we read Definition 3.5, no preideal element can be central because of (ii) and no other element because of (i). E.g., 1 is not central because there is no chance to decompose 5 ($5 = 0 \oplus 5 = 3 \oplus 2$, but $5 \not\leq_2 1$ and neither 3 nor 2 are under 1, even with respect to $\leq_1$); similarly, 2 is not a central element, e. g. because we cannot decompose 4, etc. We see that this structure is not upwards directed and not any two elements are compatible; by Proposition 3.9, I is a compatible center, thus the set $\{(a, b) : a = 3k+1, b = 3k+2, k \in \mathbb{N}_0\}$ expresses the incompatible pairs of elements.

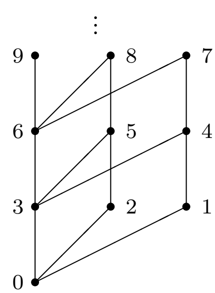

Fig. 2.

Example 4.3. A follow-up numerical example, infinite this time, can be obtained considering the real semiclosed unit interval $E = [0, 1)$ and its rationals $I = \mathbb{Q} \cap [0, 1)$. Using Theorem 3.4, one can easily see that there are no atoms in $(E, \oplus, 0)$. There are also no nontrivial central elements, neither for $+$, nor for $\oplus$ structure; we show it using an argument based on (ii) of the definition. Let $a \in (0, 1)$ be a central element. Then we can choose a real number $a_1 > 0$ such that $a_1 <_1 \min\{\frac{a}{2}, \frac{1-a}{2}\}$ and $\frac{a}{2} + a_1 \in \mathbb{Q}$. Then if we set $x = y := \frac{a}{2} + a_1$, we obtain $x, y \leq_2 a$ and $x \perp_2 y$, however $(\frac{a}{2} + a_1) \oplus (\frac{a}{2} + a_1) \not\leq_2 a$.

We note that yet another infinite example can be obtained when we take $E = \mathbb{R}^+$ and set $I = \mathbb{N} \subset \mathbb{R}$. Generalized effect algebra $(E, \oplus, 0)$ will have in this case a similar structure as in the last two examples, but with an infinite number of "branches".

Example 4.4. Now we consider the set of real functions with the values in the closed unit interval ($E = [0, 1]^{\mathbb{R}}$) and its subset $I = \chi$ of characteristic functions. The order of functions in E is accepted to be pointwise. Then I is a preideal of E which is not linearly ordered, so we cannot decide if $(E, \oplus, 0)$ has RDP by the Theorem 3.3. Anyway, we show that this GEA possesses RDP. Indeed, if $f \leq_2 g \oplus h$, then $g = \chi_A$ for some set A and either $f = \chi_B$ for some set B (which is in general independent of A) or f is of this form:

$$f(x) = \begin{cases} h(x) & \text{if } g(x) = 0 \text{ and } h(x) \neq 1 \\ 0 \text{ or } 1 & \text{if } g(x) = 0 \text{ and } h(x) = 1 \\ 0 \text{ or } 1 & \text{if } g(x) = 1 \end{cases}$$

In both cases, we set functions g_1, h_1 like this:

$$g_1(x) = \begin{cases} f(x) & \text{if } g(x) = 1 \\ 0 & \text{if } g(x) = 0 \end{cases}$$

$$h_1(x) = \begin{cases} 0 & \text{if } g(x) = 1 \\ f(x) & \text{if } g(x) = 0 \end{cases}$$

and we have $g_1 \leq_2 g, h_1 \leq_2 h$, and $f = g_1 \oplus h_1$, as required.

There are evidently no atoms in $(E, +, 0)$, but by b) from Theorem 3.4, every characteristic function f with only one point x such that $f(x) = 1$, is an atom in $(E, \oplus, 0)$, which is easily seen.

On the other hand, every characteristic function is a central element in $(E, +, 0)$, but there are no central elements in $(E, \oplus, 0)$ (except the zero function). Indeed, having any characteristic function c, every function f can be decomposed into the sum of two functions g and h such that:

$$g(x) = \begin{cases} f(x) & \text{if } c(x) = 1 \\ 0 & \text{if } c(x) = 0 \end{cases}$$

$$h(x) = \begin{cases} 0 & \text{if } c(x) = 1 \\ f(x) & \text{if } c(x) = 0 \end{cases}$$

Then we have $f = g + h$, $g \leq_1 c$ and $h \perp_1 c$. Properties (ii) and (iii) can be verified similarly. Moreover, because of (ii), there can be no other function central except characteristic ones.

The statement that there are no central elements in $(E, \oplus, 0)$ needs a bit of proof, which we do by contradiction on the first condition ((i) in Definition 3.5), while conditions (ii) and (iii) are satisfied for all functions. Let c be a central element. We are going to find a function f which we cannot decompose into $f = g \oplus h$ such that $g \leq_2 c$ and $h \perp_2 c$.

We consider two cases, according to if there are some points x where $c(x) = 0$, or there are none. In the first case we proceed as follows: let us choose the point x_1, $c(x_1) = 0$ and a function f such that $f(x_1) \neq 0, 1$. While $g \leq_2 c$, $g(x_1)$ must be 0 and thus $h(x_1) = f(x_1) \neq 0, 1$. Therefore g is that of the two functions which is characteristic ($g \oplus h$ exists). From this fact it follows that c must be characteristic, too (for $h \perp_2 c$). Let us now take a point x_2, $c(x_2) = 1$ (if there is not such a point, c is the zero function, which we are not interested in now) and let $f(x_2) \neq 0, 1$. Then $g(x_2)$ has to be 0 because $f(x_2) = g(x_2) \oplus h(x_2)$ and so $h(x_2) = f(x_2)$. But this contradicts $h(x_2) \perp_2 c(x_2)$.

Now let there be no x in which $c(x) = 0$. Then surely c is not in I (the only one could be the identically 1 function, which is evidently not a central element) and so $h \in I$ because $h \perp_2 c$. But then it must be constantly 0, since c is everywhere different from 0. Therefore $f = g$ and $g \leq_2 c$. So that if we consider any function f which is not under c, there is no way to decompose it by c and that is the end of proof.

Remark 4.5. Similar results as in the last example may be obtained considering the set of Hilbert space effects (self-adjoint operators between the zero and the identity operator) with its subset consisting of projectors. For such selection of E and I we get a GEA where atoms are one-dimensional projections. The other attributes and details of this example are left to an active reader.

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