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A NEW FAMILY OF TRIVARIATE PROPER QUASI–COPULAS

Manuel Úbeda-Flores

In this paper, we provide a new family of trivariate proper quasi-copulas. As an application, we show that W^3 – the best-possible lower bound for the set of trivariate quasi-copulas (and copulas) – is the limit member of this family, showing how the mass of W^3 is distributed on the plane $x+y+z=2$ of $[0,1]^3$ in an easy manner, and providing the generalization of this result to n dimensions.

Keywords: copula, mass distribution, quasi-copula

AMS Subject Classification: 62H05, 60E05

1. INTRODUCTION

Let n be a natural number such that $n \geq 2$. An n-dimensional *copula* (briefly, n-copula) is the restriction to $[0,1]^n$ of a continuous n-variate distribution function whose univariate margins are uniform on $[0,1]$. Equivalently, an n-copula is a function $C\colon [0,1]^n \to [0,1]$ which satisfies the following conditions:

(C1) *boundary conditions*: for any $(u_1, u_2, \ldots, u_n)$ in $[0,1]^n$ it holds that $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0$ and $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i \in \{1, 2, \ldots, n\}$;

(C2) *the n-increasing property*: for every $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in [0,1]^n$, and each *$n$-box* B in $[0,1]^n$, i. e., $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, we have that $V_C(B) = \sum \operatorname{sgn}(c_1, c_2, \ldots, c_n) \cdot C(c_1, c_2, \ldots, c_n) \geq 0$ – $V_C(B)$ is defined as the *C-volume* of B –, where the sum is taken over all the *vertices* $(c_1, c_2, \ldots, c_n)$ of B (i. e., each c_k is equal to either a_k or b_k) and $\operatorname{sgn}(c_1, c_2, \ldots, c_n)$ is 1 if $c_k = a_k$ for an even number of k's, and -1 if $c_k = a_k$ for an odd number of k's.

The importance of copulas as a tool for statistical analysis and modeling stems largely from the observation that the joint distribution H of a random vector $(X_1, X_2, \ldots, X_n)$ with respective one-dimensional margins $F_1, F_2, \ldots, F_n$ can be expressed by

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \quad (x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n,$$

where C is an n-copula that is uniquely determined on $\operatorname{Range} F_1 \times \operatorname{Range} F_2 \times \cdots \times \operatorname{Range} F_n$. For a complete survey about copulas, see [14, 23, 24].

Alsina et al. [1] introduced the notion of *quasi-copula* in order to show that a certain class of operations on univariate distribution functions can, or cannot, be derived from corresponding operations on random variables defined on the same probability space (see also [19]). Cuculescu and Theodorescu [4] have given the characterization of an n-dimensional quasi-copula (or n-quasi-copula) as a function $Q\colon [0,1]^n \to [0,1]$ which satisfies condition (C1) of n-copulas, but instead of condition (C2), the weaker conditions:

(Q1) *monotonicity*: Q is nondecreasing in each variable;

(Q2) *Lipschitz condition*: for any $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ in $[0,1]^n$, it holds that $|Q(u_1, u_2, \ldots, u_n) - Q(v_1, v_2, \ldots, v_n)| \le \sum_{i=1}^n |u_i - v_i|$.

We will refer to $V_Q(B)$ – the Q-volume of B – as the mass accumulated by Q on B. Every n-quasi-copula Q satisfies the inequalities

$$\begin{aligned} W^n(u_1, u_2, \ldots, u_n) &= \max\left(0, \sum_{i=1}^n u_i - n + 1\right) \le Q(u_1, u_2, \ldots, u_n) \\ &\le \min(u_1, u_2, \ldots, u_n) = M^n(u_1, u_2, \ldots, u_n) \end{aligned}$$

for every $(u_1, u_2, \ldots, u_n)$ in $[0,1]^n$. While every n-copula is an n-quasi-copula, there exist *proper* n-quasi-copulas, i. e., n-quasi-copulas which are not n-copulas. For any $n \ge 2$, M^n is an n-copula; but W^n is an n-copula if and only $n = 2$, and a proper n-quasi-copula for $n \ge 3$.

One of the most important applications of quasi-copulas in statistics is the following result ([15, 17, 21]): Every pointwise ordered set of copulas has a least upper bound and greatest lower bound in the set of quasi-copulas. Of interest are sets of copulas of random variables with a specific statistical property (see [10, 11, 17, 18]). Furthermore, since quasi-copulas are a special type of binary aggregation operators satisfying the Lipschitz condition (Q2) (see [3]), these functions are becoming popular in fuzzy set theory (for instance, see [2, 8, 9, 12]).

In the literature, we cannot find many families of proper n-quasi-copulas when $n \ge 3$ – for some examples (different from W^n), see [7, 16, 22]. Recently, the mass distribution associated with a 3-quasi-copula and the differences with respect to the bivariate case – we recall that the (positive) mass of W^2 is distributed uniformly in $[0,1]^2$ on the segment which joins the points $(0,1)$ to $(1,0)$, and the (infinite positive and infinite negative) mass of W^3 is distributed on the plane $x + y + z = 2$ of $[0,1]^3$ – have been studied in [7, 13]. Our purpose is to provide a new family of proper 3-quasi-copulas whose bivariate margins are 2-copulas – moreover, we construct the least upper bound and the greatest lower bound in the set of quasi-copulas with those margins. As an application, we prove that W^3 is the limit member of this new family, showing how the mass of W^3 is distributed on the plane $x + y + z = 2$ of $[0,1]^3$ in an easy manner. In the last section, we provide the generalization of this problem to n dimensions.

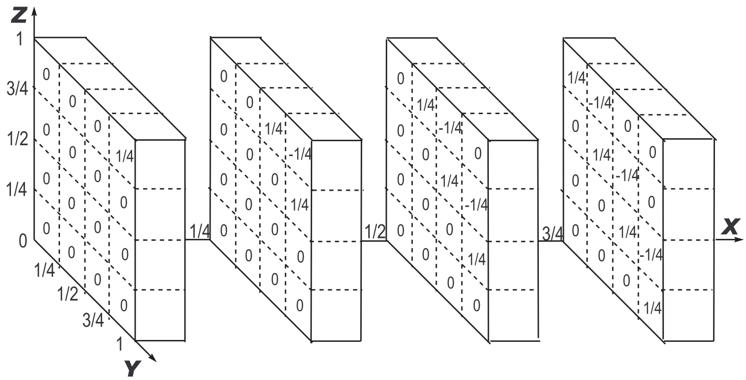

Fig. 1. Mass distribution of the trivariate function in Section 2 for $m = 4$.

2. A NEW FAMILY OF PROPER 3-QUASI-COPULAS

Let m be a natural number such that $m \geq 2$. We divide $[0,1]^3$ into m^3 3-boxes (or cubes, in this case), namely:

$$B_{i_1 i_2 i_3} = \left[\frac{i_1 - 1}{m}, \frac{i_1}{m}\right] \times \left[\frac{i_2 - 1}{m}, \frac{i_2}{m}\right] \times \left[\frac{i_3 - 1}{m}, \frac{i_3}{m}\right],$$

for all $i_1, i_2, i_3 = 1, 2, \ldots, m$. Now, we distribute $1/m$ of (positive) mass uniformly on each cube $B_{i_1 i_2 i_3}$ such that $i_1 + i_2 + i_3 = 2m + 1$; $-1/m$ of (negative) mass uniformly on each cube $B_{i_1 i_2 i_3}$ such that $i_1 + i_2 + i_3 = 2m + 2$; and 0 on the remaining cubes. It can be easily computed that there are $m(m+1)/2$ cubes with positive mass, and $m(m-1)/2$ cubes with negative mass; and the sum of positive mass is $(m+1)/2$, and the sum of negative mass is $-(m-1)/2$. Therefore, we have the amount of 1 of positive mass on $[0,1]^3$ (see Figure 1 for this construction in the case $m = 4$).

Note that if we project this construction on the planes $x = 1$, $y = 1$ and $z = 1$, we obtain a construction (similar on the three planes) with $1/m$ of (positive) mass distributed uniformly on each square of the form $R_{i_1(m-i_1+1)}$, for $i_1 = 1, 2, \ldots, m$, where

$$R_{i_1 i_2} = \left[\frac{i_1 - 1}{m}, \frac{i_1}{m}\right] \times \left[\frac{i_2 - 1}{m}, \frac{i_2}{m}\right],$$

for all $i_1, i_2 = 1, 2, \ldots, m$; and 0 on $[0,1]^2 \setminus R_{i_1(m-i_1+1)}$.

If (u_1, u_2, u_3) is a point in $[0,1]^3$, and $Q_m(u_1, u_2, u_3)$ is the mass spread on $[0, u_1] \times [0, u_2] \times [0, u_3]$, then Q_m is a proper 3-quasi-copula – whose three bivariate margins are 2-copulas –, as the following result shows.

Theorem 2.1. For each natural number $m \geq 2$, let $Q_m \colon [0,1]^3 \to [0,1]$ be the function defined by

$$\begin{aligned} &Q_m(u_1,u_2,u_3) \qquad (1)\\ &= \begin{cases} 0, & (u_1,u_2,u_3) \in B_1,\\ m^2 \prod_{j=1}^{3}\left(u_j - \frac{i_j-1}{m}\right), & (u_1,u_2,u_3) \in B_2,\\ m\sum_{k=1}^{3}\prod_{\substack{j=1\\ j\neq k}}^{3}\left(u_j - \frac{i_j-1}{m}\right) - m^2\prod_{j=1}^{3}\left(u_j - \frac{i_j-1}{m}\right), & (u_1,u_2,u_3) \in B_3,\\ u_1+u_2+u_3-2, & \text{otherwise,} \end{cases} \end{aligned}$$

where $B_1 = \{B_{i_1i_2i_3} : i_1+i_2+i_3 \leq 2m\}$, $B_2 = \{B_{i_1i_2i_3} : i_1+i_2+i_3 = 2m+1\}$, and $B_3 = \{B_{i_1i_2i_3} : i_1+i_2+i_3 = 2m+2\}$. Then, Q_m is a proper 3-quasi-copula for every $m \geq 2$ whose three bivariate margins (which are the same) are the 2-copula $C_m^{(2)}$ given by

$$C_m^{(2)}(v_1,v_2) = \begin{cases} 0, & (v_1,v_2) \in R_1,\\ m\prod_{j=1}^{2}\left(v_j - \frac{i_j-1}{m}\right), & (v_1,v_2) \in R_2,\\ v_1+v_2-1, & \text{otherwise,} \end{cases} \qquad (2)$$

where $R_1 = \{R_{i_1i_2} : i_1+i_2 \leq m\}$ and $R_2 = \{R_{i_1i_2} : i_1+i_2 = m+1\}$.

Proof. Suppose m is a fixed natural number such that $m \geq 2$, and let (u_1,u_2,u_3) be a point in $[0,1]^3$. First, we show that Q_m is well-defined. Let $B_{i_1i_2i_3} \in B_2$ and $B_{j_1j_2j_3} \in B_3$ be two cubes in $[0,1]^3$ such that $i_2 = j_2$ and $i_3 = j_3$ (all the other cases can be proved in a similar manner). Then we have that $j_1 = 1 + i_1$. Since

$$\begin{aligned} Q_m(u_1,u_2,u_3) &= m^2\left(u_1 - \frac{i_1-1}{m}\right)\prod_{k=2}^{3}\left(u_k - \frac{j_k-1}{m}\right),\\ &\qquad (u_1,u_2,u_3) \in \left[\frac{i_1-1}{m},\frac{i_1}{m}\right]\times\left[\frac{j_2-1}{m},\frac{j_2}{m}\right]\times\left[\frac{j_3-1}{m},\frac{j_3}{m}\right], \end{aligned}$$

in particular, we obtain that

$$Q_m\left(\frac{i_1}{m},u_2,u_3\right) = m^2\left(\frac{i_1}{m} - \frac{i_1-1}{m}\right)\prod_{k=2}^{3}\left(u_k - \frac{j_k-1}{m}\right) = m\prod_{k=2}^{3}\left(u_k - \frac{j_k-1}{m}\right);$$

and since

$$Q_m(u_1,u_2,u_3) \;=\; m\sum_{k=1}^{3}\prod_{\substack{j=1\\ j\neq k}}^{3}\Big(u_j-\frac{i_j-1}{m}\Big)-m^2\prod_{j=1}^{3}\Big(u_j-\frac{i_j-1}{m}\Big),$$

$$(u_1,u_2,u_3)\in\prod_{k=1}^{3}\Big[\frac{j_k-1}{m},\frac{j_k}{m}\Big],$$

in particular, we obtain that

$$Q_m\Big(\frac{i_1}{m},u_2,u_3\Big)=Q_m\Big(\frac{j_1-1}{m},u_2,u_3\Big)=m\prod_{k=2}^{3}\Big(u_k-\frac{j_k-1}{m}\Big).$$

To prove the boundary conditions, suppose $u_2=u_3=1$ (the cases $u_1=u_2=1$ and $u_1=u_3=1$ use similar arguments) in a cube $B_{i_1i_2i_3}\in B_3$ (all the remaining cases can be proved in a similar manner). Thus $i_2=i_3=m$, and hence $i_1=2$. Then, we obtain that

$$\begin{aligned}Q_m(u_1,1,1) \;&=\; m\Big[2\Big(u_1-\frac{1}{m}\Big)\Big(1-\frac{m-1}{m}\Big)+\Big(1-\frac{m-1}{m}\Big)^2\Big]\\ &\quad -m^2\Big(u_1-\frac{1}{m}\Big)\Big(1-\frac{m-1}{m}\Big)^2=u_1.\end{aligned}$$

In what follows, let (u_1',u_2,u_3) and (u_1,u_2,u_3) be two points in a cube $B_{i_1i_2i_3}$ such that $u_1'>u_1$ (the case $u_1'=u_1$ is trivial in the following). We now check that Q_m is nondecreasing in the first variable and satisfies the Lipschitz condition (Q2) in the same variable (the cases for the other two variables can be proved in a similar manner) and in each cube $B_{i_1i_2i_3}$. We consider two cases (the remaining cases are trivial).

(i) Suppose $B_{i_1i_2i_3}\in B_2$. Then we have

$$Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)=m^2(u_1'-u_1)\prod_{j=2}^{3}\Big(u_j-\frac{i_j-1}{m}\Big).$$

It is trivial that $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\geq 0$. On the other hand, we have that $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\leq u_1'-u_1$ if, and only if, $m^2\cdot\prod_{j=2}^{3}(u_j-(i_j-1)/m)\leq 1$. Since $0\leq u_j-(i_j-1)/m\leq 1/m$, for $j=2,3$, the result follows.

(ii) Suppose now $B_{i_1i_2i_3}\in B_3$. Then, we have that

$$\begin{aligned}&Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\\ =\;& m(u_1'-u_1)\Big[\sum_{j=2}^{3}\Big(u_j-\frac{i_j-1}{m}\Big)-m\prod_{j=2}^{3}\Big(u_j-\frac{i_j-1}{m}\Big)\Big].\end{aligned}$$

Thus, $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\geq 0$ if, and only if,

$$m\prod_{j=2}^{3}\left(u_j-\frac{i_j-1}{m}\right)\leq\sum_{j=2}^{3}\left(u_j-\frac{i_j-1}{m}\right). \tag{3}$$

Suppose $u_2-(i_2-1)/m>0$ and $u_3-(i_3-1)/m>0$ (the cases with the equality are trivial), then inequality (3) is equivalent to $m\leq\sum_{j=2}^{3}(u_j-(i_j-1)/m)^{-1}$. Since $u_2\in((i_2-1)/m,i_2/m]$, we have that $u_2\leq i_2/m=(i_2-1)/m+1/m$, thus $u_2-(i_2-1)/m\leq 1/m$ (and similarly for u_3); whence the result follows.

On the other hand, we have that $Q_m(u_1',u_2,u_3)-Q_m(u_1,u_2,u_3)\leq u_1'-u_1$ holds if, and only if, $m\prod_{j=2}^{3}(u_j-(i_j-1)/m)\geq\sum_{j=2}^{3}(u_j-(i_j-1)/m)+1/m$. Since $\prod_{j=2}^{3}(u_j-(i_j-1)/m)\geq 0$, i. e., $u_2u_3-u_2i_3/m-u_3i_2/m+i_2i_3/m^2\geq 0$, we have that $u_2u_3-u_2(i_3-1)/m-u_3(i_2-1)/m+(i_2-1)(i_3-1)/m^2\geq u_2/m+u_3/m-i_2/m^2-i_3/m^2+1/m^2$; whence the result follows.

Thus, we have proved that Q_m is a 3-quasi-copula. Now, since

$$\begin{aligned}
& V_{Q_m}\left(\left[\frac{1}{m},\frac{2}{m}\right]\times\left[\frac{m-1}{m},1\right]\times\left[\frac{m-1}{m},1\right]\right)\\
= {} & Q_m\left(\frac{2}{m},1,1\right)-Q_m\left(\frac{1}{m},1,1\right)-Q_m\left(\frac{2}{m},1,\frac{m-1}{m}\right)\\
& -Q_m\left(\frac{2}{m},\frac{m-1}{m},1\right)+Q_m\left(\frac{2}{m},\frac{m-1}{m},\frac{m-1}{m}\right)\\
& +Q_m\left(\frac{1}{m},1,\frac{m-1}{m}\right)+Q_m\left(\frac{1}{m},\frac{m-1}{m},1\right)\\
& -Q_m\left(\frac{1}{m},\frac{m-1}{m},\frac{m-1}{m}\right)=\frac{2}{m}-\frac{3}{m}=-\frac{1}{m},
\end{aligned}$$

we conclude that Q_m is a proper 3-quasi-copula.

Finally, since (as it is easy to check) the bivariate margins – or of higher dimension – of any n-quasi-copula are quasi-copulas, the three bivariate margins of Q_m – i. e., $Q_m(u_1,u_2,1)$, $Q_m(u_1,1,u_3)$ and $Q_m(1,u_2,u_3)$ – given by (2) are 2-copulas since the mass (only positive) of $C_m^{(2)}$ is distributed uniformly on $[0,1]^2$, which completes the proof. □

From Theorem 2.1, we first note that the 2-copulas given by (2) are a special type of orthogonal grid constructions of copulas studied in [6] with W^2 as background copula, and Π^2 – the copula of independent random variables, i. e., $\Pi^2(u,v)=uv$ for all (u,v) in $[0,1]^2$ – as foreground copula.

We also observe that there does not exist a 3-copula whose three bivariate margins are $C_m^{(2)}(u_1,u_2)$, $C_m^{(2)}(u_1,u_3)$ and $C_m^{(2)}(u_2,u_3)$ – this is related to the problem of the *compatibility* of three 2-copulas (for more details, see [5, 20]). The following result shows this fact.

Proposition 2.1. For any natural number $m\geq 2$, there does not exist a 3-copula whose three bivariate margins are the 2-copula $C_m^{(2)}$ given by (2).

P r o o f. Suppose C is a 3-copula whose three bivariate margins are $C_m^{(2)}$. Let B be the 3-box given by $B = [1/2, 1]^3$. Then we have that

$$\begin{aligned} V_C(B) &= C(1,1,1) - C\left(\frac{1}{2},1,1\right) - C\left(1,\frac{1}{2},1\right) - C\left(1,1,\frac{1}{2}\right) + C\left(\frac{1}{2},\frac{1}{2},1\right) \\ &\quad + C\left(\frac{1}{2},1,\frac{1}{2}\right) + C\left(1,\frac{1}{2},\frac{1}{2}\right) - C\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \\ &= 1 - \frac{3}{2} + 3\cdot C_m^{(2)}\left(\frac{1}{2},\frac{1}{2}\right) - C\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right). \end{aligned}$$

If m is even, it is easy to check that $V_C(B) = -1/2$ for every $m \geq 2$; and if m is odd, we have that $V_C(B) = (1-m)/(2m) < 0$ for every $m \geq 3$. In both cases we obtain a contradiction; therefore, C is not a 3-copula, which completes the proof. □

We also note that Q_m is not the unique proper 3-quasi-copula whose three bivariate margins are $C_m^{(2)}$ (for methods of constructing Lipschitz aggregation operators, see [2]). In fact, for any natural number $m \geq 2$, and given $C_m^{(2)}(u_1,u_2)$, $C_m^{(2)}(u_1,u_3)$ and $C_m^{(2)}(u_2,u_3)$, $(u_1,u_2,u_3) \in [0,1]^3$, we can construct an infinite number of proper 3-quasi-copulas whose three bivariate margins are $C_m^{(2)}$, as the following example shows.

Example 2.1. For every (u_1,u_2,u_3) in $[0,1]^3$, consider the function Q given by

$$Q(u_1,u_2,u_3) = \lambda\cdot Q_U(u_1,u_2,u_3) + (1-\lambda)\cdot Q_L(u_1,u_2,u_3),$$

where

$$Q_U(u_1,u_2,u_3) = \min(C_m^{(2)}(u_1,u_2), C_m^{(2)}(u_1,u_3), C_m^{(2)}(u_2,u_3))$$

and

$$\begin{aligned} &Q_L(u_1,u_2,u_3) \\ = \;&\max(0, C_m^{(2)}(u_1,u_2)+u_3-1, C_m^{(2)}(u_1,u_3)+u_2-1, C_m^{(2)}(u_2,u_3)+u_1-1), \end{aligned}$$

with $\lambda \in [0,1]$. Q_L and Q_U are two proper 3-quasi-copulas – whose three bivariate margins are $C_m^{(2)}$ – which satisfy the inequalities $Q_L(u_1,u_2,u_3) \leq Q_m(u_1,u_2,u_3) \leq Q_U(u_1,u_2,u_3)$ for every (u_1,u_2,u_3) in $[0,1]^3$ (see [22]). Observe that $Q_L(u_1,u_2,u_3) \neq Q_m(u_1,u_2,u_3) \neq Q_U(u_1,u_2,u_3)$ for some (u_1,u_2,u_3) in $[0,1]^3$ and for every $m \geq 2$. For instance, if i is a real number such that $3i = 2m+1$, after some elementary algebra we have that

$$Q_m\left(\frac{i}{m},\frac{i}{m},\frac{i}{m}\right) = \frac{1}{m} < \frac{m+2}{3m} = Q_U\left(\frac{i}{m},\frac{i}{m},\frac{i}{m}\right)$$

for any $m \geq 2$. Moreover, if we suppose that $i_1 = 1$ and $i_2 = i_3 = m$, then we have that

$$Q_L\left(\frac{1}{2m},1-\frac{1}{2m},1-\frac{1}{2m}\right) = 0 < \frac{1}{8m} = Q_m\left(\frac{1}{2m},1-\frac{1}{2m},1-\frac{1}{2m}\right)$$

for every $m \geq 2$.

3. APPROXIMATION OF W^3

In this section we show that W^3 is the limit member of the family of the proper 3-quasi-copulas defined by (1).

Theorem 3.1. Let $\varepsilon > 0$. For m sufficiently large, there exists a proper 3-quasi-copula Q_m given by (1) such that $|Q_m(u_1,u_2,u_3) - W^3(u_1,u_2,u_3)| < \varepsilon$ for all (u_1,u_2,u_3) in $[0,1]^3$.

Proof. Let m be a natural number such that $m \geq 6/\varepsilon$. We first prove that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{i_1+i_2+i_3}{m} - 2\right),$$

for every $i_1, i_2, i_3 = 1, 2, \ldots, m$. For that, we consider the following four cases:

(i) If $i_1 + i_2 + i_3 < 2m$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = 0 \quad \text{and} \quad W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{i_1+i_2+i_3}{m} - 2\right) = 0.$$

(ii) If $i_1 + i_2 + i_3 = 2m + 1$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = m^2 \prod_{j=1}^{3}\left(\frac{i_j}{m} - \frac{i_j - 1}{m}\right) = \frac{1}{m}$$

and

$$W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{2m+1}{m} - 2\right) = \frac{1}{m}.$$

(iii) If $i_1 + i_2 + i_3 = 2m + 2$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = 3m\left(\frac{1}{m}\right)^2 - m^2\left(\frac{1}{m}\right)^3 = \frac{2}{m}$$

and

$$W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{2m+2}{m} - 2\right) = \frac{2}{m}.$$

(iv) If $i_1 + i_2 + i_3 > 2m + 2$, then we have that

$$Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \frac{i_1+i_2+i_3}{m} - 2$$

and

$$W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) = \max\left(0, \frac{i_1+i_2+i_3}{m} - 2\right) = \frac{i_1+i_2+i_3}{m} - 2.$$

Now, let (u_1, u_2, u_3) be a point in $[0,1]^3$. We have $|u_1 - i_1/m| < 1/m$, $|u_2 - i_2/m| < 1/m$, and $|u_3 - i_3/m| < 1/m$ for some (i_1, i_2, i_3). Then

$$\begin{aligned} |Q_m(u_1,u_2,u_3) - W^3(u_1,u_2,u_3)| \leq & \left| Q_m(u_1,u_2,u_3) - Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right)\right| \\ & + \left| Q_m\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) - W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right)\right| \\ & + \left| W^3\left(\frac{i_1}{m},\frac{i_2}{m},\frac{i_3}{m}\right) - W^3(u_1,u_2,u_3)\right| \\ \leq & \; 2\left|u_1 - \frac{i_1}{m}\right| + 2\left|u_2 - \frac{i_2}{m}\right| + 2\left|u_3 - \frac{i_3}{m}\right| < \frac{6}{m} \leq \varepsilon, \end{aligned}$$

which completes the proof. □

As a consequence of Theorem 3.1, for m sufficiently large ($m \to \infty$), the mass of W^3 is distributed on the plane $x + y + z = 2$ of $[0,1]^3$ with subsets with arbitrarily large W^3-volume and subsets with arbitrarily small W^3-volume (see also [13, 14]).

4. CONCLUSION

In this paper, we have defined a new family of proper 3-quasi-copulas for which W^3 is the limit member of that family. Although our study is restricted to the trivariate case – for the sake of simplicity –, similar results can be obtained in higher dimensions – with a tedious algebra – by defining families of proper n-quasi-copulas in a similar manner. Let m be a natural number such that $m \geq 2$, and suppose $n \geq 3$. We divide $[0,1]^n$ into m^n n-boxes, namely:

$$B_{i_1 i_2 \ldots i_n} = \left[\frac{i_1 - 1}{m}, \frac{i_1}{m}\right] \times \left[\frac{i_2 - 1}{m}, \frac{i_2}{m}\right] \times \cdots \times \left[\frac{i_n - 1}{m}, \frac{i_n}{m}\right],$$

for all $i_1, i_2, \ldots, i_n = 1, 2, \ldots, m$. Now, we distribute $1/m$ of (positive) mass uniformly on each n-box $B_{i_1 i_2 \ldots i_n}$ such that $i_1 + i_2 + \cdots + i_n = (n-1)m + 1$; $-1/m$ of (negative) mass uniformly on each n-box $B_{i_1 i_2 \ldots i_n}$ such that $i_1 + i_2 + \cdots + i_n = (n-1)m + 2$; and 0 on the remaining n-boxes. For example, if $n = 4$, the number of 4-boxes with positive mass is $\sum_{i=2}^{m+1} \binom{i}{2}$, and the number of 4-boxes with negative mass is $\sum_{i=2}^{m+1} \binom{i}{2} - m$; then, the amount of positive and negative mass can be easily computed. Therefore, W^n – whose (infinite positive and infinite negative) mass is distributed on the set $\{(x_1, x_2, \ldots, x_n) \in [0,1]^n \,|\, x_1 + x_2 + \cdots + x_n = n - 1\}$ – is the member limit of this family of proper n-quasi-copulas.

Finally, we note that the family introduced in this paper (and its generalization to n-dimensions) could be much interesting in applications, especially in the construction of aggregation operators to fitting a data set.

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