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Krishnan Balachandran; S. Karthikeyan; Jeong-Hoon Kim

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CONTROLLABILITY OF SEMILINEAR STOCHASTIC INTEGRODIFFERENTIAL SYSTEMS

K. Balachandran, S. Karthikeyan and J.-H. Kim

In this paper we study the approximate and complete controllability of stochastic integrodifferential system in finite dimensional spaces. Sufficient conditions are established for each of these types of controllability. The results are obtained by using the Picard iteration technique.

Keywords: Controllability, approximate controllability, stochastic integrodifferential system, Picard iteration

AMS Subject Classification: 93B05

1. INTRODUCTION

The problem of controllability of linear deterministic system is well documented. It is well known that controllability of deterministic equations are widely used in analysis and the design of control system. Any control system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals. In many dynamical systems, it is possible to steer the dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls; that is there are systems which are completely controllable. If the system cannot be controlled completely then different types of controllability can be defined such as approximate, null, local null and local approximate null controllability.

The controllability of nonlinear deterministic systems in finite dimensional space has been extensively studied by several authors, see [1,5] and references therein. Controllability of linear stochastic systems in finite dimensional spaces has been studied by Dobov and Mordukhovich [3], Enrhardt and Kliemann [4], Mahmudov [9], Mahmudov and Denker [8] and Zabczyk [15]. There are very few works about controllability of nonlinear stochastic systems. In [14], the authors introduced the definitions of stochastic ϵ-controllability and controllability with probability and established sufficient conditions for stochastic controllability of a class of nonlinear systems. In [6], using a stochastic Lyapunov-like approach, sufficient conditions for stochastic ϵ-controllability are formulated. Balachandran and Dauer [2] and Mahmudov and Zorlu [12] studied the controllability of nonlinear stochastic systems.

The problem of controllability of a linear stochastic system of the form

$$\left.\begin{aligned} \mathrm{d}x(t) &= [Ax(t) + Bu(t)]\,\mathrm{d}t + \tilde{\sigma}(t)\,\mathrm{d}w(t), \quad t \in [0,T] \\ x(0) &= x_0, \end{aligned}\right\} \tag{1}$$

has been studied by various authors [11,14], where $\tilde{\sigma} : [0,T] \to R^{n\times n}$.

Mahmudov [10, 11] studied approximate controllability of non-linear stochastic system when nonlinear f and σ are uniformly bounded and satisfy the Lipschiz condition. Recently, Mahmudov and Zorlu [13] investigated the approximate and complete controllability of the following semilinear stochastic system

$$\mathrm{d}x(t) = [Ax(t) + Bu(t) + f(t, x(t), u(t))]\,\mathrm{d}t + \sigma(t, x(t), u(t))\,\mathrm{d}w(t)$$

with non-Lipschitz coefficients when f and σ depends on control u. They established the results by using the Picard type approximation.

In this paper we shall study the approximate and complete controllability of the following semilinear stochastic integrodifferential system

$$\left.\begin{aligned} \mathrm{d}x(t) &= \Big[Ax(t) + Bu(t) + f(t, x(t), u(t)) + \int_0^t g(t, s, x(s), u(s))\,\mathrm{d}s\Big]\,\mathrm{d}t \\ &\qquad + \sigma(t, x(t), u(t))\,\mathrm{d}w(t), \quad t \in [0,T] \\ x(0) &= x_0, \end{aligned}\right\} \tag{2}$$

where A and B are matrices of dimensions $n \times n, n \times m$ respectively, $g : [0,T] \times [0,T] \times R^n \times R^m \to R^n, f : [0,T] \times R^n \times R^m \to R^n, \sigma : [0,T] \times R^n \times R^m \to R^{n\times n}$ and w is an n-dimensional Wiener process. The results generalize the results of [13].

2. PRELIMINARIES

In this paper we use the following notations:

- $(\Omega, \mathcal{F}, P)$:= The probability space with probability measure P on Ω
- $\{\mathcal{F}_t | t \in [0,T]\}$:= The filtration generated by $\{w(s) : 0 \le s \le t\}$ and $\mathcal{F} = \mathcal{F}_T$.
- $L_2(\Omega, \mathcal{F}_T, R^n)$:= The Hilbert space of all $\mathcal{F}_T$-measurable square integrable variables with values in R^n.
- $L_2^{\mathcal{F}}([0,T], R^n)$:= The Hilbert space of all square integrable and $\mathcal{F}_t$-measurable processes with values in R^n.
- $C([0,T], L_2(\Omega, \mathcal{F}, P, X))$:= The Banach space of continuous maps from $[0,T]$ into $L_2(\Omega, \mathcal{F}, P, X)$ satisfying the condition $\sup_{t\in[0,T]} \mathsf{E}\|x(t)\|^2 < \infty$.
- X_s := The Banach space with norm topology given by $\|x\|_s^2 = \sup_{t\in[0,s]} \mathsf{E}\|x(t)\|^2$ which is a closed subspace of $C([0,T], L_2(\Omega, \mathcal{F}, P, X))$ consisting of measurable and $\mathcal{F}_t$-adapted processes $x(t)$.
- U_s := The Banach space with norm topology given by $\|u\|_s^2 = \sup_{t\in[0,s]} \mathsf{E}\|u(t)\|^2$ which is a closed subspace of $C([0,T], L_2(\Omega, \mathcal{F}, P, U))$ consisting of measurable and $\mathcal{F}_t$-adapted processes $u(t)$.
- $\mathcal{L}(X, Y)$:= The space of all linear bounded operators from a Banach space X to a Banach space Y.
- Denote $S(t) = \exp(At)$.

Now let us introduce the following operators and sets.

1. The operator $L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}([0,T],R^m), L_2(\Omega,\mathcal{F}_T,R^n))$ is defined by
$$L_0^T u = \int_0^T S(T-s)Bu(s)\,\mathrm{d}s.$$
Clearly the adjoint $(L_0^T)^* : L_2(\Omega,\mathcal{F}_T,R^n) \to L_2^{\mathcal{F}}([0,T],R^m)$ is defined by
$$(L_0^T)^* z = B^* S^*(T-t)\mathsf{E}\{z|\mathcal{F}_t\}.$$

2. The controllability matrix $\Gamma_s^T \in \mathcal{L}(R^n,R^n)$
$$\Gamma_s^T = \int_s^T S(T-t)BB^*S^*(T-t)\,\mathrm{d}t, \quad 0 \le s < t$$
and the resolvent operator
$$R(\alpha,\Gamma_s^T) = (\alpha I + \Gamma_s^T)^{-1}, \quad 0 \le s \le T.$$

3. Set of all states attainable from x_0 in time $t > 0$
$$\mathcal{R}_t(x_0) = \{x(t;x_0,u) : u(\cdot) \in L_2(\Omega,\mathcal{F}_T,R^n)\}$$
where $x(t,x_0,u)$ is the solution of (2) corresponding to $x_0 \in R^n$, $u(\cdot) \in L_2(\Omega,\mathcal{F}_T,R^n)$.

Now for our convenience, let us introduce the following notations:
$$\begin{aligned} M_B &= \|B\|, \quad M_S = \max\{\|S(t)\| : t \in [0,T]\}, \\ M_\Gamma &= \max\{\|\Gamma_s^T\| : s,t \in [0,T]\}. \end{aligned}$$

Definition 2.1. The stochastic system (2) is approximately controllable on $[0,T]$ if
$$\overline{\mathcal{R}_T(x_0)} = L_2(\Omega,\mathcal{F}_T,R^n)$$
that is, if it is possible to steer the system from the initial point x_0 to within a distance $\epsilon > 0$ from all the final points in the state space $L_2(\Omega,\mathcal{F}_T,R^n)$ at time T.

Definition 2.2. The stochastic system (2) is completely controllable on $[0,T]$ if
$$\mathcal{R}_T(x_0) = L_2(\Omega,\mathcal{F}_T,R^n),$$
that is, if all the points in $L_2(\Omega,\mathcal{F}_T,R^n)$ can be reached from the point x_0 at time T.

We assume the following conditions on the problem:

(H1) The functions f, g and σ satisfies the Lipschitz condition and there exist constants $L_1, L_2 > 0$ for $x_1, x_2 \in X$, $u_1, u_2 \in U$ and $0 \le s \le t \le T$
$$\begin{aligned} & \Big\| f(t,x_1,u_1) - f(t,x_2,u_2) \Big\|^2 + \|\sigma(t,x_1,u_1) - \sigma(t,x_2,u_2)\|^2 \\ \le\ & L_1(\|x_1 - x_2\|^2 + \|u_1 - u_2\|^2) \\ & \Big\| \int_0^t g(t,s,x_1(s),u_1(s)) - g(t,s,x_2(s),u_2(s))\,\mathrm{d}s \Big\|^2 \\ \le\ & L_2(\|x_1 - x_2\|^2 + \|u_1 - u_2\|^2) \end{aligned}$$

(H2) The functions f, g and σ are continuous and there exists a constant $L > 0$ such that,

$$\|f(t,x,u)\|^2 + \left\| \int_0^t g(t,s,x,u)\,\mathrm{d}s \right\|^2 + \|\sigma(t,x,u)\|^2 \le L(\|x\|^2 + \|u\|^2 + 1)$$

for all $t \in [0,T]$ and all $(x,u) \in X \times U$.

(H2)′ The functions f, g and σ are continuous and there exists a constant $M_f > 0$ such that

$$\|f(t,x,u)\|^2 + \left\| \int_0^t g(t,s,x,u)\,\mathrm{d}s \right\|^2 + \|\sigma(t,x,u)\|^2 \le M_f$$

for all $t, s \in [0,T]$ and all $(x,u) \in X \times U$.

(H3) The linear system (1) is approximately controllable.

(H4) The linear system (1) is completely controllable.

(H5) A is non-negative and self-adjoint.

(H6) BB^* is positive, that is there exists $\gamma > 0$ such that $\langle BB^*x, x\rangle \ge \gamma\|x\|^2$.

(AC) $\|\alpha R(\alpha, \Gamma_0^T)\| \to 0$ as $\alpha \to 0^+$.

Note that the assumptions (AC), (H3) and (H4) are equivalent, see [9]. The following lemmas whose proof can be found in [13] give a formula for a control steering the state x_0 to some neighborhood of an arbitrary point h.

Lemma 2.1. For arbitrary $f(\cdot) \in L_2^{\mathcal{F}}([0,T], R^n)$, $\sigma(\cdot) \in L_2^{\mathcal{F}}([0,T], R^{n\times n})$, $g(\cdot, t) \in L_2^{\mathcal{F}}([0,T], R^n)$, $h \in L_2(\Omega, \mathcal{F}, R^n)$ the control

$$\begin{aligned} u^\alpha(t) \;&=\; B^*S^*(T-t)(\alpha I + \Gamma_0^T)^{-1}(\mathsf{E}h - S(T)x_0) \\ &\quad - B^*S^*(T-t)\int_0^t (\alpha I + \Gamma_r^T)^{-1} S(T-r) f(r)\,\mathrm{d}r \\ &\quad - B^*S^*(T-t)\int_0^t (\alpha I + \Gamma_r^T)^{-1} (S(T-r)\sigma(r) - \varphi(r))\,\mathrm{d}w(r) \\ &\quad - B^*S^*(T-t)\int_0^t (\alpha I + \Gamma_r^T)^{-1} S(T-r) \left(\int_0^r g(r,s)\,\mathrm{d}s \right) \mathrm{d}r \end{aligned} \tag{3}$$

transfers the system

$$\begin{aligned} x(t) \;&=\; S(t)x_0 + \int_0^t S(t-s)Bu(s)\,\mathrm{d}s + \int_0^t S(t-s)f(s)\,\mathrm{d}s \\ &\quad + \int_0^t S(t-s)\sigma(s)\,\mathrm{d}w(s) + \int_0^t \int_0^s S(t-s) g(s,\tau)\,\mathrm{d}\tau\mathrm{d}s \end{aligned} \tag{4}$$

from $x_0 \in R^n$ to some neighbourhood of h at time T and

$$\begin{aligned} x_\alpha(T) \;&=\; h - \alpha(\alpha I + \Gamma_0^T)^{-1}(\mathsf{E}h - S(T)x_0) \\ &\quad + \int_0^T \alpha(\alpha I + \Gamma_r^T)^{-1} S(T-r) f(r)\,\mathrm{d}r \end{aligned}$$

$$+\int_0^T \alpha(\alpha I+\Gamma_r^T)^{-1}(S(T-r)\sigma(r)-\varphi(r))\,\mathrm{d}w(r)$$
$$+\int_0^T \alpha(\alpha I+\Gamma_r^T)^{-1}S(T-r)\left(\int_0^r g(r,s)\,\mathrm{d}s\right)\mathrm{d}r$$

where h has the following representation $h=\mathsf{E}h+\int_0^T\varphi(r)\,\mathrm{d}w(r)$, see [7].

Lemma 2.2. Let Assumptions (H4), (H5) and (H6) hold. Then there exists $C>0$ such that for all $g(\cdot)\in L_2^{\mathcal{F}}(0,T;R^n)$ the following inequality holds

$$\lim_{t\to T^-}\mathsf{E}\int_0^t\left\|\Gamma_r^T S^*(T-t)(\Gamma_r^T)^{-1}S(t-r)g(r)\right\|^2\mathrm{d}r\le C\int_0^T\mathsf{E}\|g(r)\|^2\,\mathrm{d}r. \qquad (5)$$

3. CONTROLLABILITY RESULTS

In this section we derive some controllability conditions for the semilinear stochastic integrodifferential system (2) by using the Picard approximation. In [8, 9] it is shown that complete controllability and approximate controllability of the linear system (1) coincide. But this may not always be true for semilinear stochastic integrodifferential systems.

In order to apply the Picard approximation we have to introduce the nonlinear operator $\Phi_\alpha,\ \alpha>0$ from $X_T\times U_T$ to $X_T\times U_T$ which is defined by

$$\Phi_\alpha(x,u)=(z,w) \qquad (6)$$

where

$$\begin{aligned}
z(t) &= S(t)x_0+\int_0^t S(t-r)Bw(r)\,\mathrm{d}r+\int_0^t S(t-r)f(r,x(r),u(r))\,\mathrm{d}r\\
&\quad+\int_0^t S(t-r)\Big[\int_0^r g(r,\tau,x(\tau),u(\tau))\,\mathrm{d}\tau\Big]\mathrm{d}r\\
&\quad+\int_0^t S(t-r)\sigma(r,x(r),u(r))\,\mathrm{d}w(r),\\
w(t) &= B^*S^*(T-t)\left[(\alpha I+\Gamma_0^T)^{-1}(\mathsf{E}h-S(T)x_0)+\int_0^t(\alpha I+\Gamma_r^T)^{-1}\varphi(r)\,\mathrm{d}w(r)\right]\\
&\quad-B^*S^*(T-t)\int_0^t(\alpha I+\Gamma_r^T)^{-1}S(T-r)f(r,x(r),u(r))\,\mathrm{d}r\\
&\quad-B^*S^*(T-t)\int_0^t(\alpha I+\Gamma_r^T)^{-1}S(T-r)\Big[\int_0^r g(r,\tau,x(\tau),u(\tau))\,\mathrm{d}\tau\Big]\mathrm{d}r\\
&\quad-B^*S^*(T-t)\int_0^t(\alpha I+\Gamma_r^T)^{-1}S(T-r)\sigma(r,x(r),u(r))\,\mathrm{d}w(r),
\end{aligned}$$

and $\varphi\in L_2^{\mathcal{F}}([0,T],R^{n\times n})$ comes from the representation $h=\mathsf{E}h+\int_0^T\varphi(r)\,\mathrm{d}w(r)$ of $h\in L_2(\Omega,\mathcal{F},R^n)$. It will be shown that the system (2) is approximately controllable if for all $\alpha > 0$ there exists a fixed point of the operator Φ_α. To show this we employ the Picard type approximations to (6).

$$\begin{aligned} x_0(t) &= S(t)x_0 \\ x_{n+1}(t) &= S(t)x_0 + \int_0^t S(t-r)Bu_{n+1}(r)\,\mathrm{d}r + \int_0^t S(t-r)f(r, x_n(r), u_n(r))\,\mathrm{d}r \\ &\quad + \int_0^t S(t-r)\Big[\int_0^r g(r,\tau,x_n(\tau),u_n(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}r \\ &\quad + \int_0^t S(t-r)\sigma(r,x_n(r),u_n(r))\,\mathrm{d}w(r) \qquad (7) \\ u_0(t) &= B^*S^*(T-t)\left[(\alpha I + \Gamma_0^T)^{-1}(\mathsf{E}h - S(T)x_0) + \int_0^t (\alpha I + \Gamma_r^T)^{-1}\varphi(r)\,\mathrm{d}w(r)\right] \\ u_{n+1}(t) &= B^*S^*(T-t)\left[(\alpha I + \Gamma_0^T)^{-1}(\mathsf{E}h - S(T)x_0) + \int_0^t (\alpha I + \Gamma_r^T)^{-1}\varphi(r)\,\mathrm{d}w(r)\right] \\ &\quad - B^*S^*(T-t)\int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)f(r,x_n(r),u_n(r))\,\mathrm{d}r \\ &\quad - B^*S^*(T-t)\int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)\Big[\int_0^r g(r,\tau,x_n(\tau),u_n(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}r \\ &\quad - B^*S^*(T-t)\int_0^t (\alpha I + \Gamma_r^T)^{-1}S(T-r)\sigma(r,x_n(r),u_n(r))\,\mathrm{d}w(r). \qquad (8) \end{aligned}$$

Lemma 3.1. Under the conditions (H1), (H2) the operator Φ_α is well defined and there exist $M_T(\alpha), k_1(\alpha), k_2(\alpha) > 0$ such that if $(x_1, u_1), (x_2, u_2) \in X_T \times U_T$ then

$$\begin{aligned} \|\Phi_\alpha(x_1,u_1) - \Phi_\alpha(x_2,u_2)\|_t^2 \le M_T(\alpha)(L_1+L_2)\Big\{&\int_0^t \Big(\sup_{0\le r\le s} \mathsf{E}\|x_1(r)-x_2(r)\|^2\Big)\,\mathrm{d}s \\ &+ \int_0^t \Big(\sup_{0\le r\le s} \mathsf{E}\|u_1(r)-u_2(r)\|^2\Big)\,\mathrm{d}s\Big\} \end{aligned}$$

and

$$\|\Phi_\alpha(x_1,u_1)\|_t^2 \le k_1(\alpha) + k_2(\alpha)LT\Big\{\sup_{0\le r\le s}\mathsf{E}\|x_1(r)\|^2 + \sup_{0\le r\le s}\mathsf{E}\|u_1(r)\|^2 + 1\Big\},$$

for each $t \in [0,T]$, where

$$\begin{aligned} M_T(\alpha) &= \max\Big\{4M_S^2T + \frac{3}{\alpha^2}M_S^4M_B^2T + \frac{12}{\alpha^2}M_S^6M_B^4T^2, 4M_S^2 + \frac{3}{\alpha^2}M_S^4M_B^2 + \frac{12}{\alpha^2}M_S^6M_B^4T, \\ &\qquad 4M_S^2T^2 + \frac{3}{\alpha^2}M_S^4M_B^2T^2 + \frac{12}{\alpha^2}M_S^6M_B^4T^3\Big\}, \\ k_1(\alpha) &= 5M_S^2\mathsf{E}\|x_0\|^2 + \frac{40}{\alpha^2}M_S^4M_B^4T\Big\{2\|\mathsf{E}h\|^2 + 2M_S^2\mathsf{E}\|x_0\| + \int_0^T \mathsf{E}\|\varphi(r)\|^2\,\mathrm{d}r\Big\}, \\ k_2(\alpha) &= \Big(5M_S^2 + \frac{4}{\alpha^2}M_S^4M_B^2 + \frac{20}{\alpha^2}M_S^6M_B^4T\Big)\max\{T^2, T, 1\}. \end{aligned}$$

P r o o f. Let us consider

$$\|\Phi_\alpha(x_1,u_1)-\Phi_\alpha(x_2,u_2)\|_t^2=\sup_{0\le s\le t}\mathsf{E}\|z_1(s)-z_2(s)\|^2+\sup_{0\le s\le t}\mathsf{E}\|w_1(s)-w_2(s)\|^2$$

$$\begin{aligned}
\le\ & 4M_S^2M_B^2\int_0^t \mathsf{E}\|w_1(s)-w_2(s)\|^2\,\mathrm{d}s\\
&+4M_S^2T\mathsf{E}\int_0^t \|f(r,x_1(r),u_1(r))-f(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+4M_S^2\mathsf{E}\int_0^t \|\sigma(r,x_1(r),u_1(r))-\sigma(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+4M_S^2T^2\mathsf{E}\Big\|\int_0^r (g(r,\tau,x_1(\tau),u_1(\tau))-g(r,\tau,x_2(\tau),u_2(\tau)))\,\mathrm{d}\tau\Big\|^2\\
&+\frac{3}{\alpha^2}M_S^4M_B^2T^2\mathsf{E}\int_0^t \|f(r,x_1(r),u_1(r))-f(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+\frac{3}{\alpha^2}M_S^4M_B^2\mathsf{E}\int_0^t \|\sigma(r,x_1(r),u_1(r))-\sigma(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+\frac{3}{\alpha^2}M_S^4M_B^2T^2\mathsf{E}\Big\|\int_0^r (g(r,\tau,x_1(\tau),u_1(\tau))-g(r,\tau,x_2(\tau),u_2(\tau)))\,\mathrm{d}\tau\Big\|^2\\
\le\ & \left(4M_S^2T+\frac{3}{\alpha^2}M_S^4M_B^2T+\frac{12}{\alpha^2}M_S^6M_B^4T^2\right)\\
&\times\int_0^t \mathsf{E}\|f(r,x_1(r),u_1(r))-f(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+\left(4M_S^2+\frac{3}{\alpha^2}M_S^4M_B^2+\frac{12}{\alpha^2}M_S^6M_B^4T\right)\\
&\times\int_0^t \mathsf{E}\|\sigma(r,x_1(r),u_1(r))-\sigma(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+\left(4M_S^2T^2+\frac{3}{\alpha^2}M_S^4M_B^2T^2+\frac{12}{\alpha^2}M_S^6M_B^4T^3\right)\\
&\times\mathsf{E}\Big\|\int_0^r (g(r,\tau,x_1(\tau),u_1(\tau))-g(r,\tau,x_2(\tau),u_2(\tau)))\,\mathrm{d}\tau\Big\|^2\\
\le\ & M_T(\alpha)\bigg\{\int_0^t \mathsf{E}\|f(r,x_1(r),u_1(r)))-f(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+\int_0^t \mathsf{E}\|\sigma(r,x_1(r),u_1(r))-\sigma(r,x_2(r),u_2(r))\|^2\,\mathrm{d}r\\
&+\mathsf{E}\Big\|\int_0^r (g(r,\tau,x_1(\tau),u_1(\tau))-g(r,\tau,x_2(\tau),u_2(\tau)))\,\mathrm{d}\tau\Big\|^2\bigg\}\\
\le\ & M_T(\alpha)(L_1+L_2)\bigg\{\int_0^t\left(\sup_{0\le r\le s}\mathsf{E}\|x_1(r)-x_2(r)\|^2\right)\mathrm{d}s\\
&+\int_0^t\left(\sup_{0\le r\le s}\mathsf{E}\|u_1(r)-u_2(r)\|^2\right)\mathrm{d}s\bigg\}.
\end{aligned}$$

Observe that standard computations yield,

$$
\begin{aligned}
\|\Phi_\alpha(x_1,u_1)\|_t^2 &= \sup_{0\le s\le t} \mathsf{E}\|z_1(s)\|^2 + \sup_{0\le s\le t} \mathsf{E}\|w_1(s)\|^2 \\
\le\ & 5M_S^2\mathsf{E}\|x_0\|^2 + 5M_S^2M_B^2\int_0^t \mathsf{E}\|w(s)\|^2 s + 5M_S^2T\mathsf{E}\int_0^t \|f(r,x_1(r),u_1(r))\|^2\,\mathrm{d}r \\
& +5M_S^2\mathsf{E}\int_0^t \|\sigma(r,x_1(r),u_1(r))\|^2\,\mathrm{d}r + 5M_S^2T^2\mathsf{E}\Big\|\int_0^r g(r,\tau,x_1(\tau),u_1(\tau))\,\mathrm{d}r\Big\|^2 \\
& +4M_S^2M_B^2\left\{\frac{2}{\alpha^2}(2\|\mathsf{E}h\|^2 + 2M_S^2\mathsf{E}\|x_0\|^2) + \frac{2}{\alpha^2}\int_0^t \mathsf{E}\|\varphi(r)\|^2\,\mathrm{d}r\right\} \\
& +\frac{4}{\alpha^2}M_S^4M_B^2T\int_0^t \mathsf{E}\|f(r,x_1(r),u_1(r))\|^2\,\mathrm{d}r + \frac{4}{\alpha^2}M_S^4M_B^2\int_0^t \mathsf{E}\|\sigma(r,x_1(r),u_1(r)\|^2\,\mathrm{d}r \\
& +\frac{4}{\alpha^2}M_S^4M_B^2T^2\mathsf{E}\Big\|\int_0^r g(r,\tau,x_1(\tau),u_1(\tau))\,\mathrm{d}\tau\Big\|^2 \\
\le\ & 5M_S^2\mathsf{E}\|x_0\|^2 + \frac{40}{\alpha^2}M_S^4M_B^4T\left\{2\|\mathsf{E}h\|^2 + 2M_S^2\mathsf{E}\|x_0\| + \int_0^T \mathsf{E}\|\varphi(r)\|^2\,\mathrm{d}r\right\} \\
& + \left(5M_S^2T + \frac{4}{\alpha^2}M_S^4M_B^2T + \frac{20}{\alpha^2}M_S^6M_B^4T^2\right)\mathsf{E}\int_0^t \|f(r,x_1(r),u_1(r))\|^2\,\mathrm{d}r \\
& + \left(5M_S^2 + \frac{4}{\alpha^2}M_S^4M_B^2 + \frac{20}{\alpha^2}M_S^6M_B^4T\right)\mathsf{E}\int_0^t \|\sigma(r,x_1(r),u_1(r))\|^2\,\mathrm{d}r \\
& + \left(5M_S^2T^2 + \frac{4}{\alpha^2}M_S^4M_B^2T^2 + \frac{20}{\alpha^2}M_S^6M_B^4T^3\right)\mathsf{E}\Big\|\int_0^r g(r,\tau,x_1(\tau),u_1(\tau))\,\mathrm{d}\tau\Big\|^2 \\
\le\ & k_1(\alpha) + k_2(\alpha)LT\left\{\sup_{0\le r\le s}\mathsf{E}\|x_1(r)\|^2 + \sup_{0\le r\le s}\mathsf{E}\|u_1(r)\|^2 + 1\right\}. \qquad \square
\end{aligned}
$$

Lemma 3.2. Under the conditions (H1), (H2) the sequence (x_n, u_n) is bounded in $X_T \times U_T$.

P r o o f. By Lemma 3.1 for any $n \ge 0$ we have

$$
\begin{aligned}
\|(x_{n+1},u_{n+1})\|^2 &= \sup_{0\le s\le t}\mathsf{E}\|x_{n+1}(s)\|^2 + \sup_{0\le s\le t}\mathsf{E}\|u_{n+1}(s)\|^2 \\
&\le k_1 + k_2LT\left\{\sup_{0\le r\le s}\mathsf{E}\|x_n(r)\|^2 + \sup_{0\le r\le s}\mathsf{E}\|u_n(r)\|^2 + 1\right\} \qquad (9)
\end{aligned}
$$

where k_1, k_2 are positive constants independent of n. Then by (9) and successive approximation, we obtain that

$$
\begin{aligned}
\|(x_{n+1},u_{n+1})\|^2 \le\ & (k_1 + k_2LT)\Big[1 + k_2LT + \cdots + k_2^nL^nT^n\Big] \\
& +(k_2LT)^{n+1}\left\{\mathsf{E}\|x_0(t)\|^2 + \mathsf{E}\|u_0(t)\|^2 + 1\right\} \\
\le\ & (k_1 + k_2LT)\Big[1 + k_2LT + k_2^2L^2T^2 + \cdots + k_2^nL^nT^n\Big] + (k_2LT)^{n+1}C_0
\end{aligned}
$$

where $C_0 = 1 + M_S^2 \mathsf{E}\|x_0\|^2 + \dfrac{2}{\alpha^2} M_S^2 M_B^2 \Big\{ 2\|Eh\|^2 + 2M_S^2 \mathsf{E}\|x_0\|^2 + \int_0^T \mathsf{E}\|\varphi(r)\|^2 \, \mathrm{d}r \Big\}$.
Thus, we have

$$\|(x_{n+1}, u_{n+1})\|^2 \leq (k_1 + k_2 LT) \frac{1 - (k_2^2 LT)^n}{1 - (k_2^2 LT)} + (k_2 LT)^{n+1} C_0. \qquad \square$$

Lemma 3.3. Under the conditions (H1), (H2) the sequence (x_n, u_n) is a Cauchy sequence in $X_T \times U_T$.

Proof. Let us take

$$\begin{aligned} r_n(t) &= \sup_{m \geq n} \|(x_m, u_m) - (x_n, u_n)\|_t^2, \\ p_n(t) &= \sup_{m \geq n} \|x_m - x_n\|_t^2, \\ q_n(t) &= \sup_{m \geq n} \|u_m - u_n\|_t^2. \end{aligned}$$

The functions $r_n, p_n, q_n, n \geq 0$, are well defined, uniformly bounded and evidently monotone non-decreasing. Since $\{r_n(t) : n \geq 0\}, \{p_n(t) : n \geq 0\}, \{q_n(t) : n \geq 0\}$ are monotone non-increasing sequences for each $t \in [0, T]$, there exists a monotone non-decreasing function $(r(t), p(t), q(t))$ such that

$$\lim_{n \to \infty} (r_n(t), p_n(t), q_n(t)) = (r(t), p(t), q(t)).$$

By Lemma 3.1 we obtain that

$$\begin{aligned} \|\Phi_\alpha(x_m, u_m) - \Phi_\alpha(x_n, u_n)\|_t^2 \leq \; & M_T(\alpha)(L_1 + L_2) \int_0^t \Big\{ \sup_{0 \leq r \leq s} \mathsf{E}\|x_{m-1}(r) - x_{n-1}(r)\|^2 \\ & + \sup_{0 \leq r \leq s} \mathsf{E}\|u_{m-1}(r) - u_{n-1}(r)\|^2 \Big\} \, \mathrm{d}s \end{aligned}$$

from which it follows that

$$\begin{aligned} r(t) &\leq r_n(t) = p_n(t) + q_n(t) \\ &\leq M_T(\alpha)(L_1 + L_2) \int_0^t \Big\{ \sup_{0 \leq r \leq s} \mathsf{E}\|x_{m-1}(r) - x_{n-1}(r)\|^2 \\ &\qquad + \sup_{0 \leq r \leq s} \mathsf{E}\|u_{m-1}(r) - u_{n-1}(r)\|^2 \Big\} \, \mathrm{d}s \\ &= M_T(\alpha)(L_1 + L_2) \int_0^t \Big[p_{n-1}(s) + q_{n-1}(s) \Big] \, \mathrm{d}s. \end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain

$$r(t) \leq p(t) + q(t) \leq M_T(\alpha)(L_1 + L_2) \int_0^t \Big[p(s) + q(s) \Big] \, \mathrm{d}s.$$

Now if $w = p + q$, then

$$w'(t) \leq M_T(\alpha)(L_1 + L_2)[p(t) + q(t)] \leq 2M_T(\alpha)(L_1 + L_2)w(t)$$

and also we see that $w(0) = 0$. Then, by Grownwall's inequality it follows that $w(t) = 0$ for all $t \in [0, T]$. But

$$\|(x_m, u_m) - (x_n, u_n)\|_T^2 \leq p_n(T) + q_n(T) \to w(T) = 0.$$

Therefore $\|(x_m, u_m) - (x_n, u_n)\|_T^2 \to 0$ as $n, m \to \infty$. □

Theorem 3.1. Under the conditions (H1), (H2) the operator (6) has a unique fixed point.

Proof. By Lemma 3.3 the sequence (x_n, u_n) is Cauchy in $X_T \times U_T$. The completeness of $X_T \times U_T$ implies the existence of a process $(x, u) \in X_T \times U_T$ such that

$$\lim_{n\to\infty} \|(x_n, u_n) - (x, u)\|_T^2 = 0.$$

Hence taking the limit in (7) we see that (x, u) is a fixed point of Φ_α. Further, if $(x_1, u_1), (x_2, u_2) \in X_T \times U_T$ are two fixed points of Φ_α, then Lemma 3.1 would imply that

$$\begin{aligned}\|\Phi_\alpha(x_1, u_1) - \Phi_\alpha(x_2, u_2)\|_t^2 &\leq M_T(\alpha)(L_1 + L_2)\Big\{\int_0^t \Big(\sup_{0\leq r\leq s} \mathsf{E}\|x_1(r) - x_2(r)\|^2 \\ &\quad + \sup_{0\leq r\leq s} \mathsf{E}\|u_1(r) - u_2(r)\|^2\Big)\,\mathrm{d}s\Big\}\end{aligned}$$

So as in the proof Lemma 3.3 we obtain that

$$\|\Phi_\alpha(x_1, u_1) - \Phi_\alpha(x_2, u_2)\|_T^2 = 0$$

Consequently $(x_1, u_1) = (x_2, u_2)$ in $X_T \times U_T$. Hence Φ_α has a unique fixed point.□

If $\alpha = 0$ the nonlinear operator Φ_0 is defined by

$$\Phi_0(x, u) = (z, w) \tag{10}$$

where

$$\begin{aligned}z(t) &= S(t)x_0 + \int_0^t S(t - r)Bw(r)\,\mathrm{d}r + \int_0^t S(t - r)f(r, x(r), u(r))\,\mathrm{d}r \\ &\quad + \int_0^t S(t - r)\sigma(r, x(r), u(r))\,\mathrm{d}w(r) + \int_0^t S(t - r)\Big[\int_0^r g(r, \tau, x(\tau), u(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}r, \\ w(t) &= B^*S^*(T - t)\left[(\Gamma_0^T)^{-1}(\mathsf{E}h - S(T)x_0) + \int_r^t (\Gamma_0^T)^{-1}\varphi(r)\,\mathrm{d}w(r)\right] \\ &\quad - B^*S^*(T - t)\int_0^t (\Gamma_r^T)^{-1}S(T - r)f(r, x(r), u(r))\,\mathrm{d}r\end{aligned}$$

$$-B^*S^*(T-t)\int_0^t (\Gamma_r^T)^{-1}S(T-r)\sigma(r,x(r),u(r))\,\mathrm{d}w(r)$$
$$-B^*S^*(T-t)\int_0^t (\Gamma_r^T)^{-1}S(T-r)\Big[\int_0^r g(r,\tau,x(\tau),u(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}r.$$

Theorem 3.2. Assume hypotheses (H1), (H2) and (H4) hold. Then the operator Φ_0 has a fixed point.

Proof. The proof is similar to that of Theorem 3.1. Note that here we need to use estimation (5) from Lemma 2.2. □

Theorem 3.3. Assume hypotheses (H1), (H2)′ and (H3) are satisfied. Then the system (2) is approximately controllable.

Proof. Let (x^α, u^α) be a fixed point of Φ_α in $X_T \times U_T$. By Lemma 2.1, x^α satisfies the following equality

$$\begin{aligned} x^\alpha(T) &= h-\alpha(\alpha I+\Gamma_0^T)^{-1}(\mathsf{E}h-S(T)x_0)\\ &\quad+\int_0^T \alpha(\alpha I+\Gamma_r^T)^{-1}S(T-r)f(r,x^\alpha(r),u^\alpha(r))\,\mathrm{d}r\\ &\quad+\int_0^T \alpha(\alpha I+\Gamma_r^T)^{-1}(S(T-r)\sigma(r,x^\alpha(r),u^\alpha(r))-\varphi(r))\,\mathrm{d}w(r)\\ &\quad+\int_0^T \alpha(\alpha I+\Gamma_r^T)^{-1}S(T-r)\Big[\int_0^r g(r,\tau,x^\alpha(\tau),u^\alpha(\tau)d\tau\Big]\,\mathrm{d}r. \end{aligned} \tag{11}$$

By (11) and the assumption (H2),

$$\begin{aligned} \mathsf{E}\|x^\alpha(T)-h\|^2 &\le 5\|\alpha R(\alpha,\Gamma_0^T)(\mathsf{E}h-S(T)x_0)\|^2\\ &+5T\int_0^T \mathsf{E}\|\alpha R(\alpha,\Gamma_r^T)S(T-r)f(r,x^\alpha(r),u^\alpha(r))\|^2\,\mathrm{d}r\\ &+5\int_0^T \mathsf{E}\|\alpha R(\alpha,\Gamma_r^T)S(T-r)\sigma(r,x^\alpha(r),u^\alpha(r))\|^2\,\mathrm{d}r\\ &+5\int_0^T \mathsf{E}\|\alpha R(\alpha,\Gamma_r^T)\varphi(r)\|^2\,\mathrm{d}r\\ &+5T\int_0^T \mathsf{E}\|\alpha R(\alpha,\Gamma_r^T)S(T-r)\Big[\int_0^r g(r,\tau,x^\alpha(\tau),u^\alpha(\tau))d\tau\Big]\|^2\,\mathrm{d}r\\ &\le 5\|\alpha R(\alpha,\Gamma_r^T)\|^2\|\mathsf{E}h-S(T)x_0\|^2\\ &+5T\int_0^T \|\alpha R(\alpha,\Gamma_r^T)\|^2\mathsf{E}\|S(T-r)f(r,x^\alpha(r),u^\alpha(r))\|^2\,\mathrm{d}r\\ &+5\int_0^T \|\alpha R(\alpha,\Gamma_r^T)\|^2\mathsf{E}\|S(T-r)\sigma(r,x^\alpha(r),u^\alpha(r))\|^2\,\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&+5\int_0^T \|\alpha R(\alpha,\Gamma_r^T)\|^2\mathsf{E}\|\varphi(r)\|^2\,\mathrm{d}r \\
&+5T\int_0^T \|\alpha R(\alpha,\Gamma_r^T)\|^2\mathsf{E}\Big\|S(T-r)\Big[\int_0^r g(r,\tau,x^\alpha(\tau),u^\alpha(\tau))\,\mathrm{d}\tau\Big]\Big\|^2\,\mathrm{d}r \\
\le\ & 5\|\alpha R(\alpha,\Gamma_r^T)\|^2\|\mathsf{E}h-S(T)x_0\|^2+5M_S^2M_f(2T+1)\int_0^T \|\alpha R(\alpha,\Gamma_r^T)\|^2\,\mathrm{d}r \\
&+5\int_0^T \|\alpha R(\alpha,\Gamma_r^T)\|^2\mathsf{E}\|\varphi(r)\|^2\,\mathrm{d}r.
\end{aligned}
$$

Since $\|\alpha R(\alpha,\Gamma_r^T)\|^2\le 1$, $\|\alpha R(\alpha,\Gamma_r^T)\|^2\to 0$ as $\alpha\to 0^+$ for all $0\le r\le T$, by the Lebesque dominated convergence theorem $\mathsf{E}\|x^\alpha(T)-h\|^2\to 0$ as $\alpha\to 0^+$. This gives the approximate controllability. □

Theorem 3.4. Assume hypotheses (H1)–(H6) are satisfied. Then the system (2) is completely controllable.

Proof. By Theorem 3.2, the operator Φ_0 has a fixed point. So, the control

$$
\begin{aligned}
u_0(t) \;=\;& B^*S^*(T-t)(\Gamma_0^T)^{-1}(\mathsf{E}h-S(T)x_0) \\
&-B^*S^*(T-t)\int_0^t (\Gamma_r^T)^{-1}S(T-r)f(r,x(r),u(r))\,\mathrm{d}r \\
&-B^*S^*(T-t)\int_0^t (\Gamma_r^T)^{-1}(S(T-r)\sigma(r,x(r),u(r))-\varphi(r))\,\mathrm{d}w(r) \\
&-B^*S^*(T-t)\int_0^t (\Gamma_r^T)^{-1}S(T-r)\Big[\int_0^r g(r,\tau,x(\tau),u(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}r
\end{aligned}
$$

transfers the system (2) from x_0 to h. Hence, the theorem is proved. □

4. EXAMPLE

Consider the following semilinear stochastic integrodifferential system

$$
\left.
\begin{aligned}
\mathrm{d}x(t) &= \Big[Ax(t)+Bu(t)+f(t,x(t),u(t))+\int_0^t g(t,s,x(s),u(s))\,\mathrm{d}s\Big]\,\mathrm{d}t \\
&\qquad +\sigma(t,x(t),u(t))\,\mathrm{d}w(t),\quad t\in[0,T] \\
x(0) &= x_0,
\end{aligned}
\right\}
\tag{12}
$$

where $w(t)$ is one-dimensional Brownian motion and

$$
A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\quad B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
f(t,x(t),u(t))=\begin{bmatrix} (2+\cos x_2(t))x_1(t)+3x_2(t)+u_1(t) \\ (3+\sin x_1(t))x_2(t)+2x_1(t)+u_2(t) \end{bmatrix}
$$

$$\int_0^t g(t,s,x(s),u(s))ds = \begin{bmatrix} \int_0^t (e^{-x_1(s)} + u_1(s))\,\mathrm{d}s \\ \int_0^t e^{-s}(5x_1(s) + 3x_2(s) + u_2(s))\,\mathrm{d}s \end{bmatrix}$$

$$\sigma(t,x(t),u(t)) = \begin{bmatrix} \frac{(2t^2+1)e^{-t}}{(1+x_1(t)+u_1(t))} \\ \frac{\sin t\ \cos t\ e^{-t}}{(1+x_2(t)+u_2(t))} \end{bmatrix}$$

The corresponding iterative scheme for (12) is

$$\begin{aligned} x_{n+1}(t) \quad = \quad & S(t)x_0 + \int_0^t S(t-r)Bu_{n+1}(r)dr + \int_0^t S(t-r)f(r,x_n(r),u_n(r))\,\mathrm{d}r \\ & + \int_0^t S(t-r)\Big[\int_0^r g(r,\tau,x_n(\tau),u_n(\tau))\,\mathrm{d}\tau\Big]\,\mathrm{d}r \\ & + \int_0^t S(t-r)\sigma(r,x_n(r),u_n(r))\,\mathrm{d}w(r) \qquad (13) \end{aligned}$$

where the fundamental matrix $S(t)$ is given by

$$S(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

The controllability matrix is given by

$$\Gamma_0^T = \int_0^T S(T-t)BB^*S^*(T-t)\,\mathrm{d}t = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = TI$$

and it is nonsingular for $T > 0$. Moreover, it is easy to show that for all $(x,u) \in R^2 \times R^2$, $|f(t,x(t),u(t))|^2 \leq 75(|x|^2 + |u|^2 + 1)$, $\left|\int_0^t g(t,s,x(s),u(s))\,\mathrm{d}s\right|^2 \leq 40(T + 1)(1 + |x|^2 + |u|^2)$, $|\sigma(t,x(t),u(t))| \leq 2(2t^2+1)e^{-t}$. By defining a suitable control (8) and by applying the Picard iteration technique to (13), one can establish the approximate and complete controllability of the stochastic system (12).

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Krishnan Balachandran and Shanmugasundaram Karthikeyan, Department of Mathematics, Bharathiar University, Coimbatore – 641 046. India.
e-mails: balachandran_k@lycos.com; karthi82india@gmail.com

Jeong-Hoon Kim, Department of Mathematics, Yonsei University, Seoul 120-749. Korea.
e-mail: jhkim96@yonsei.ac.kr