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# A SECOND–ORDER STOCHASTIC DOMINANCE PORTFOLIO EFFICIENCY MEASURE

Miloš Kopa and Petr Chovanec

In this paper, we introduce a new linear programming second-order stochastic dominance (SSD) portfolio efficiency test for portfolios with scenario approach for distribution of outcomes and a new SSD portfolio inefficiency measure. The test utilizes the relationship between CVaR and dual second-order stochastic dominance, and contrary to tests in Post [14] and Kuosmanen [7], our test detects a dominating portfolio which is SSD efficient. We derive also a necessary condition for SSD efficiency using convexity property of CVaR to speed up the computation. The efficiency measure represents a distance between the tested portfolio and its least risky dominating SSD efficient portfolio. We show that this measure is consistent with the second-order stochastic dominance relation. We find out that this measure is convex and we use this result to describe the set of SSD efficient portfolios. Finally, we illustrate our results on a numerical example.

*Keywords:* stochastic dominance, CVaR, SSD portfolio efficiency measure

*AMS Subject Classification:* 91B28, 91B30

## 1. INTRODUCTION

The questions how to maximize profit and how to diversify risk has been around for centuries; however, both these questions took another dimension with the work of Markowitz [10]. In his work, Markowitz identified two main components of portfolio performance, mean reward and risk represented by variance, and by applying a simple parametric optimization model found the optimal trade-off between these two components. Unfortunately, these optimal portfolios are not consistent with expected utility maximization unless the utility is quadratic or returns are normally distributed; because of this Markowitz [11] suggested as more plausible the semivariance instead of variance. Decades later Ogryczak and Ruszczyński [12] proved that the optimal mean-semivariance portfolio is also optimal in second-order stochastic dominance sense and vice-versa.

Stochastic dominance is another possible approach to portfolio selection. In economics and finance it was introduced independently in Hadar and Russel [4], Hanoch and Levy [5], Rothschild and Stiglitz [15] and Whitmore [19].[1]

[1] for more information see Levy [8] or Levy [9].

The usual definition of stochastic dominance uses cumulative distribution function[2], but the following alternative definition fits our questions better – it has nice financial consequences with risk-averse agents, and is easier to understand in our context. We say that risky asset $X$ stochastically dominates (in the first-order) risky asset $Y$, if and only if $\mathbb{E}\,u(X) \geq \mathbb{E}\,u(Y)$ for every utility function (i. e. for every non-decreasing function). If this holds only for the concave utility function (for every risk-averter), we say that $X$ stochastically dominates $Y$ in the second-order. This can be applied in the portfolio selection problem as a search for a portfolio that no risk-averse agent would want to choose.

In this paper, we propose a new test of second-order stochastic dominance of a portfolio relative to all portfolios created from a set of assets with discrete distributions. Until 2003, stochastic dominance tests considered only pairs of assets and not the sets of assets; however, especially in finance we would like to know whether our portfolio is the best one, or whether for any risk-averse agent there exists another better portfolio. Therefore, a test for stochastic dominance efficiency was needed. In 2003, Post [14] published a linear programming procedure for testing the second-order stochastic dominance of a given portfolio relative to a given set of assets and he discussed its statistical properties. Post used a primal approach and a representative characterization of concave utility functions. Therefore, his algorithm does not identify the SSD efficient portfolio. On the other hand, linear programming algorithm works in linear space in both numbers of assets and scenarios. Our approach is thus slower, but it identifies the dominating SSD efficient portfolio. In the same year, Ruszczyński and Vanderbei [17] developed a parametric linear programming procedure for computing all efficient portfolios in the dual mean risk space (in the second-order stochastic dominance sense). They used dual approach, the same as we did, but they used another identity. Our test procedure should generate more sparse matrix and, therefore, should be quicker. Furthermore, another linear programming test for second-order stochastic dominance was presented in Kuosmanen [7]. This test is based on comparisons of cumulated returns. It identifies a dominating portfolio but this dominating portfolio need not to be SSD efficient. Moreover, the Kuosmanen test is computationally more demanding than our test.

Our approach is based on second-order stochastic dominance consistence with Conditional Value-at-Risk (shown in Ogryczak and Ruszczyński [12]), and because CVaR has a linear programming representation explored by Uryasev and Rockafellar [18], it is sufficient to solve a linear program. Another connection of risk measures and SSD relation was analyzed in [1] or [2]. We derive a LP sufficient and necessary condition for SSD efficiency. Moreover, using convexity of CVaR, a necessary condition is presented. In addition, our test identifies the dominating portfolio which is already SSD efficient. With the help of this test, we introduce a SSD portfolio inefficiency measure in the dual risk (CVaR) space. Our measure is consistent with the second-order stochastic dominance relation and it is represented by a distance between the tested portfolio and its dominating SSD efficient portfolio. If there exist more dominating SSD efficient portfolios then the least risky portfolio is considered. Since the set of SSD efficient portfolios can be non-convex, see Dybvig and Ross [3],

[2]and can be found in e. g. Levy [8] or Levy [9]

we explore the convexity of this measure. We prove that all portfolios dominated by a given portfolio form a convex set and the measure is convex on these sets.

The rest of the paper is organized as follows. A Preliminaries section with precise assumptions and definitions for a stochastic dominance relation is followed by a section dealing with CVaR. In Section 4, we state our main theorems, allowing us to test the SSD efficiency of a given portfolio and to identify a dominating portfolio which is already SSD efficient. Subsequent section defines the inefficiency measure in dual risk (CVaR) space and presents the convexity results. This section is followed by the numerical illustration in Section 6.

## 2. PRELIMINARIES

For two random variables $X_1$ and $X_2$ with respective cumulative probability distributions functions $F_1(x)$, $F_2(x)$ we say that $X_1$ *dominates* $X_2$ *by second-order stochastic dominance*: $X_1 \succeq_{\mathrm{SSD}} X_2$ if

$$\mathbb{E}_{F_1} u(x) - \mathbb{E}_{F_2} u(x) \geq 0$$

for every $u \in U_2$ where $U_2$ denotes the set of all concave utility functions such that these expected values exist. The corresponding strict dominance relation $\succ_{\mathrm{SSD}}$ is defined in the usual way: $X_1 \succ_{\mathrm{SSD}} X_2$ if and only if $X_1 \succeq_{\mathrm{SSD}} X_2$ and $X_2 \not\succeq_{\mathrm{SSD}} X_1$. According to Russel and Seo [16], $u \in U_2$ may be represented by simple utility functions in the following sense:

$$\mathbb{E}_{F_1} u(x) - \mathbb{E}_{F_2} u(x) \geq 0 \quad \forall\, u \in U_2 \Longleftrightarrow \mathbb{E}_{F_1} u(x) - \mathbb{E}_{F_2} u(x) \geq 0 \quad \forall\, u \in V$$

where $V = \{u_\eta(x) : \eta \in \mathbb{R}\}$ and $u_\eta(x) = \min\{x - \eta, 0\}$.

Set

$$F_i^{(2)}(t) = \int_{-\infty}^{t} F_i(x)\,\mathrm{d}x \qquad i = 1, 2.$$

The following necessary and sufficient conditions for the second-order stochastic dominance relation were proved in Hanoch and Levy [5].

**Lemma 1.** Let $F_1(x)$ and $F_2(x)$ be the cumulative distribution functions of $X_1$ and $X_2$. Then

- $X_1 \succeq_{\mathrm{SSD}} X_2 \Longleftrightarrow F_1^{(2)}(t) \leq F_2^{(2)}(t)\ \forall\, t \in \mathbb{R}$
- $X_1 \succ_{\mathrm{SSD}} X_2 \Longleftrightarrow F_1^{(2)}(t) \leq F_2^{(2)}(t)\ \forall\, t \in \mathbb{R}$ with at least one strict inequality.

Lemma 1 can be used as an alternative definition of the second-order stochastic dominance relation.

Consider now the quantile model of stochastic dominance as in Ogryczak and Ruszczyński [12]. The *first quantile function* $F_X^{(-1)}$ corresponding to a real random variable $X$ is defined as the left continuous inverse of its cumulative probability distribution function $F_X$:

$$F_X^{(-1)}(v) = \min\{u : F_X(u) \geq v\}. \tag{1}$$

The *second quantile function* $F_X^{(-2)}$ is defined as

$$\begin{aligned} F_X^{(-2)}(p) &= \int_{-\infty}^{p} F_X^{(-1)}(t)\,\mathrm{d}t && \text{for } \ 0 < p \leq 1 \\ &= 0 && \text{for } \ p = 0 \\ &= +\infty && \text{otherwise.} \end{aligned}$$

The function $F_X^{(-2)}$ is convex and it is well defined for any random variable $X$ satisfying the condition $\mathbb{E}\,|X| < \infty$. An interpretation of this function will be given in Section 3.

**Lemma 2.** For every random variable $X$ with $\mathbb{E}\,|X| < \infty$ we have:

(i) $$F_X^{(-2)}(p) = \sup_{\nu}\{\nu p - \mathbb{E}\,\max(\nu - X, 0)\}$$

(ii) $$X_1 \succeq_{\mathrm{SSD}} X_2 \Longleftrightarrow \frac{F_1^{(-2)}(p)}{p} \geq \frac{F_2^{(-2)}(p)}{p} \qquad \forall\, p \in \langle 0, 1\rangle.$$

These properties follow from the Fenchel duality relation between $F_X^{(2)}$ and $F_X^{(-2)}$. For the entire proof of Lemma 2 and more details about dual stochastic dominance see Ogryczak and Ruszczyński [12].

## 3. CVaR FOR SCENARIO APPROACH

Let $Y$ be a random loss variable corresponding to the return described by random variable $X$, i. e. $Y = -X$. We assume that $\mathbb{E}\,|Y| < \infty$. For a fixed level $\alpha$, the *value-at-risk* (VaR) is defined as the $\alpha$-quantile of the cumulative distribution function $F_Y$:

$$\mathrm{VaR}_\alpha(Y) = F_Y^{(-1)}(\alpha). \tag{2}$$

We follow Pflug [13] in defining *conditional value-at-risk* (CVaR) as the solution of the optimization problem

$$\mathrm{CVaR}_\alpha(Y) = \min_{a\in\mathbb{R}}\left\{a + \frac{1}{1-\alpha}\mathbb{E}\,[Y-a]^+\right\} \tag{3}$$

where $[x]^+ = \max(x, 0)$. This problem has always a solution and one of the minimizers is $\mathrm{VaR}_\alpha(Y)$, see Pflug [13] for the proof and more details. It was shown in

Uryasev and Rockafellar [18] that the CVaR can be also defined as the conditional expectation of Y, given that $Y > \mathrm{VaR}_\alpha(Y)$, i. e.

$$\mathrm{CVaR}_\alpha(Y) = \mathbb{E}\,(Y \,|\, Y > \mathrm{VaR}_\alpha(Y)). \tag{4}$$

If we use $-Y$ and $1-\alpha$ instead of $X$ and $p$, respectively, we can directly from the definition of CVaR and Lemma 2 derive:

$$\frac{F_X^{(-2)}(p)}{p} = -\mathrm{CVaR}_\alpha(Y),$$

and consequently

$$X_1 \succeq_{\mathrm{SSD}} X_2 \Longleftrightarrow \mathrm{CVaR}_\alpha(Y_1) \le \mathrm{CVaR}_\alpha(Y_2) \quad \forall\, \alpha \in \langle 0,1\rangle. \tag{5}$$

From now on, let us assume that $Y$ is a discrete random variable which takes scenarios $y^t, t = 1,\dots,T$ with equal probabilities. Following Rockafellar and Uryasev [18] and Pflug [13], (3) can be rewritten as a linear programming problem:

$$\begin{array}{rrcl} \mathrm{CVaR}_\alpha(Y) & = & \min\limits_{a,w_t} & a + \dfrac{1}{(1-\alpha)T}\displaystyle\sum_{t=1}^{T} w_t \\ \text{s. t.} \quad w_t & \ge & y_t - a & \\ w_t & \ge & 0. & \end{array} \tag{6}$$

Let $y^{[k]}$ be the $k$th smallest element among $y^1, y^2, \dots, y^T$, i. e. $y^{[1]} \le y^{[2]} \le \dots \le y^{[T]}$. The optimal solution of (6) is derived in the following theorem.

**Theorem 3.** If $\alpha \in \left\langle \frac{k}{T}, \frac{k+1}{T} \right\rangle$ and $\alpha \neq 1$ then

$$\mathrm{CVaR}_\alpha(Y) = y^{[k+1]} + \frac{1}{(1-\alpha)T} \sum_{i=k+1}^{T} (y^{[i]} - y^{[k+1]}) \tag{7}$$

for $k = 0, 1, \dots, T-1$ and $\mathrm{CVaR}_1(Y) = y^{[T]}$.

P r o o f. Consider a random variable $Y$ which takes values $y^t, t = 1,\dots,T$ with probabilities $p_1, p_2, \dots, p_T$. For a chosen $\alpha$ define $j_\alpha$ such that

$$\alpha \in \left\langle \sum_{j=1}^{j_\alpha - 1} p_j, \sum_{j=1}^{j_\alpha} p_j \right) .$$

Then the following formula was proved in Rockafellar and Uryasev [18]:

$$\mathrm{CVaR}_\alpha(Y) = \frac{1}{1-\alpha}\left[ \left( \sum_{j=1}^{j_\alpha} p_j - \alpha \right) y^{[j_\alpha]} + \sum_{j=j_\alpha+1}^{T} p_j y^{[j]} \right].$$

Since $p_t = 1/T$, $t = 1, \dots, T$ we set: $j_\alpha = k + 1$ and the theorem follows. □

Combining Theorem 3 with (5) we obtain the necessary and sufficient condition for the second-order stochastic dominance. This conditions can be more easily verified than the general conditions in Lemma 1, Lemma 2 or (5).

**Theorem 4.** Let $Y_1 = -X_1$ and $Y_2 = -X_2$ be discrete random variables which take values $y_1^t$ and $y_2^t$, $t = 1, \dots, T$, respectively, with equal probabilities. Then

$$X_1 \succeq_{\mathrm{SSD}} X_2 \Longleftrightarrow \mathrm{CVaR}_\alpha(Y_1) \le \mathrm{CVaR}_\alpha(Y_2) \quad \forall\, \alpha \in \left\{0, \frac{1}{T}, \frac{2}{T}, \dots, \frac{T-1}{T}\right\}. \tag{8}$$

P r o o f. Let $\alpha_k = k/T$, $k = 0, 1, \dots, T-2$. Lemma 1 implies:

$$\mathrm{CVaR}_{\beta_1}(Y_i) = \mathrm{CVaR}_{\beta_2}(Y_i), \quad i = 1, 2 \quad \text{for all} \quad \beta_1, \beta_2 \in \left\langle \frac{T-1}{T}, 1 \right\rangle.$$

Thus it suffices to show that if

$$\mathrm{CVaR}_{\alpha_k}(Y_1) \le \mathrm{CVaR}_{\alpha_k}(Y_2) \tag{9}$$

and

$$\mathrm{CVaR}_{\alpha_{k+1}}(Y_1) \le \mathrm{CVaR}_{\alpha_{k+1}}(Y_2) \tag{10}$$

then it holds for all $\alpha \in \langle \alpha_k, \alpha_{k+1} \rangle$. To obtain a contradiction, suppose that (9) and (10) holds and there exists $\widetilde{\alpha} \in \langle \alpha_k, \alpha_{k+1} \rangle$ such that $\mathrm{CVaR}_{\widetilde{\alpha}}(Y_1) > \mathrm{CVaR}_{\widetilde{\alpha}}(Y_2)$. From continuity of CVaR in $\alpha$ there exists $\alpha^1 \in \langle \alpha_k, \alpha_{k+1} \rangle$ and $\alpha^2 \in \langle \alpha_k, \alpha_{k+1} \rangle$, $\alpha^1 \ne \alpha^2$ such that

$$\mathrm{CVaR}_{\alpha^1}(Y_1) = \mathrm{CVaR}_{\alpha^1}(Y_2) \tag{11}$$

$$\mathrm{CVaR}_{\alpha^2}(Y_1) = \mathrm{CVaR}_{\alpha^2}(Y_2). \tag{12}$$

Substituting (7) into (11) and (12) we conclude that $\alpha^1 = \alpha^2$, contrary to $\alpha^1 \ne \alpha^2$, and the proof is complete. □

## 4. SSD PORTFOLIO EFFICIENCY CRITERIA

Consider a random vector $\boldsymbol{r} = (r_1, r_2, \dots, r_N)'$ of returns of $N$ assets and $T$ equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \\ \vdots \\ \boldsymbol{x}^T \end{pmatrix}$$

where $\boldsymbol{x}^t = (x_1^t, x_2^t, \dots, x_N^t)$ is the $t$-th row of matrix $X$. Without loss of generality we can assume that the columns of $X$ are linearly independent. In addition to the individual choice alternatives, the decision maker may also combine the alternatives into a portfolio. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$ for a vector of portfolio weights and the portfolio possibilities are given by

$$\Lambda = \{\boldsymbol{\lambda} \in R^N \mid \mathbf{1}'\boldsymbol{\lambda} = 1, \quad \lambda_n \geq 0, \ n = 1, 2, \ldots, N\}.$$

The tested portfolio is denoted by $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_N)'$.

**Definition 1.** A given portfolio $\boldsymbol{\tau} \in \Lambda$ is *SSD inefficient* if and only if there exists portfolio $\boldsymbol{\lambda} \in \Lambda$ such that $\boldsymbol{r}'\boldsymbol{\lambda} \succ_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}$. Otherwise, portfolio $\boldsymbol{\tau}$ is *SSD efficient.*

This definition classifies portfolio as SSD efficient if and only if no other portfolio is better for all risk averse and risk neutral decision makers.

In Post [14] and Kuosmanen [7], the SSD portfolio efficiency tests based on applications of Lemma 1 were introduced. We will derive sufficient and necessary conditions for SSD efficiency of $\boldsymbol{\tau}$ based on quantile model of second order stochastic dominance, in particular the relationship between CVaR and SSD will be used.

We start with necessary condition using the following theorem. To simplify the notation, set $\Gamma = \left\{0, \frac{1}{T}, \frac{2}{T}, \ldots, \frac{T-1}{T}\right\}$.

**Theorem 5.** Let $\alpha_k \in \Gamma$ and

$$d^* = \max_{\lambda_n} \sum_{k=0}^{T-1} \sum_{n=1}^{N} \lambda_n \left[\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau}) - \mathrm{CVaR}_{\alpha_k}(-r_n)\right] \tag{13}$$

$$\text{s. t.} \ \sum_{n=1}^{N} \lambda_n \left[\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau}) - \mathrm{CVaR}_{\alpha_k}(-r_n)\right] \geq 0, \quad k = 0, 1, \ldots, T-1, \ \lambda \in \Lambda.$$

If $d^* > 0$ then $\boldsymbol{\tau}$ is SSD inefficient. Optimal solution $\boldsymbol{\lambda}^*$ of (13) is an SSD efficient portfolio such that $\boldsymbol{r}'\boldsymbol{\lambda}^* \succ_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}$.

P r o o f. If $d^* > 0$ then there is a feasible solution $\boldsymbol{\lambda}$ of problem (13) satisfying

$$\sum_{n=1}^{N} \lambda_n \left[\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau}) - \mathrm{CVaR}_{\alpha_k}(-r_n)\right] \geq 0, \quad \forall\, \alpha_k \in \Gamma$$

where at least one strict inequality holds. For this $\boldsymbol{\lambda}$ we have

$$\sum_{n=1}^{N} \lambda_n \mathrm{CVaR}_{\alpha_k}(-r_n) \leq \mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau}), \quad \forall\, \alpha_k \in \Gamma$$

with at least one strict inequality. From the convexity of CVaR we obtain

$$\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\lambda}) \leq \sum_{n=1}^{N} \lambda_n \mathrm{CVaR}_{\alpha_k}(-r_n) \quad \forall\, \alpha_k \in \Gamma.$$

Hence
$$\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\lambda}) \le \mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau}) \quad \forall\, \alpha_k \in \Gamma$$
with at least one strict inequality and the rest of the proof follows from Theorem 4. □

The power of necessary condition in Theorem 5 depends on correlation between random variables $r_n$, $n = 1, 2, \ldots, N$ and portfolio $\boldsymbol{\tau}$ can be SSD inefficient even if (13) has no feasible solution or $d^* = 0$. If $d^* = 0$ then two possibilities may occur:

(1) Problem (13) has a unique solution $\boldsymbol{\lambda}^* = \boldsymbol{\tau}$. If this is the case then $\boldsymbol{\tau}$ is SSD efficient.

(2) Problem (13) has an optimal solution $\boldsymbol{\lambda}^* \neq \boldsymbol{\tau}$. In this case, $\boldsymbol{\tau}$ is SSD inefficient and $\boldsymbol{r}'\boldsymbol{\lambda}^* \succ_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}$. Moreover, $\boldsymbol{\lambda}^*$ is an SSD efficient portfolio.

The situation when $d^* = 0$, $\boldsymbol{\lambda}^* \neq \boldsymbol{\tau}$ and $\boldsymbol{\tau}$ is SSD efficient would imply $X\boldsymbol{\lambda}^* = X\boldsymbol{\tau}$ which contradicts the assumption of linearly independent columns of $X$.

If problem (13) has no feasible solution then we can employ the following necessary and sufficient condition for SSD efficiency.

**Theorem 6.** Let $\alpha_k \in \Gamma$ and
$$D^*(\boldsymbol{\tau}) = \max_{D_k, \lambda_n, b_k} \sum_{k=0}^{T-1} D_k \tag{14}$$
s. t.
$$\begin{aligned}
\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau}) - b_k - \frac{1}{1-\alpha_k}\mathbb{E}\max(-\boldsymbol{r}'\boldsymbol{\lambda} - b_k, 0) &\ge D_k, \; k = 0, 1, \ldots, T-1\\
D_k &\ge 0, \quad k = 0, 1, \ldots, T-1\\
\boldsymbol{\lambda} &\in \Lambda.
\end{aligned}$$
If $D^*(\boldsymbol{\tau}) > 0$ then $\boldsymbol{\tau}$ is SSD inefficient and $\boldsymbol{r}'\boldsymbol{\lambda}^* \succ_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}$. Otherwise, $D^*(\boldsymbol{\tau}) = 0$ and $\boldsymbol{\tau}$ is SSD efficient.

Proof. Let $\boldsymbol{\lambda}^*$, $b_k^*$, $k=0, 1, \ldots, T-1$ be an optimal solution of (14). If $D^*(\boldsymbol{\tau})>0$ then
$$b_k^* + \frac{1}{1-\alpha_k}\mathbb{E}\max(-\boldsymbol{r}'\boldsymbol{\lambda}^* - b_k^*, 0) \le \mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau}) \;\; \forall\, \alpha_k \in \Gamma \tag{15}$$
where at least one inequality holds strict. Since from the definition of CVaR we have
$$\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\lambda}^*) = \min_{b_k}\left\{b_k + \frac{1}{1-\alpha_k}\mathbb{E}\max(-\boldsymbol{r}'\boldsymbol{\lambda}^* - b_k, 0)\right\}$$
we conclude from (15) that
$$\mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\lambda}^*) \le \mathrm{CVaR}_{\alpha_k}(-\boldsymbol{r}'\boldsymbol{\tau})$$
with at least one strict inequality. Hence $\boldsymbol{r}'\boldsymbol{\lambda}^* \succ_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}$ and $\boldsymbol{\tau}$ is SSD inefficient.

If $D^*(\boldsymbol{\tau}) = 0$ then problem (14) has unique optimal solution: $\boldsymbol{\lambda}^* = \boldsymbol{\tau}$, because the presence of another optimal solution contradicts the assumption of linearly independent columns of $X$. Thus there is no strictly dominating portfolio and hence $\boldsymbol{\tau}$ is SSD efficient. Since $\boldsymbol{\tau}$ is always a feasible solution of (14), $D^*$ can not be negative and the proof is complete. □

Nonlinear program (14) has $N+2T+1$ constraints and $N+2T$ variables. Inspired by (6) and following Pflug [13], Rockafellar and Uryasev [18], it can be rewritten as a linear programming problem with $2T(T+1)+N+1$ constraints and $T(T+2)+N$ variables:

$$D^*(\boldsymbol{\tau}) = \max_{D_k,\lambda_n,b_k,w_k^t} \sum_{k=1}^{T} D_k \tag{16}$$

s. t.

$$\begin{aligned}
\mathrm{CVaR}_{\frac{k-1}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}) - b_k - \frac{1}{(1-\frac{k-1}{T})T}\sum_{t=1}^{T} w_k^t &\geq D_k, && k = 1,2,\ldots,T\\
w_k^t &\geq -\mathbf{x}^t\boldsymbol{\lambda} - b_k, && t,k = 1,2,\ldots,T\\
w_k^t &\geq 0, && t,k = 1,2,\ldots,T\\
D_k &\geq 0, && k = 1,2,\ldots,T\\
\boldsymbol{\lambda} &\in \Lambda.
\end{aligned}$$

Using (16) instead of (14) in Theorem 6 we obtain a linear programming criterion for SSD efficiency.

This sufficient and necessary condition requires to solve a smaller linear program than it is in the Kuosmanen test. Furthermore, contrary to the Post and the Kuosmanen test, it identifies SSD efficient dominating portfolio as a by-product.

## 5. A SSD PORTFOLIO INEFFICIENCY MEASURE

Inspired by Post [14] and Kopa and Post [6], $D^*(\boldsymbol{\tau})$ from (14) or (16) can be considered as a measure of inefficiency of portfolio $\boldsymbol{\tau}$, because it expresses the distance between a given tested portfolio and its dominating SSD efficient portfolio. If there exist more dominating SSD efficient portfolios then the least risky portfolio, measured by CVaR, is considered. To be able to compare SSD inefficiency of two portfolios we need to consider such a measure, which is "consistent" with SSD relation.

**Definition 2.** Let $\xi$ be a measure of SSD portfolio inefficiency. We say that $\xi$ is consistent with SSD if and only if

$$\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2 \Rightarrow \xi(\boldsymbol{\tau}_2) \geq \xi(\boldsymbol{\tau}_1)$$

for any $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \Lambda$.

The property of consistency guarantees that if a given portfolio is worse than the other one for every risk averse investor then it has larger measure of inefficiency. Let $\Lambda^*(\boldsymbol{\tau}) \in \Lambda$ be a set of optimal solutions $\boldsymbol{\lambda}^*$ of (14) or (16).

**Theorem 7.**

(i) The measure of SSD portfolio inefficiency $D^*$ given by either (14) or (16) is consistent with SSD.

(ii) If $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$ and both $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ are SSD inefficient then

$$D^*(\boldsymbol{\tau}_2) = D^*(\boldsymbol{\tau}_1) + \sum_{k=1}^{T} \left[\mathrm{CVaR}_{\frac{k-1}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k-1}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1)\right].$$

(iii) If $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$ then

$$D^*(\boldsymbol{\tau}_2) \geq D^*(\boldsymbol{\tau}_1) + \sum_{k=1}^{T} \left[\mathrm{CVaR}_{\frac{k-1}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k-1}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1)\right].$$

Proof. Applying Theorem 4, if $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$ then

$$\sum_{k=1}^{T} \left[\mathrm{CVaR}_{\frac{k-1}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k-1}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1)\right] \geq 0.$$

Hence it suffices to prove (ii) and (iii).

Let $\boldsymbol{r}'\boldsymbol{\tau}_1$ be SSD inefficient. It is easily seen that (14) can be rewritten in the following way:

$$\begin{aligned} D^*(\boldsymbol{\tau}) = \max_{\lambda_n} & \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda})\right] \\ \text{s.t.}\ & \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}) \geq 0, \quad k = 0,1,\ldots,T-1 \\ & \boldsymbol{\lambda} \in \Lambda. \end{aligned} \tag{17}$$

Let $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\boldsymbol{\tau}_1)$, $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_2) \in \Lambda^*(\boldsymbol{\tau}_2)$. Using Theorem 4 and $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$,

$$\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1) \geq 0 \quad k = 0,1,\ldots,T-1.$$

Since the sum of these differences does not depend on the choice of $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1)$, the dominating portfolio $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1)$ is also an optimal solution of (14) when deriving $D^*(\boldsymbol{\tau}_2)$, i.e. $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\boldsymbol{\tau}_2)$. Hence

$$\begin{aligned} D^*(\boldsymbol{\tau}_2) &= \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_2))\right] \\ &= \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1)\right] \\ &\quad + \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1))\right] \\ &= D^*(\boldsymbol{\tau}_1) + \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1)\right] \end{aligned}$$

what completes the proof of (ii).

Let $\boldsymbol{r}'\boldsymbol{\tau}_1$ be SSD efficient. From Theorem 6, we have $D^*(\boldsymbol{\tau}_1) = 0$. According to (17),

$$\begin{aligned} D^*(\boldsymbol{\tau}_2) = \max_{\lambda_n} \sum_{k=0}^{T-1} &\left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda})\right] \\ \text{s.t.}\ \ \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}) &\geq 0, \quad k = 0, 1, \ldots, T-1 \\ \boldsymbol{\lambda} &\in \Lambda. \end{aligned}$$

Since $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$, portfolio $\boldsymbol{\tau}_1$ is a feasible solution of (17). Hence

$$D^*(\boldsymbol{\tau}_2) \geq \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1)\right]$$

and combining it with (ii), the proof is complete. □

Since SSD relation is not complete, i.e. there exist incomparable pairs of portfolios, the strict inequality of values of any portfolio inefficiency measure can not imply SSD relation. Also for the measure $D^*$ some pair of portfolios $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ can be found such that $D^*(\boldsymbol{\tau}_2) \geq D^*(\boldsymbol{\tau}_1)$ and $\boldsymbol{r}'\boldsymbol{\tau}_1 \not\succeq_{SSD} \boldsymbol{r}'\boldsymbol{\tau}_2$.

In the following theorem, a convexity property of portfolio inefficiency measure $D^*$ is analyzed.

**Theorem 8.** Let $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3 \in \Lambda$.

(i) If $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$ then

$$D^*(\eta\boldsymbol{\tau}_1 + (1-\eta)\boldsymbol{\tau}_2) \leq \eta D^*(\boldsymbol{\tau}_1) + (1-\eta) D^*(\boldsymbol{\tau}_2)$$

for any $\eta \in \langle 0, 1\rangle$.

(ii) If $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$ and $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_3$ then

$$\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'(\eta\boldsymbol{\tau}_2 + (1-\eta)\boldsymbol{\tau}_3)$$

and

$$D^*(\eta\boldsymbol{\tau}_2 + (1-\eta)\boldsymbol{\tau}_3) \leq \eta D^*(\boldsymbol{\tau}_2) + (1-\eta) D^*(\boldsymbol{\tau}_3)$$

for any $\eta \in \langle 0, 1\rangle$.

Proof. (i) Applying Lemma 1 for equiprobable scenario approach, we obtain

$$\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2 \Rightarrow \boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\mathrm{SSD}} \boldsymbol{r}'\left(\eta\boldsymbol{\tau}_1 + (1-\eta)\boldsymbol{\tau}_2\right) \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$$

for any $\eta \in \langle 0,1\rangle$. By analogy to the proof of previous theorem, if $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\boldsymbol{\tau}_1)$ then $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\boldsymbol{\tau}_2)$ and $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\eta\boldsymbol{\tau}_1 + (1-\eta)\boldsymbol{\tau}_2)$. Hence

$$\begin{aligned}
D^*(\eta\boldsymbol{\tau}_1 + (1-\eta)\boldsymbol{\tau}_2) &= \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}\left(-\boldsymbol{r}'[\eta\boldsymbol{\tau}_1 + (1-\eta)\boldsymbol{\tau}_2]\right) \\
&= \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1)) \\
D^*(\boldsymbol{\tau}_1) &= \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_1) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1))\right] \\
D^*(\boldsymbol{\tau}_2) &= \sum_{k=0}^{T-1} \left[\mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\tau}_2) - \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1))\right].
\end{aligned}$$

Combining it with convexity of CVaR, we obtain

$$\begin{aligned}
D^*(\eta\boldsymbol{\tau}_1 + (1-\eta)\boldsymbol{\tau}_2) &= \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}\left(-\boldsymbol{r}'[\eta\boldsymbol{\tau}_1 + (1-\eta)\boldsymbol{\tau}_2]\right) \\
&\quad - \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1)) \\
&\leq \eta \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}\left(-\boldsymbol{r}'\boldsymbol{\tau}_1\right) + (1-\eta) \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}\left(-\boldsymbol{r}'\boldsymbol{\tau}_2\right) \\
&\quad - \eta \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1)) \\
&\quad - (1-\eta) \sum_{k=0}^{T-1} \mathrm{CVaR}_{\frac{k}{T}}(-\boldsymbol{r}'\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1)) \\
&\leq \eta D^*(\boldsymbol{\tau}_1) + (1-\eta) D^*(\boldsymbol{\tau}_2).
\end{aligned}$$

(ii) Applying Lemma 1 for scenario approach, we obtain:

$$\boldsymbol{r}'\boldsymbol{\tau} \succeq_{\mathrm{SSD}} \boldsymbol{r}'\boldsymbol{\lambda} \Longleftrightarrow \sum_{t=1}^{T} (\boldsymbol{x}^t\boldsymbol{\tau} - \boldsymbol{x}^t\boldsymbol{\lambda}) \geq 0 \quad \forall\, t = 1,2,\ldots,T. \tag{18}$$

Hence

$$\begin{aligned}
\sum_{t=1}^{T} (\boldsymbol{x}^t\boldsymbol{\tau}_1 - \boldsymbol{x}^t\boldsymbol{\tau}_2) &\geq 0 \quad \forall\, t = 1,2,\ldots,T \\
\sum_{t=1}^{T} (\boldsymbol{x}^t\boldsymbol{\tau}_1 - \boldsymbol{x}^t\boldsymbol{\tau}_3) &\geq 0 \quad \forall\, t = 1,2,\ldots,T
\end{aligned}$$

and therefore

$$\sum_{t=1}^{T} (\boldsymbol{x}^t\boldsymbol{\tau}_1 - \eta\boldsymbol{x}^t\boldsymbol{\tau}_2 - (1-\eta)\boldsymbol{x}^t\boldsymbol{\tau}_3) \geq 0 \quad \forall\, t = 1,2,\ldots,T$$

for any $\eta \in \langle 0, 1\rangle$. Thus, according to Lemma 1,

$$\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\text{SSD}} \boldsymbol{r}'(\eta\boldsymbol{\tau}_2 + (1-\eta)\boldsymbol{\tau}_3) \quad \text{for any } \eta \in \langle 0, 1\rangle.$$

Similarly to the proof of previous theorem, if $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\boldsymbol{\tau}_1)$ then $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\boldsymbol{\tau}_2)$, $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\boldsymbol{\tau}_3)$ and $\boldsymbol{\lambda}^*(\boldsymbol{\tau}_1) \in \Lambda^*(\eta\boldsymbol{\tau}_2 + (1-\eta)\boldsymbol{\tau}_3)$ for any $\eta \in \langle 0, 1\rangle$ and the rest of the proof follows by analogy to (i). □

Let $I(\boldsymbol{\tau})$ be a set of all portfolios whose returns are SSD dominated by return of $\boldsymbol{\tau}$, i. e.

$$I(\boldsymbol{\tau}) = \{\boldsymbol{\lambda} \in \Lambda \,|\, \boldsymbol{r}'\boldsymbol{\tau} \succeq_{\text{SSD}} \boldsymbol{r}'\boldsymbol{\lambda}\}.$$

Theorem 8 shows that $I(\boldsymbol{\tau})$ is convex and $D^*$ is convex on $I(\boldsymbol{\tau})$ for any $\boldsymbol{\tau} \in \Lambda$. Both these properties are consequences of convexity of CVaR. The following example illustrates these results and we stress the fact that the set of SSD efficient portfolios is not convex.

## 6. NUMERICAL EXAMPLE

Consider three assets with three scenarios:

$$X = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & 7 & 5 \end{pmatrix}.$$

It is easy to check that $\boldsymbol{\lambda}_1 = (1,0,0)'$, $\boldsymbol{\lambda}_2 = (0,1,0)'$ and $\boldsymbol{\lambda}_3 = (0,0,1)'$ are SSD efficient. Let $\boldsymbol{\tau}_1 = \boldsymbol{\lambda}_3$, $\boldsymbol{\tau}_2 = (\frac{1}{2}, \frac{1}{2}, 0)'$ and let $\boldsymbol{\tau}_3 = (\frac{1}{3}, \frac{2}{3}, 0)'$. Then $X\boldsymbol{\tau}_2 = (-\frac{1}{2}, \frac{1}{2}, \frac{9}{2})$ and according to (18), $\boldsymbol{r}'\boldsymbol{\tau}_1 \succ_{\text{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_2$. Hence the set of SSD efficient portfolios is not convex. Similarly, $\boldsymbol{r}'\boldsymbol{\tau}_1 \succ_{\text{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_3$ and $\boldsymbol{r}'\boldsymbol{\tau}_1 \succeq_{\text{SSD}} \boldsymbol{r}'\boldsymbol{\tau}_1$. Applying Theorem 8, a set of convex combinations of $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$ is a subset of $I(\boldsymbol{\tau}_1)$. We will show that $I(\boldsymbol{\tau}_1)$ consists only of convex combinations of $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_3$, i. e.

$$I(\boldsymbol{\tau}_1) = \left\{ \boldsymbol{\lambda} \in \Lambda \,|\, \lambda = \eta_1\boldsymbol{\tau}_1 + \eta_2\boldsymbol{\tau}_2 + \eta_3\boldsymbol{\tau}_3,\ \eta_i \geq 0,\ i = 1,2,3, \sum_{i=1}^{3} \eta_i = 1 \right\}.$$

Substituting into (18) we can see that only portfolios $\boldsymbol{\lambda} \in \Lambda$ satisfying the following system of inequalities can be included in $I(\boldsymbol{\tau}_1)$:

$$\begin{aligned} -\lambda_2 &\leq 0 \\ \lambda_1 - \lambda_2 &\leq 0 \\ 3\lambda_1 + 6\lambda_2 + 5(1-\lambda_1-\lambda_2) &\leq 5. \end{aligned}$$

The graphical solution of this system is illustrated in the following Figure 1 and we can see that the set of portfolios which returns are SSD dominated by return of portfolio $\boldsymbol{\tau}_1$ is equal to the set of all convex combinations of portfolios $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$. Points A, B and C correspond to portfolios $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$, $\boldsymbol{\tau}_1$, respectively.

As was shown in Theorem 8 (ii), SSD portfolio inefficiency measure $D^*$ is convex on $I(\boldsymbol{\tau}_1)$. Figure 2 shows the graph of $D^*$ on $I(\boldsymbol{\tau}_1)$. Since $\boldsymbol{\tau}_1$ is SSD efficient, $D^*(\boldsymbol{\tau}_1) = 0$ and $D^*(\boldsymbol{\tau}) > 0$ for all $\boldsymbol{\tau} \in I(\boldsymbol{\tau}_1)\backslash\{\boldsymbol{\tau}_1\}$.

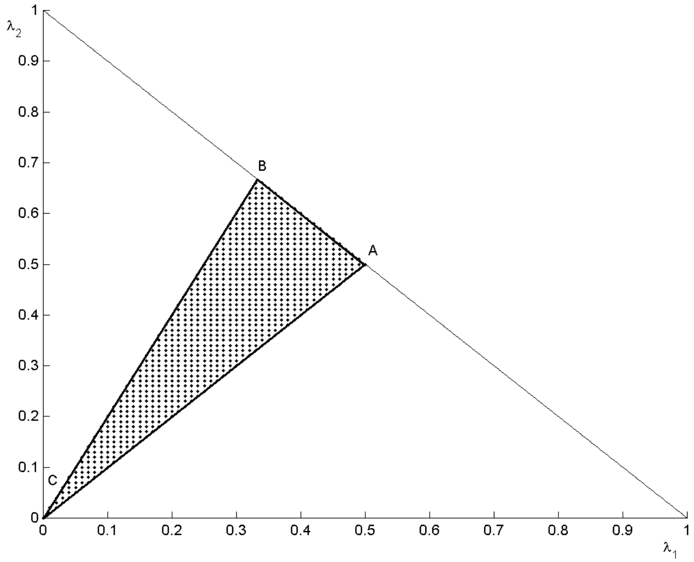


**Fig. 1.** The set $I(\boldsymbol{\tau}_1)$ of portfolios whose returns are SSD dominated by return of portfolio $\boldsymbol{\tau}_1 = (0, 0, 1)$.

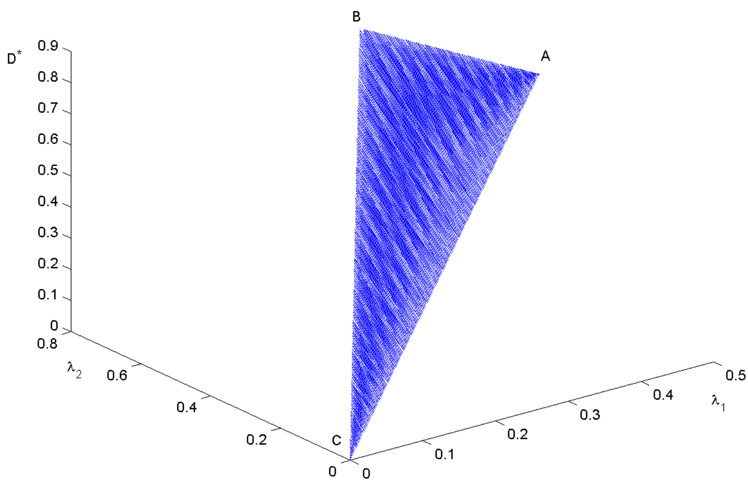


**Fig. 2.** The graph of $D^*$ on $I(\boldsymbol{\tau}_1)$.

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## REFERENCES

[1] E. De Giorgi: Reward-risk portfolio selection and stochastic dominance. J. Banking Finance *29* (2005), 895–926.

[2] E. De Giorgi and T. Post: Second order stochastic dominance, reward-risk portfolio selection and the CAPM. J. Financial Quantitative Analysis, to appear.

[3] P. H. Dybvig and S. A. Ross: Portfolio efficient sets. Econometrica *50* (1982), 6, 1525–1546.

[4] J. Hadar and W. R. Russell: Rules for ordering uncertain prospects. Amer. Econom. Rev. *59* (1969), 1, 25–34.

[5] G. Hanoch and H. Levy: The efficiency analysis of choices involving risk. Rev. Econom. Stud. *36* (1969), 335–346.

[6] M. Kopa and T. Post: A portfolio optimality test based on the first-order stochastic dominance criterion. J. Financial Quantitative Analysis, to appear.

[7] T. Kuosmanen: Efficient diversification according to stochastic dominance criteria. Management Sci. *50* (2004), 10, 1390–1406.

[8] H. Levy: Stochastic dominance and expected utility: Survey and analysis. Management Sci. *38* (1992), 4, 555–593.

[9] H. Levy: Stochastic Dominance: Investment Decision Making Under Uncertainty. Second edition. Springer Science, New York 2006.

[10] H. M. Markowitz: Portfolio Selection. J. Finance *7* (1952), 1, 77–91.

[11] H. M. Markowitz: Portfolio Selection: Efficient Diversification in Investments. Wiley, New York 1959.

[12] W. Ogryczak and A. Ruszczyński: Dual stochastic dominance and related mean-risk models. SIAM J. Optim. *13* (2002), 60–78.

[13] G. Ch. Pflug: Some remarks on the value-at-risk and the conditional value-at-risk. In: Probabilistic Constrained Optimization: Methodology and Applications (S. Uryasev, ed.), Kluwer Academic Publishers, Norwell MA 2000, pp. 278–287.

[14] T. Post: Empirical tests for stochastic dominance efficiency. J. Finance *58* (2003), 1905–1932.

[15] M. Rothschild and J. E. Stiglitz: Rules for ordering uncertain prospects. J. Econom. Theory *2* (1969), 225–243.

[16] W. R. Russell and T. K. Seo: Representative sets for stochastic dominance rules. In: Studies in the Economics of Uncertainty (T. B. Fomby and T. K. Seo, eds.), Springer-Verlag, New York 1989, pp. 59–76.

[17] A. Ruszczyński and R. J. Vanderbei: Frontiers of stochastically nondominated portfolios. Econometrica *71* (2003), 4, 1287–1297.

[18] S. Uryasev and R. T. Rockafellar: Conditional value-at-risk for general loss distributions. J. Banking Finance *26* (2002), 1443–1471.

[19] G. A. Whitmore: Third degree stochastic dominance. Amer. Econom. Rev. *60* (1970), 457–459.

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