Kybernetika

Milena Bieniek; Dominik Szynal

On entropies for random partitions of the unit segment

Kybernetika, Vol. 44 (2008), No. 1, 75--94

Persistent URL: http://dml.cz/dmlcz/135835

Terms of use:

© Institute of Information Theory and Automation AS CR, 2008

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

KYBERNETIKA — VOLUME 44 (2008), NUMBER 1, PAGES 75 – 94

ON ENTROPIES FOR RANDOM PARTITIONS OF THE UNIT SEGMENT

Milena Bieniek and Dominik Szynal

We prove the complete convergence of Shannon's, paired, genetic and α-entropy for random partitions of the unit segment. We also derive exact expressions for expectations and variances of the above entropies using special functions.

Keywords: paired entropy, genetic entropy, α-entropy, random partitions, complete convergence

AMS Subject Classification: 94A17, 62G30, 60F15

1. INTRODUCTION

Entropy is a measure of uncertainty or information. Shannon's entropy

$$H(p_0, \dots, p_k) = -\sum_{i=0}^{k} p_i \log p_i$$

(cf. [21]) for probabilities $p_0, \dots, p_k$, $\sum_{i=0}^{k} p_i = 1$, is the most common used measure of randomness. There are known many generalizations of this entropy (cf. [18]). Burbea and Rao [3] introduced ϕ-entropy defined as

$$D_k^{\phi} = \sum_{i=0}^{k} \phi\left(p_i\right)$$

where $(p_0, \dots, p_k)$ is a probability distribution and ϕ is a twice differentiable real function on $(0, 1)$. Special cases of ϕ-entropy are

- Shannon's entropy if $\phi(x) = -x \log x$,
- paired entropy if $\phi(x) = -x \log x - (1 - x) \log(1 - x)$,
- genetic entropy if $\phi(x) = x - x^2 - x^2(1 - x)^2$ (cf. [17]).

Menendez et al. [18] generalized ϕ-entropy and defined a family of (h, ϕ)-entropies,

$$D_k^{(h,\phi)} = h\left(\sum_{i=0}^{k} \phi\left(p_i\right)\right),$$

where $h : \mathbb{R} \to \mathbb{R}$ and $\phi : (0,1) \to \mathbb{R}$ are twice differentiable real functions. Examples of (h, ϕ)-entropies are as follows

- α degree entropy if $\phi(x) = x^\alpha$ and $h(x) = \frac{x-1}{2^{1-\alpha}-1}$, $\alpha > 0$, $\alpha \neq 1$ (cf. [14]),
- α order entropy if $\phi(x) = x^\alpha$ and $h(x) = \frac{\log x}{1-\alpha}$, $\alpha > 0$, $\alpha \neq 1$ (cf. [20]).

The Shannon entropy of spacings is the quantity

$$D_k^S = -\sum_{j=0}^{k} Y_{j,k} \log Y_{j,k},$$

where

$$Y_{j,k} = X_{j+1:k} - X_{j:k}, \quad 0 \leq j \leq k,$$

and $0 \leq X_{1:k} \leq \ldots \leq X_{k:k} \leq 1$, $X_{0:k} = 0$ and $X_{k+1:k} = 1$, are order statistics of a sample $(X_1, \ldots, X_n)$ from a distribution F. The asymptotic behaviour of Shannon's entropy D_k^S, when F is the uniform distribution, was studied in [7], [22] and [23]. Goldstein [7] proved that the sequence $(D_k^S - \log(k+1))$ converges to $\gamma - 1$ in probability as $k \to \infty$, where $\gamma = 0.577215\ldots$ is the Euler–Masheroni constant. Slud [23] showed that the convergence holds almost surely. Shao and Jimenez [22] proved that if the spacings come from a continuous distribution F then the almost sure convergence of D_k^S to $\gamma - 1$ characterizes uniform distribution among continuous distributions. Some related problems were investigated in Ekstörm [10], Hall [11], [12] and Misra [19]. In that papers limit theorems and tests of uniformity for sums of mth spacings were considered. We are interested in the complete convergence of Shannon's entropy of spacings D_k^S.

Recall that a sequence (X_n) converges completely to X $(X_n \xrightarrow{c} X)$ if for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \Pr\left(|X_n - X| > \varepsilon\right) < \infty \quad \text{(cf. [16])}.$$

We consider also the complete convergence of paired, genetic and α-entropy of spacings. Namely, we investigate the asymptotic behaviour of D_k^S,

$$\begin{aligned} D_k^P &= -\sum_{j=0}^{k} \left(Y_{j,k} \log Y_{j,k} + (1 - Y_{j,k}) \log(1 - Y_{j,k})\right) \\ D_k^G &= \sum_{j=0}^{k} \left(Y_{j,k}(1 - Y_{j,k}) - Y_{j,k}^2 (1 - Y_{j,k})^2\right) \end{aligned}$$

and

$$D_k^\alpha = \frac{1}{2^{1-\alpha} - 1} \left(\sum_{j=0}^{k} Y_{j,k}^\alpha - 1 \right), \quad \alpha > 0, \quad \alpha \neq 1,$$

where $Y_{j,k}$ are uniform spacings, i. e. F is uniform distribution on $(0,1)$.

The paper is organized as follows. In Section 2 we present definitions and auxiliary results containing some formulae for sums and integrals. The explicit expressions for expectations and variances of the above entropies are given in Section 3. Finally, Section 4 is devoted to the complete convergence of D_k^S, D_k^P, D_k^G and D_k^α.

2. PRELIMINARIES

Denote by $H_k^{(r)}$, $r \in \mathbb{N}$, $k \in \mathbb{N}$, the harmonic number of order r, i. e.

$$H_k^{(r)} = \sum_{i=1}^{k} \frac{1}{i^r}, \quad r \geq 1$$

(cf. [9]). For simplicity we write $H_k := H_k^{(1)}$. For $r > 1$ we use Riemann ζ-function and Hurwitz generalized ζ-function defined, respectively, by

$$\zeta(r) = H_\infty^{(r)} = \sum_{i=1}^{\infty} \frac{1}{i^r}, \quad \zeta(r;q) = \zeta(r) - H_{q-1}^{(r)} = \sum_{i=0}^{\infty} \frac{1}{(i+q)^r}, \quad q \geq 1.$$

We also use the relation between the harmonic numbers, Psi (or Digamma) function

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x)$$

and the derivatives of Psi function (or Polygamma functions).
Here $\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t$, $x > 0$, is the Gamma function. It is known that

$$H_k := H_k^{(1)} = \gamma + \psi(k+1) \tag{1}$$

(cf. [8]) and for $r \geq 2$

$$H_k^{(r)} = \frac{(-1)^r}{(r-1)!} \left(\psi^{(r-1)}(1) - \psi^{(r-1)}(k+1) \right)$$

as

$$\zeta(r;q) = \frac{(-1)^r}{(r-1)!} \psi^{(r-1)}(q).$$

By $B(x,y)$ we denote Beta function, i. e.

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t, \quad x > 0, \quad y > 0. \tag{2}$$

$(\lambda)_r$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_r := \frac{\Gamma(\lambda + r)}{\Gamma(r)} = \begin{cases} 1 & r = 0, \\ \lambda(\lambda+1)\ldots(\lambda+r-1) & r \in \mathbb{N}. \end{cases}$$

Also let $\beta(x)$ be the function defined by

$$\beta(x) := \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) \right] = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+x}, \quad x > 0,$$

and

$$\beta'(x) = -\sum_{j=0}^{\infty} \frac{(-1)^j}{(j+x)^2} \quad \text{(cf. [8])}.$$

We need the following lemmas.

Lemma 1. The following summation formulae hold true

$$\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^{j+1}}{j}=\psi(k+1)+\gamma \quad \text{(cf. [8], 0.155.4)}, \tag{3}$$

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}H_{j+r}=-B(k,r+1), \quad r\in\mathbb{N} \quad \text{(cf. [25])}, \tag{4}$$

$$\sum_{j=1}^{k+2}\frac{(-1)^{j-1}}{j^2}=\frac{\pi^2}{12}+(-1)^{k}\beta'(k+3) \quad \text{(cf. [13], 5.12.50)}, \tag{5}$$

$$\sum_{j=1}^{k+1}\frac{H_j}{j}=\frac{1}{2}\left(\psi(k+2)+\gamma\right)^2+\frac{1}{2}\left(\frac{\pi^2}{6}-\psi'(k+2)\right) \quad \text{(cf. [9], 6.71)}, \tag{6}$$

$$\sum_{j=1}^{\infty}\frac{1}{j(k+1+j)^2}=\frac{\psi(k+2)+\gamma}{(k+1)^2}-\frac{\psi'(k+2)}{k+1}, \quad k\in\mathbb{N} \quad \text{(cf. [13], 6.1.82)}. \tag{7}$$

Lemma 2.

$$\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^j}{(k+1-j)(j+1)}=\frac{(-1)^k\left(\psi(k+3)+\gamma\right)}{k+2}+\frac{1}{(k+2)^2}, \tag{8}$$

$$\sum_{j=0}^{k+1}\binom{k+2}{j}\frac{(-1)^jH_j}{k+2-j}=(-1)^{k+1}\left(\left(\gamma+\psi(k+3)\right)^2-\psi'(k+3)\right)+2\beta'(k+3), \tag{9}$$

$$\begin{aligned}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(k-j)^2(j+1)(j+2)}&=\frac{(-1)^k\left(\gamma+\psi(k+3)-1\right)}{k(k^2-1)(k+2)}\\&\quad+\frac{1}{(k^2-1)(k+1)}-\frac{1}{k(k-1)(k+2)^2},\end{aligned} \tag{10}$$

$$\begin{aligned}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^jH_{j+2}}{(k-j)^2(j+1)(j+2)}&=-\frac{2(k+2)\beta'(k+3)+1}{k(k^2-1)(k+2)^2}+\frac{1}{(k-1)(k+1)^2}\\&\quad+\frac{(-1)^k\left(\left(\gamma+\psi(k+3)\right)^2-\left(\gamma+\psi(k+3)\right)-\psi'(k+3)\right)}{k(k^2-1)(k+2)}.\end{aligned} \tag{11}$$

P r o o f. To prove (8) we use

$$\frac{1}{(k+1-j)(j+1)}=\frac{1}{k+2}\left(\frac{1}{k+1-j}+\frac{1}{j+1}\right)$$

and next by (3) and (1)

$$\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^j}{(k+1-j)(j+1)}=\frac{1}{k+2}\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^j}{k+1-j}$$
$$+\frac{1}{k+2}\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^j}{j+1}+\frac{(-1)^k}{(k+2)^2}$$
$$=\frac{(-1)^k}{k+2}\sum_{j=1}^{k+1}\binom{k+1}{j}\frac{(-1)^{j+1}}{j}+\frac{1+(-1)^k}{(k+2)^2}=\frac{(-1)^k\left(\psi(k+3)+\gamma\right)}{k+2}+\frac{1}{(k+2)^2}.$$

Now we prove (9). Let

$$S_k:=\sum_{j=0}^{k}\binom{k+1}{j}\frac{(-1)^jH_j}{k+1-j}.$$

Using

$$\binom{k+2}{j}=\binom{k+1}{j}+\binom{k+1}{j-1},\quad j=0,\dots,k+1,\quad \binom{k}{-1}=0,$$

we get

$$S_{k+1}=\sum_{j=0}^{k+1}\binom{k+1}{j}\frac{(-1)^jH_j}{k+2-j}+\sum_{j=0}^{k+1}\binom{k+1}{j-1}\frac{(-1)^jH_j}{k+2-j}$$
$$=\frac{\sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^jH_j}{k+2}-\frac{(-1)^kH_{k+2}}{k+2}-\sum_{j=0}^{k}\frac{\binom{k+1}{j}(-1)^j}{(j+1)(k+1-j)}-S_k.$$

Then by (4) and (8) we get the recurrence relation

$$S_{k+1}=-\frac{2}{(k+2)^2}-\frac{2(-1)^kH_{k+2}}{k+2}-S_k,\quad k=0,1,\dots$$

where $S_0=0$. Hence

$$S_{k+1}=2(-1)^k\sum_{j=1}^{k+2}\frac{(-1)^{j-1}}{j^2}-2(-1)^k\sum_{j=1}^{k+2}\frac{H_j}{j},$$

which by (6) and (5) gives (9).

Now consider (10). We see that

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(j+1)(j+2)(k-j)^2}=\frac{1}{k(k^2-1)(k+2)}\sum_{j=2}^{k+1}\binom{k+2}{j}(-1)^j\frac{k+1-j}{k+2-j},$$

and changing the order of summation we have

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(k-j)^2(j+1)(j+2)}=\frac{(-1)^k}{k(k^2-1)(k+2)}\sum_{j=1}^{k}\binom{k+2}{j}(-1)^j\frac{j-1}{j}$$
$$=\frac{(-1)^k}{k(k^2-1)(k+2)}\left(\sum_{j=0}^{k+2}\binom{k+2}{j}(-1)^j-1+\sum_{j=1}^{k+2}\binom{k+2}{j}\frac{(-1)^{j+1}}{j}\right)$$
$$+\frac{1}{(k-1)(k+1)^2}-\frac{1}{k(k-1)(k+2)^2}.$$

Finally using (3) we get (10).

Now we prove (11). Applying the same evaluations as above we see that

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^jH_{j+2}}{(j+1)(j+2)(k-j)^2}=\frac{1}{k(k^2-1)(k+2)}\sum_{j=0}^{k-2}\binom{k+2}{j+2}(-1)^jH_{j+2}\frac{k-1-j}{k-j}$$
$$=\frac{1}{k(k^2-1)(k+2)}\left(\sum_{j=1}^{k+2}\binom{k+2}{j}(-1)^jH_j-(-1)^kH_{k+2}-\sum_{j=1}^{k+1}\binom{k+2}{j}\frac{(-1)^jH_j}{k+2-j}\right)$$
$$+\frac{1}{(k-1)(k+1)^2},$$

which after using (4) and (9) gives (11). □

Corollary 1.

$$\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j\,(H_{j+2}-1)}{(j+1)(j+2)(k-j)^2}=(-1)^k\frac{\left((\gamma+\psi(k+3)-1)^2-\psi'(k+3)\right)}{k(k^2-1)(k+2)}$$
$$-\frac{2\beta'(k+3)}{k(k^2-1)(k+2)}+\frac{1}{(k^2-1)(k+2)^2}. \tag{12}$$

Lemma 3.

$$\int_0^1 x^{p-1}\left(\log\frac{1}{x}\right)^q\mathrm{d}x=\frac{1}{p^{q+1}}\Gamma(q+1),\quad p>0,\quad q>-1\quad (\text{cf. [8], 3.653.2}), \tag{13}$$

and for $p>1$, $q>1$

$$\int_0^1 x^{p-1}(1-x)^{q-1}\log\frac{1}{x}\,\mathrm{d}x=B(p,q)(\psi(p+q)-\psi(p))\quad(\text{cf. [8], 3.628.1}), \tag{14}$$

$$\int_0^1 x^p\log\frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x=\frac{\psi(p+2)+\gamma}{(p+1)^2}-\frac{\psi'(p+2)}{p+1}, \tag{15}$$

for $r \in \mathbb{N}$ and $q \in \mathbb{N}$

$$\int_0^1 x^{r-1}\Big(\log\frac{1}{1-x}\Big)^q \mathrm{d}x$$
$$= \frac{q!}{r}\sum_{a_1+2a_2+\ldots+qa_q=q}\prod_{i=1}^{q}\frac{(-1)^{ia_i}(\psi^{(i-1)}(1)-\psi^{(i-1)}(r+1))}{a_i!(i!)^{a_i}} \quad \text{(cf. [2])},$$

where the summation is over all $a_i \in \mathbb{N}$, $i = 1,\ldots,q$, and $c(j,q)$ are the unsigned Stirling numbers of the first kind. In particular

$$\int_0^1 x^{r-1}\left(\log\frac{1}{1-x}\right)^2 \mathrm{d}x = \frac{(\psi(r+1)+\gamma)^2 + \frac{\pi^2}{6} - \psi'(r+1)}{r}. \tag{16}$$

P r o o f. We prove (15). Using the expansion

$$\log\frac{1}{1-x} = \sum_{j=1}^{\infty}\frac{x^j}{j}, \qquad |x|<1,$$

and (13) we get

$$\int_0^1 x^r \log\frac{1}{x}\log\frac{1}{1-x}\,\mathrm{d}x = \sum_{j=1}^{\infty}\frac{1}{j(j+r+1)^2},$$

which by (7) proves (15). □

3. MEAN AND VARIANCE OF ENTROPY FOR RANDOM PARTITIONS

In this section we present formulae for the expectation and variance of Shannon's, paired, genetic and α-entropy of spacings. In what follows we use Darling's (cf. [4]) moment formulae for the statistic $\sum_{j=0}^{k} f(Y_{j,k})$, which are given by

$$\mathrm{E}\left(\sum_{j=0}^{k} f(Y_{j,k})\right) = k(k+1)\int_0^1 (1-x)^{k-1} f(x)\,\mathrm{d}x, \tag{17}$$

$$\begin{aligned}\mathrm{E}\left(\sum_{j=0}^{k} f(Y_{j,k})\right)^2 &= k(k+1)\int_0^1 (1-x)^{k-1} f^2(x)\,\mathrm{d}x \\ &\quad + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2} f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned} \tag{18}$$

and

$$\begin{aligned}\mathrm{E}\left(\sum_{j=0}^{k} f(Y_{j,k})\sum_{j=0}^{k} g(Y_{j,k})\right) &= k(k+1)\int_0^1 (1-x)^{k-1} f(x)g(x)\,\mathrm{d}x \\ &+ k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2} f(x)g(y)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned} \tag{19}$$

for any real functions f and g such that the above moments exist.

First we consider D_k^S. The results regarding the moments of Shanon's entropy of spacings are partially known (cf. [7, 23]) but we establish explicit formulae in terms of Polygamma functions.

Proposition 1. The expectation and the variance of D_k^S are given by

$$\mathrm{E}D_k^S = \psi(k+2) + \gamma - 1, \tag{20}$$

$$\operatorname{var} D_k^S = \frac{\pi^2 - 6}{3(k+2)} - \zeta(2, k+2). \tag{21}$$

P r o o f. Using (17) and (14) with $f(x) = -x \log x$ we get

$$\begin{aligned} \mathrm{E}D_k^S &= k(k+1) \int_0^1 x(1-x)^{k-1} \log \frac{1}{x} \, \mathrm{d}x \\ &= k(k+1) B(2,k) \left(\psi(k+2) - \psi(2)\right) \quad \text{(cf. [7, 23])} \end{aligned}$$

which gives (20).

Now by (18) we have

$$\begin{aligned} \mathrm{E}\left(D_k^S\right)^2 &= k(k+1) \int_0^1 (1-x)^{k-1} x^2 \log^2 x \, \mathrm{d}x \\ &\quad + k^2(k^2-1) \int_0^1 \int_0^{1-x} (1-x-y)^{k-2} x \log xy \log y \, \mathrm{d}y \, \mathrm{d}x, \end{aligned}$$

and substituting $y = (1-x)t$ in the second integral we get

$$\begin{aligned} \mathrm{E}\left(D_k^S\right)^2 &= k(k+1) \int_0^1 \left(x^{k+1} \log^2 \frac{1}{1-x} - 2x^k \log^2 \frac{1}{1-x} + x^{k-1} \log^2 \frac{1}{1-x} \right) \mathrm{d}x \\ &+ k^2(k^2-1) \int_0^1 t(1-t)^{k-2} \, \mathrm{d}t \left(\int_0^1 x^k \log \frac{1}{x} \log \frac{1}{1-x} \, \mathrm{d}x - \int_0^1 x^{k+1} \log \frac{1}{x} \log \frac{1}{1-x} \, \mathrm{d}x \right) \\ &+ k^2(k^2-1) \int_0^1 x(1-x)^k \log \frac{1}{x} \, \mathrm{d}x \int_0^1 t(1-t)^{k-2} \log \frac{1}{t} \, \mathrm{d}t, \end{aligned}$$

Therefore by (2), (16) and (14) we obtain

$$\mathrm{E}\left(D_k^S\right)^2 = \left(\gamma + \psi(k+2) - 1\right)^2 - \zeta(2, k+2) + \frac{\pi^2 - 6}{3(k+2)},$$

which with (20) leads to (21). □

Since

$$\zeta(2, k+2) \geq \frac{1}{(k+2)(k+3)} + \frac{1}{(k+3)(k+4)} + \frac{1}{(k+4)(k+5)} + \ldots = \frac{1}{k+2}$$

we get

Corollary 2. $$\operatorname{var} D_k^S \le \frac{\pi^2-9}{3(k+2)}.$$

For paired entropy D_k^P we have

Proposition 2. The expectation and the variance of D_k^P are given by

$$\mathrm{E}D_k^P = \psi(k+1)+\gamma, \tag{22}$$

$$\operatorname{var} D_k^P = \frac{\pi^2-6}{3(k+2)} - \zeta(2,k+2) - \frac{k}{k+2}\zeta\left(2,\frac{k+2}{2}\right) + \frac{k(2k+3)}{(k+1)^2(k+2)}. \tag{23}$$

P r o o f. Write $\overline{D}_k^S = -\sum_{j=0}^{k}(1-Y_{j,k})\log(1-Y_{j,k})$. Then using (17) with $f(x) = -(1-x)\log(1-x)$, we have

$$\mathrm{E}\overline{D}_k^S = k(k+1)\int_0^1 (1-x)^k \log\frac{1}{1-x}\,\mathrm{d}x = \frac{k}{k+1}. \tag{24}$$

Since $\mathrm{E}D_k^P = \mathrm{E}D_k^S + \mathrm{E}\overline{D}_k^S$ then by (20) we get (22).

Now using (18) we have

$$\mathrm{E}\left(\overline{D}_k^S\right)^2 = k(k+1)\int_0^1 (1-x)^{k+1}\log^2(1-x)\,\mathrm{d}x + k^2(k^2-1)$$
$$\cdot \int_0^1\int_0^{1-x}(1-x-y)^{k-2}(1-x)\log(1-x)(1-y)\log(1-y)\,\mathrm{d}y\,\mathrm{d}x := A_k + B_k,$$

say. By (13)

$$A_k := k(k+1)\int_0^1 (1-x)^{k+1}\log^2(1-x)\,\mathrm{d}x = \frac{2k(k+1)}{(k+2)^3}.$$

Next setting $t = 1-y$ in B_k we have

$$B_k = k^2(k^2-1)\left(\int_0^1\int_0^1 (t-x)^{k-2}(1-x)\log(1-x)t\log t\,\mathrm{d}t\,\mathrm{d}x\right.$$
$$\left. - \int_0^1\int_0^x (t-x)^{k-2}(1-x)\log(1-x)t\log t\,\mathrm{d}t\,\mathrm{d}x\right).$$

Then using the binomial expansion for $(t-x)^{k-2}$ in the first integral and substituting $t = zx$ in the second we get

$$B_k = k^2(k^2-1)\sum_{j=0}^{k-2}\binom{k-2}{j}(-1)^j\int_0^1 x^j(1-x)\log\frac{1}{1-x}\,\mathrm{d}x\int_0^1 t^{k-j-1}\log\frac{1}{t}\,\mathrm{d}t$$
$$-(-1)^k k^2(k^2-1)\left(\int_0^1 x^k(1-x)\log(1-x)\log x\,\mathrm{d}x\int_0^1 z(1-z)^{k-2}\,\mathrm{d}z\right.$$
$$\left. + \int_0^1 x(1-x)^k\log\frac{1}{x}\,\mathrm{d}x\int_0^1 z(1-z)^{k-2}\log\frac{1}{z}\,\mathrm{d}z\right).$$

Applying (2), (14) and (15) we see that

$$B_k = k^2(k^2-1)\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j\,(H_{j+2}-1)}{(j+1)(j+2)(k-j)^2} - \frac{(-1)^k k}{k+2}\Big((\gamma+\psi(k+3)-1)^2-\psi'(k+3)\Big).$$

Hence by (12) we get

$$B_k = \frac{k^2}{(k+2)^2} - \frac{2k}{k+2}\beta'(k+3).$$

Therefore
$$\mathrm{E}\left(\overline{D}_k^S\right)^2 = -\frac{2k}{k+2}\beta'(k+3) + \frac{k(k^2+4k+2)}{(k+2)^3},$$

which by (24) and the equality $\beta'(k+3) = -\beta'(k+2) - \frac{1}{(k+2)^2}$ gives

$$\mathrm{var}\,\overline{D}_k^S = \frac{2k}{k+2}\beta'(k+2) + \frac{k}{(k+1)^2(k+2)}.$$

Now using (19) with $f(x) = -(1-x)\log(1-x)$ and $g(x) = -x\log x$ we get

$$\mathrm{E}\overline{D}_k^S D_k^S = k(k+1)\int_0^1 (1-x)^k x\log x\log(1-x)\,\mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2}(1-x)\log(1-x)y\log y\,\mathrm{d}y\mathrm{d}x := C_k + E_k,$$

say. For C_k we have

$$C_k := k(k+1)\int_0^1 (1-x)^k x\log x\log(1-x)\,\mathrm{d}x = k(k+1)\int_0^1 (x^k\log x\log(1-x) - x^{k+1}\log x\log(1-x))\,\mathrm{d}x,$$

which by (15) gives

$$C_k = \frac{k(2k+3)}{(k+1)(k+2)^2}\left(\gamma+\psi(k+2)\right) - \frac{2k(k+1)}{(k+2)^3} - \frac{k}{k+2}\zeta(2,k+2).$$

Substituting $y = (1-x)t$ in E_k we get

$$E_k = k^2(k^2-1)\left(\int_0^1 (1-x)^{k+1}\log\frac{1}{1-x}\,\mathrm{d}x\int_0^1 t(1-t)^{k-2}\log\frac{1}{t}\,\mathrm{d}t + \int_0^1 (1-x)^{k+1}\log^2\frac{1}{1-x}\,\mathrm{d}z\int_0^1 t(1-t)^{k-2}\,\mathrm{d}t\right),$$

which by (13) and (14) gives

$$E_k = \frac{k(k+1)}{(k+2)^2}\left(\gamma+\psi(k+2)\right) - \frac{k(k^2+2k+2)}{(k+2)^3}.$$

Therefore

$$\mathrm{E}\overline{D}_k^S D_k^S = \frac{k}{k+1}\left(\gamma+\psi(k+2)\right) - \frac{k}{k+2}\zeta(2,k+2) - \frac{k}{k+2},$$

and

$$\operatorname{cov}(\overline{D}_k^S, D_k^S) = -\frac{k}{k+2}\zeta(2,k+2) + \frac{k}{(k+1)(k+2)}.$$

Hence

$$\begin{aligned}\operatorname{var} D_k^P &= \operatorname{var}\overline{D}_k^S + \sigma^2 D_k^S + 2\operatorname{cov}\left(\overline{D}_k^S, D_k^S\right)\\ &= \frac{\pi^2-6}{3(k+2)} - \zeta(2,k+2) - \frac{2k}{k+2}\left(\zeta(2,k+2)-\beta'(k+2)\right) + \frac{k(2k+3)}{(k+1)^2(k+2)}\end{aligned}$$

which proves (23) after using the equality

$$\zeta(2,k+2) - \beta'(k+2) = \sum_{j=0}^{\infty}\frac{1+(-1)^j}{(j+k+2)^2} = \frac{1}{2}\zeta\left(2, \frac{k+2}{2}\right). \qquad \square$$

Corollary 3.
$$\operatorname{var} D_k^P \le \frac{\pi^2-7}{3(k+2)}.$$

P r o o f. Since $-\zeta(2,k+2) \le -\frac{1}{k+2}$ and $-\zeta\left(2,\frac{k+2}{2}\right) \le -\frac{2}{k+2}$ then

$$\operatorname{var} D_k^P \le \frac{\pi^2-7}{3(k+2)} - \frac{(2k-1)(k^2-1)+3}{3(k+1)^2(k+2)^2},$$

which proves the required assertion. $\square$

Remark 1. For $k \ge 7$
$$\operatorname{var} D_k^P \le \frac{\pi^2-8}{3(k+2)}.$$

Now we present the moments of D_k^G.

Proposition 3. The expectation and the variance of D_n^G are given by

$$\mathrm{E}D_k^G = \frac{k(k^2+5k+10)}{(k+2)_3}, \tag{25}$$

$$\operatorname{var} D_k^G = \frac{16k\left(k^6+15k^5+103k^4+435k^3+1282k^2+2700k+3240\right)}{(k+2)_3(k+2)_7}. \tag{26}$$

P r o o f. Using (17) with $f(x) = x(1-x) - x^2(1-x)^2$ write $\mathrm{E}D_k^G = \mathrm{E}\hat{D}_k^G + \mathrm{E}\overline{D}_k^G$, where

$$\mathrm{E}\hat{D}_k^G := k(k+1)\int_0^1 x(1-x)^k\,\mathrm{d}x = k(k+1)B(2,k+1) = \frac{k}{k+2}$$

and

$$\mathrm{E}\overline{D}_k^G := -k(k+1)\int_0^1 x^2(1-x)^{k+1}\,\mathrm{d}x = -k(k+1)B(3,k+2) = -\frac{2k(k+1)}{(k+2)_3},$$

which gives (25).

Next using (18) with $f(x) = x(1-x)$

$$\mathrm{E}\left(\hat{D}_k^G\right)^2 = k(k+1)\int_0^1 x^2(1-x)^{k+1}\,\mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2}x(1-x)y(1-y)\,\mathrm{d}y\,\mathrm{d}x.$$

Making the substitution $y = (1-x)t$ in the second integral we get

$$\mathrm{E}\left(\hat{D}_k^G\right)^2 = k(k+1)\int_0^1 x^2(1-x)^{k+1}\,\mathrm{d}x + k^2(k^2-1) \cdot \int_0^1\int_0^1 x(1-x)^{k+1}t(1-t)^{k-2}(1-(1-x)t)\,\mathrm{d}t\,\mathrm{d}x = \frac{k(k^2+5k+2)}{(k+2)_3}.$$

Similarly, from (18) and (19), we get

$$\mathrm{E}\left(\overline{D}_k^G\right)^2 = \frac{4k(k^3+18k^2+59k+18)}{(k+3)_6} \quad \text{and} \quad \mathrm{E}\,\hat{D}_k^G\overline{D}_k^G = -\frac{2k(k^2+8k+3)}{(k+3)_4}.$$

Hence

$$\operatorname{var}\hat{D}_k^G = \frac{4k}{(k+2)(k+2)_3},$$

$$\operatorname{var}\overline{D}_k^G = \frac{4k(k^6-3k^5-59k^4+147k^3+1714k^2+2520k+864)}{(k+2)_3(k+2)_7}$$

and

$$\operatorname{cov}\left(\hat{D}_k^G,\overline{D}_k^G\right) = \frac{4k(k^2-7k-6)}{(k+2)(k+2)_5}.$$

Finally (26) we obtain from

$$\operatorname{var}D_k^G = \operatorname{var}\hat{D}_k^G + \operatorname{var}\overline{D}_k^G + 2\operatorname{cov}\left(\hat{D}_k^G,\overline{D}_k^G\right). \qquad \square$$

Corollary 4.

$$\operatorname{var}D_k^G \le \frac{16}{k^3}$$

For α-entropy D_k^α we get

Proposition 4. The expectation and the variance of D_k^α are given by

$$\mathrm{E}D_k^\alpha = \frac{1}{2^{1-\alpha}-1}\left(k(k+1)B(\alpha+1,k)-1\right), \tag{27}$$

$$\operatorname{var} D_k^\alpha = \frac{k(k+1)}{(2^{1-\alpha}-1)^2}\big(B(2\alpha+1,k)(1+\alpha k B(\alpha+1,\alpha)) - k(k+1)B^2(\alpha+1,k)\big). \quad (28)$$

P r o o f. Using (17) and (18) we get

$$\mathrm{E}\sum_{j=0}^{k} Y_{j,k}^\alpha = k(k+1)\int_0^1 (1-x)^{k-1}x^\alpha\,\mathrm{d}x$$

and

$$\mathrm{E}\left(\sum_{j=0}^{k} Y_{j,k}^\alpha\right)^2 = k(k+1)\int_0^1 (1-x)^{k-1}x^{2\alpha}\,\mathrm{d}x + k^2(k^2-1)\int_0^1\int_0^{1-x}(1-x-y)^{k-2}x^\alpha y^\alpha\,\mathrm{d}y\mathrm{d}x,$$

which gives (27) and (28). □

Corollary 5.

$$\mathrm{E}D_k^\alpha \sim \frac{1}{2^{1-\alpha}-1}\left(\frac{\Gamma(\alpha+1)}{k^{\alpha-1}}-1\right), \quad (29)$$

$$\operatorname{var} D_k^\alpha \sim \frac{\Gamma(2\alpha+1)-\left(\alpha^2+1\right)\Gamma^2(\alpha+1)}{(2^{1-\alpha}-1)^2 k^{2\alpha-1}} \quad \text{(cf. [4])}. \quad (30)$$

P r o o f. Using the formula

$$\frac{\Gamma(k)}{\Gamma(k+\beta)} = \frac{1}{k^\beta} - \frac{\beta(\beta-1)}{2k^{\beta+1}} + o\left(\frac{1}{k^{\beta+1}}\right), \quad \beta\geq 0, \quad k\to\infty$$

(cf. [4], [24], p. 67, 3.31) in (27) and (28) we get the desired assertions. □

4. ASYMPTOTIC PROPERTIES

Let $U_0, U_1, \ldots, U_k$ be exponential distributed random variables with mean 1. It is known that

$$Y_{j,k} \overset{d}{=} \frac{U_j}{\sum_{i=0}^k U_i} \quad \text{for} \quad 0\leq j\leq k \quad (31)$$

(cf. [6]), and the equality holds in distribution. Slud in [23] established the rate of the almost sure convergence of the sequence $(D_k^S - \log(k+1))$. Using the above representation and the law of iterated logarithm he proved that

$$\log(k+1) - D_k^S + \gamma - 1 = O\left((\log\log k/k)^{\frac{1}{2}}\right) \quad \text{a.s.,} \quad k\to\infty.$$

We prove the complete convergence of that sequence.

Theorem 1. $$D_k^S - \log(k+1) \xrightarrow{c} \gamma - 1, \quad k \to \infty.$$

Proof. Let $\varepsilon > 0$. Using (31) we see that

$$\Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) = \Pr\left(\left|\log\frac{\sum_{j=0}^k U_j}{k+1} + \frac{\sum_{j=0}^k U_j \log\frac{1}{U_j}}{\sum_{j=0}^k U_j} - \gamma + 1\right| > \varepsilon\right)$$

$$\le \Pr\left(\left|\log\frac{\sum_{j=0}^k U_j}{k+1}\right| > \frac{\varepsilon}{2}\right) + \Pr\left(\left|\frac{\sum_{j=0}^k U_j \log\frac{1}{U_j}}{k+1} - \gamma + 1\right| > \frac{\varepsilon\sum_{j=0}^k U_j}{4(k+1)}\right)$$

$$+ \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)}\frac{1}{k+1}\sum_{j=0}^k U_j\right).$$

Now let $0 < \delta < 1$. Then

$$\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)}\frac{1}{k+1}\sum_{j=0}^k U_j\right)$$

$$\le \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)}\frac{1}{k+1}\sum_{j=0}^k U_j, \left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| \le \delta\right)$$

$$+ \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon}{4(1-\gamma)}\frac{1}{k+1}\sum_{j=0}^k U_j, \left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right)$$

$$\le \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \frac{\varepsilon(1-\delta)}{4(1-\gamma)}\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right).$$

Similarly

$$\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon}{4}\frac{1}{k+1}\sum_{j=0}^k U_j\right)$$

$$\le \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon(1-\delta)}{4}\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right).$$

Also by the Theorem of Hsu and Robbins (cf. [5], [16])

$$\sum_{k=1}^\infty \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon(1-\delta)}{4}\right) < \infty,$$

and

$$\sum_{k=1}^\infty \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^k U_j - 1\right| > \delta\right) < \infty.$$

Moreover, we see that

$$\sum_{k=1}^{\infty} \Pr\left(\left|\log \frac{\sum_{j=0}^{k} U_j}{k+1}\right| > \frac{\varepsilon}{2}\right) \leq \sum_{k=1}^{\infty} \Pr\left(\left|\frac{\sum_{j=0}^{k} U_j}{k+1} - 1\right| > e^{\frac{\varepsilon}{2}} - 1\right) + \sum_{k=1}^{\infty} \Pr\left(\left|\frac{\sum_{j=0}^{k} U_j}{k+1} - 1\right| > 1 - e^{-\frac{\varepsilon}{2}}\right) < \infty,$$

which ends the proof. □

Additional information can be obtained from Heyde's theorem [15], who proved that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} \Pr(|S_n - n\mu| > n\varepsilon) = \sigma^2, \tag{32}$$

where S_n is the sum of n i.i.d. random variables with mean μ and variance var. Here we obtain

Corollary 6.

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} \Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) \leq 18+16\left((1-\gamma)^2 + (2-\gamma)^2 + \frac{\pi^2}{3}\right).$$

Proof. Letting $\delta = \varepsilon$ in the inequalities of Theorem 1 we get

$$\Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) \leq 2 \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > \varepsilon\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > \frac{\varepsilon(1-\varepsilon)}{4(1-\gamma)}\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j \log\frac{1}{U_j} - \gamma + 1\right| > \frac{\varepsilon(1-\varepsilon)}{4}\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > e^{\frac{\varepsilon}{2}} - 1\right) + \Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j - 1\right| > 1 - e^{-\frac{\varepsilon}{2}}\right).$$

which by (32) gives

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{k=1}^{\infty} \Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) \leq (2 + 16(1-\gamma)^2)\mathrm{var}\, U_1 + 16\mathrm{var}\,(U_1 \log U_1).$$

and after using $\mathrm{var}\, U_1 = 1$ and $\mathrm{var}\,(U_1 \log U_1) = (2-\gamma)^2 + \frac{\pi^2}{3} + 1$ we complete the proof. □

Theorem 2. $$D_k^P - \log(k+1) \xrightarrow{c} \gamma, \quad k \to \infty.$$

P r o o f. Since $D_k^P = D_k^S + \overline{D}_k^S$ then it is enough to show $\overline{D}_k^S \xrightarrow{c} 1$. Using the inequality
$$(1-x)x \le (1-x)\log\frac{1}{1-x} \le x, \quad x < 1,$$
we have
$$\sum_{j=0}^{k} Y_{j,k} - \sum_{j=0}^{k} Y_{j,k}^2 \le \overline{D}_k^S = -\sum_{j=0}^{k}(1-Y_{j,k})\log(1-Y_{j,k}) \le \sum_{j=0}^{k} Y_{j,k}.$$
Hence for any given $\varepsilon > 0$
$$\sum_{k=1}^{\infty}\Pr\left(\left|\overline{D}_k^S - 1\right| > \varepsilon\right) \le \sum_{k=1}^{\infty}\Pr\left(\sum_{j=0}^{k} Y_{j,k}^2 > \varepsilon\right) \le \frac{16}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{16\zeta(3)}{\varepsilon^2} < \infty,$$
which ends the proof. □

Now taking into account that
$$\Pr\left(\left|D_k^P - \log(k+1) - \gamma\right| > 2\varepsilon\right) = \Pr\left(\left|D_k^S - \log(k+1) - \gamma + \overline{D}_k^S\right| > 2\varepsilon\right)$$
$$\le \Pr\left(\left|D_k^S - \log(k+1) - \gamma + 1\right| > \varepsilon\right) + \Pr\left(\left|\overline{D}_k^S - 1\right| > \varepsilon\right)$$
we get

Corollary 7.
$$\lim_{\varepsilon\to 0}\varepsilon^2\sum_{k=1}^{\infty}\Pr\left(\left|D_k^P - \log(k+1) - \gamma\right| > \varepsilon\right) \le 72+64\left((1-\gamma)^2 + (2-\gamma)^2 + \frac{\pi^2}{3} + \zeta(3)\right).$$

Theorem 3. $$D_k^G \xrightarrow{c} 1, \quad k \to \infty.$$

P r o o f. Let $\varepsilon > 0$. If $k \to \infty$ then $\mathrm{E}D_k^G \to 1$ and by Chebyshev's inequality and (26)
$$\sum_{k=1}^{\infty}\Pr\left(\left|D_k^G - \mathrm{E}D_k^G\right| > \varepsilon\right) \le \sum_{k=1}^{\infty}\frac{\operatorname{var} D_k^G}{\varepsilon^2} \le \frac{16}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{16\zeta(3)}{\varepsilon^2} < \infty,$$
which implies the theorem. □

Remark 2. Note that by (26)
$$\lim_{\varepsilon\to 0}\varepsilon^2\sum_{k=1}^{\infty} k\Pr\left(\left|D_k^G - \mathrm{E}D_k^G\right| > \varepsilon\right) \le \sum_{k=1}^{\infty}\frac{16}{k^2} = \frac{8\pi^2}{3}$$

In the proof of complete convergence of α-entropy we use the following theorem of Baum and Katz [1].

Theorem 4. (cf. Baum and Katz [1]) Let $\frac{1}{2} < \alpha \leq 1$ and $\{X_k, k \geq 1\}$ be the i.i.d. random variables. If $\mathrm{E}|X_k|^{\frac{2}{\alpha}} < \infty$, $\mathrm{E}X_k = \mu$ and $S_n = X_1 + \ldots + X_n$ then for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \Pr(|S_n - n\mu| > n^{\alpha}\varepsilon) < \infty.$$

Theorem 5. For $\alpha > \frac{1}{2}$

$$D_k^{\alpha} - (k+1)^{1-\alpha}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1} \xrightarrow{c} \frac{1}{1-2^{1-\alpha}}, \quad k \to \infty.$$

Proof. Let $\varepsilon > 0$. If $\alpha > 1$ then by (29) $\mathrm{E}D_k^{\alpha} \to \dfrac{1}{1-2^{1-\alpha}}$, $k \to \infty$. Using Chebyshev's inequality and (30)

$$\sum_{k=1}^{\infty} \Pr\left(|D_k^{\alpha} - \mathrm{E}D_k^{\alpha}| > \varepsilon\right) \leq \sum_{k=1}^{\infty} \frac{\operatorname{var} D_k^{\alpha}}{\varepsilon^2} \leq \sum_{k=1}^{\infty} \frac{C}{k^{2\alpha-1}} < \infty,$$

which implies the theorem.

Now let $\alpha \in \left(\frac{1}{2}, 1\right)$. Using (31) we see that

$$\Pr\left(\left|D_k^{\alpha} - (k+1)^{1-\alpha}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1} - \frac{1}{1-2^{1-\alpha}}\right| > \varepsilon\right) = \Pr\left((k+1)^{1-\alpha}\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j^{\alpha}\right.\right.$$
$$\left.\left. -\Gamma(\alpha+1)\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right| > \varepsilon\left(2^{1-\alpha}-1\right)\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right).$$

Now let $0 < \delta < 1$ and $\varepsilon_1 = \varepsilon\left(2^{1-\alpha}-1\right)$. Then

$$\Pr\left((k+1)^{1-\alpha}\left|\frac{1}{k+1}\sum_{j=0}^{k} U_j^{\alpha} - \Gamma(\alpha+1)\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right| > \varepsilon_1\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha}\right)$$
$$\leq \Pr\left(\left|\sum_{j=0}^{k}\left(U_j^{\alpha} - \Gamma(\alpha+1)\right)\right| > \varepsilon_1\frac{1-\delta}{2}(k+1)^{\alpha}\right) + \Pr\left(\left|\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha} - 1\right| > \delta\right)$$
$$+ \Pr\left((k+1)^{1-\alpha}\left|\left(\frac{1}{k+1}\sum_{j=0}^{k} U_j\right)^{\alpha} - 1\right| > \frac{\varepsilon_1(1-\delta)}{2\Gamma(\alpha+1)}\right).$$

Using Theorem 4 we see that

$$\sum_{k=1}^{\infty} \Pr\left(\left|\sum_{j=0}^{k}\left(U_j^{\alpha} - \Gamma(\alpha+1)\right)\right| > \frac{\varepsilon_1(1-\delta)}{2}(k+1)^{\alpha}\right) < \infty,$$

and by Theorem of Hsu and Robbins (cf. [5, 16])

$$\sum_{k=1}^{\infty}\Pr\left(\left|\left(\frac{1}{k+1}\sum_{j=0}^{k}U_j\right)^{\alpha}-1\right|>\delta\right)\le\sum_{k=1}^{\infty}\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k}U_j-1\right|>(\delta+1)^{\frac{1}{\alpha}}-1\right)$$
$$+\sum_{k=1}^{\infty}\Pr\left(\left|\frac{1}{k+1}\sum_{j=0}^{k}U_j-1\right|>1-(1-\delta)^{\frac{1}{\alpha}}\right)<\infty.$$

Now let $\varepsilon_2=\frac{\varepsilon_1(1-\delta)}{2\Gamma(\alpha+1)}$. Then

$$\Pr\left((k+1)^{1-\alpha}\left|\left(\frac{1}{k+1}\sum_{j=0}^{k}U_j\right)^{\alpha}-1\right|>\varepsilon_2\right)\le\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>(k+1)\right.$$
$$\left.\cdot\left(\left(1+\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}-1\right)\right)+\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>(k+1)\left(1-\left(1-\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}\right)\right).$$

Since $\left(\left(1+\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}-1\right)\sim\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha-1}$ and $\left(1-\left(1-\frac{\varepsilon_2}{(k+1)^{1-\alpha}}\right)^{\frac{1}{\alpha}}\right)\sim\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha-1}$ then

$$\sum_{k=1}^{\infty}\Pr\left((k+1)^{1-\alpha}\left|\left(\frac{1}{k+1}\sum_{j=0}^{k}U_j\right)^{\alpha}-1\right|>\varepsilon_2\right)$$
$$\le\sum_{k=1}^{\infty}\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha}\right)+\sum_{k=1}^{\infty}\Pr\left(\left|\sum_{j=0}^{k}(U_j-1)\right|>\frac{\varepsilon_2}{\alpha}(k+1)^{\alpha}\right)<\infty,$$

by Theorem 4. The proof is complete. □

Moreover for $0<\alpha\le\frac{1}{2}$ we get the following statement

Theorem 6. If $0<\alpha\le\frac{1}{2}$ then

$$k^{\alpha-1}D_k^{\alpha}\xrightarrow{a.s.}\frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1},\quad k\to\infty.\tag{33}$$

P r o o f. We see that $k^{\alpha-1}\mathrm{E}D_k^{\alpha}\sim\frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}$. Now by Chebyshev's inequality and (30)

$$\Pr\left(\left|k^{\alpha-1}D_k^{\alpha}-k^{\alpha-1}\mathrm{E}D_k^{\alpha}\right|>\varepsilon\right)\le\frac{\operatorname{var}D_k^{\alpha}k^{2\alpha-2}}{\varepsilon^2}\le\frac{C}{k}.$$

But for every k there exists an integer $m = m(k)$ with $m^2 < k \le (m+1)^2$. Hence $0 < k - m^2 \le 2m$ and $k \to \infty$ implies $m \to \infty$. Moreover

$$\sum_{m=1}^{\infty} \Pr\left(\left|m^{2(\alpha-1)} D^{\alpha}_{m^2} - m^{2(\alpha-1)} E D^{\alpha}_{m^2}\right| > \varepsilon\right) \le \sum_{m=1}^{\infty} \frac{C}{m^2} < \infty,$$

which gives

$$m^{2(\alpha-1)} D^{\alpha}_{m^2} \xrightarrow{c} \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}, \quad m \to \infty. \tag{34}$$

Since under the constraint $\sum_{j=1}^{k} x_i = 1$, $x_i \ge 0$, we have $\sum_{j=1}^{k} x_i^{\alpha} \le \frac{1}{k^{\alpha-1}}$, then

$$\left|k^{\alpha-1} D^{\alpha}_{k} - m^{2(\alpha-1)} D^{\alpha}_{m^2}\right| = \left|\left(k^{\alpha-1} - m^{2(\alpha-1)}\right) \sum_{j=0}^{m^2} Y^{\alpha}_{j,k} - k^{\alpha-1} \sum_{j=m^2+1}^{k} Y^{\alpha}_{j,k}\right|$$

$$\le \frac{k^{1-\alpha} - m^{2(1-\alpha)}}{k^{1-\alpha}} + \frac{\left(k-m^2\right)^{1-\alpha}}{k^{1-\alpha}} \le \left(1+\frac{1}{m}\right)^{2(1-\alpha)} - 1 + \left(\frac{2}{m}\right)^{1-\alpha} \quad \text{a.s..}$$

Therefore we get

$$\left|k^{\alpha-1} D^{\alpha}_{k} - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}\right| \le \left|k^{\alpha-1} D^{\alpha}_{k} - m^{2(\alpha-1)} D^{\alpha}_{m^2}\right| + \left|m^{2(\alpha-1)} D^{\alpha}_{m^2} - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}\right|$$

$$\le \left(1+\frac{1}{m}\right)^{2(1-\alpha)} - 1 + \left(\frac{2}{m}\right)^{1-\alpha} + \left|m^{2(\alpha-1)} D^{\alpha}_{m^2} - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}-1}\right|,$$

which by (34) implies (33). □

ACKNOWLEDGEMENT

The authors wish to thank anonymous referees for their useful comments and remarks which helped in preparation of the final version of the paper.

(Received September 12, 2006.)

REFERENCES

[1] L. E. Baum and M. Katz: Convergence rates in the law of large numbers. Trans. Amer. Math. Soc. *120* (1968), 108–123.
[2] M. Bieniek and D. Szynal: A contribution to results on random partitions of the segment. Internat. J. Pure and Appl. Math. *13* (2004), 3, 337–378.
[3] N. Burbea and N. Rao: Entropy differential metrics and divergence measures in probability spaces: a unified approach. J. Multivariate Anal. *12* (1982), 575–596.
[4] D. A. Darling: On a class of problems related to the random division of an interval. Ann. Math. Statist. *24* (1953), 239–253.
[5] P. Erdős: On a theorem of Hsu and Robbins. Ann. Math. Statist. *20* (1949), 286–291.
[6] W. Feller: An Introduction to Probability Theory and its Applications. Vol. II. Wiley, New York 1966.
[7] S. Goldstein: On entropy of random partitions of the segment [0, 1]. Bull. Soc. Sci. Lett. Łódź XXIV *4* (1974), 1–7.

[8] I. S. Gradstein and I. M. Ryzyk: Tables of Integrals, Sums, Series and Products. Fourth edition. Academic Press, New York – London 1965.
[9] R. L. Graham, D. E. Knuth, and O. Patashnik: Concrete Mathematics. Addison–Wesley Publishing Company Advanced Book Program, Reading, MA 1989.
[10] M. Ekstörm: Sum-functions of spacings of increasing order. J. Statist. Plann. Inference *136* (2006), 2535–2546.
[11] P. Hall: Limit theorems for sums of general functions of m-spacings. Math. Proc. Cambridge Philos. Soc. *96* (1984), 517–532.
[12] P. Hall: On power distributional tests based on sample spacings. J. Multivariate Anal. *19* (1986), 201–224.
[13] E. R. Hansen: A Table of Series and Products. Prentice-Hall, Englewood Clifts, N. J. 1975.
[14] J. Havrda and F. Charvát: Quantification method in classification process: Concept of structural α-entropy. Kybernetika *3* (1967), 30–35.
[15] C. C. Heyde: A suplement to the strong law of large numbers. J. Appl. Probab. *12* (1975), 173–175.
[16] P. L. Hsu, and H. Robbins: Complete convergence and the law of large numbers. Proc. Nat. Acad. Sci. U. S. A. *33* (1947), 25–31.
[17] B. D. H. Latter: Measures of genetic distance between indviduals and populations. Publ. Univ. Hawai, Honolulu, Genetic Structure of Populations (1973), 27–39.
[18] M. L. Menendez, D. Morales, L. Pardo, and M. Salicrú: Asymptotic distribution of $\{h, \phi\}$-entropies. Comm. Statist. – Theory Methods *22* (1993), 7, 2015–2031.
[19] N. Misra: A new test if uniformity based on overlapping sample spacings. Comm. Statist. – Theory Methods *30* (2001), 7, 1435–1470.
[20] A. Renyi: New nonadditive measures of entropy for discrete probability distributions. In: Proc. 4th Berkeley Symp. Math. Statist. and Prob. Vol. 1, 1961, pp. 547–561.
[21] C. E. Shannon: A mathematical theory of communications. Bell System Tech. J. *27* (1948), 379–425, 623–656.
[22] Y. Shao and R. Jimenez: Entropy for random partitons and its applications. J. Theoret. Probab. *11* (1998), 417–433.
[23] E. Slud: Entropy and maximal spacings for random partitions. Z. Warsch. verw. Gebiete *41* (1978), 341–352.
[24] N. M. Temme: Special Functions: An Introduction to the Classical Functions of Mathematical Physics. Wiley, New York 1996.
[25] H. M. Srivastava, S.-T. Tu, and T.-C. Wu: Some combinatorial series identities associated with the Digamma function and harmonic numbers. Appl. Math. Lett. *13* (2000), 101–106.

Milena Bieniek, Faculty of Economics, Statistic and Econometric Department, Maria Curie–Skłodowska University, Plac Marii Curie-Skłodowskiej 5, 20-031, Lublin. Poland.
e-mail: milena@ramzes.umcs.lublin.pl

Dominik Szynal, Institute of Mathematics, Maria Curie–Skłodowska University, Plac Marii Curie-Skłodowskiej 1, 20-031, Lublin. Poland.
e-mail: szynal@golem.umcs.lublin.pl