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AN EXISTENCE RESULT ON PARTITIONING OF A MEASURABLE SPACE: PARETO OPTIMALITY AND CORE

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This paper investigates the problem of optimal partitioning of a measurable space among a finite number of individuals. We demonstrate the sufficient conditions for the existence of weakly Pareto optimal partitions and for the equivalence between weak Pareto optimality and Pareto optimality. We demonstrate that every weakly Pareto optimal partition is a solution to the problem of maximizing a weighted sum of individual utilities. We also provide sufficient conditions for the existence of core partitions with non-transferable and transferable utility.

Keywords: optimal partitioning, nonatomic finite measure, nonadditive set function, Pareto optimality, core

AMS Subject Classification: 28A10, 28B05, 90C29, 91B32

1. INTRODUCTION

This paper investigates the problem of optimal partitioning of a measurable space among a finite number of individuals. Unlike previous works such as Barbanel and Zwicker [1], and Dubins and Spanier [3] who imposed additivity on preferences of each individual, we assume that preferences on a σ-field of individual i are represented by a continuous transformation of a measure with the form of a nonadditive set function u_i defined by $u_i(A_i) = f_i(\mu_i(A_i))$, where μ_i is a nonatomic finite measure of a measurable space, A_i is a measurable set and f_i is a continuous function on the range of μ_i.

We first demonstrate that if f_i is continuous for each i, then weakly Pareto optimal partitions exist; if f_i is concave for each i, then a partition is weakly Pareto optimal if and only if it is a solution to the maximization problem of a weighted sum of individual utilities for some weight vector; if f_i is continuous and strictly increasing, and each μ_i is absolutely continuous with respect to each other, then weak Pareto optimality is equivalent to Pareto optimality.

Next we introduce the concept of a cooperative game in the partitioning of a measurable space. We demonstrate that if f_i is continuous and quasi-concave for each i, then core partitions with non-transferable utility (NTU) exist; if f_i is continuous and concave for each i, then core partitions with transferable utility (TU) exist.

2. PRELIMINARIES

Let $\mu_1, \dots, \mu_n$ be finite measures of a σ-field $\mathcal{F}$ of Ω. A *partition* of Ω is an n-tuple of disjoint elements of $\mathcal{F}$ whose union is Ω. We denote the set of partitions of Ω by $\mathcal{P}_n$. The image of the n-dimensional vector-valued measure $(\mu_1, \dots, \mu_n)$ under $\mathcal{P}_n$ is given by

$$M = \{(\mu_1(A_1), \dots, \mu_n(A_n)) \in \mathbb{R}^n \mid (A_1, \dots, A_n) \in \mathcal{P}_n\}.$$

Define the set of *weakly efficient points* of M by

$$M^w = \{(a_1, \dots, a_n) \in M \mid \nexists\, (b_1, \dots, b_n) \in M : a_i < b_i \ \forall\, i = 1, \dots, n\}$$

and the set of *efficient points* of M by

$$\begin{aligned} M^* = \{(a_1, \dots, a_n) \in M \mid \nexists\, (b_1, \dots, b_n) \in M : a_i \le b_i \ \forall\, i = 1, \dots, n \\ \text{and } a_i < b_i \ \exists\, i\}. \end{aligned}$$

The following result plays a crucial role in the existence argument in the subsequent sections. The proof is found in [5].

Proposition 1. Let $\mu_1, \dots, \mu_n$ be nonatomic finite measures of a measurable space $(\Omega, \mathcal{F})$. Then it follows:

(i) M is compact and convex in $\mathbb{R}^n$.

(ii) If μ_i is absolutely continuous with respect to μ_j for each i and j, then $M^* = M^w$.

3. PARETO OPTIMAL PARTITION

Let f_i be a function on the interval $[0, \mu_i(\Omega)]$ for $i = 1, \dots, n$. Denote the finite set of individuals by $I = \{1, \dots, n\}$. Suppose that preferences of each individual on $\mathcal{F}$ are represented by a real-valued set function u_i on $\mathcal{F}$ of the form $u_i(A) = f_i(\mu_i(A))$ for each $i \in I$.

Definition 1. A partition $(A_1, \dots, A_n)$ of Ω is:

(i) *Weakly Pareto optimal* if there exists no partition $(B_1, \dots, B_n)$ such that $u_i(A_i) < u_i(B_i)$ for each $i \in I$.

(ii) *Pareto optimal* if no partition $(B_1, \dots, B_n)$ exists such that $u_i(A_i) \le u_i(B_i)$ for each $i \in I$ and $u_i(A_i) < u_i(B_i)$ for some $i \in I$.

Denote the $(n-1)$-dimensional *unit simplex* by Δ^{n-1}, that is,

$$\Delta^{n-1} = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \ge 0,\, i = 1, \dots, n \right\}.$$

Theorem 1. Let $\mu_1, \dots, \mu_n$ be nonatomic finite measures of a measurable space $(\Omega, \mathcal{F})$. Then it follows:

(i) If f_i is continuous for each $i \in I$, then there exists a weakly Pareto optimal partition.

(ii) If f_i is concave for each $i \in I$, then a partition is weakly Pareto optimal if and only if it solves the problem

$$\max \left\{ \sum_{i \in I} \alpha_i u_i(A_i) \mid (A_1, \dots, A_n) \in \mathcal{P}_n \right\} \qquad (P_\alpha)$$

for some $\alpha = (\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$.

(iii) If μ_i is absolutely continuous with respect to μ_j for each $i, j \in I$ and f_i is continuous and strictly increasing for each $i \in I$, then a partition is Pareto optimal if and only if it is weakly Pareto optimal.

Proof. (i) Let f_i be continuous for each $i \in I$. Since M is compact by Proposition 1 (i), for any given $\alpha = (\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$ the function $\sum_{i \in I} \alpha_i f_i$ attains a maximum at some point $(a_1, \dots, a_n)$ in M by the continuity of f_i. Let $(A_1, \dots, A_n)$ be a partition satisfying $\mu_i(A_i) = a_i$ for each $i \in I$. Then $(A_1, \dots, A_n)$ solves (P_α). If $(A_1, \dots, A_n)$ is not weakly Pareto optimal, then there exists a partition $(B_1, \dots, B_n)$ such that $f_i(\mu_i(A_i)) < f_i(\mu_i(B_i))$ for each $i \in I$, which implies $\sum_{i \in I} \alpha_i f_i(\mu_i(A_i)) < \sum_{i \in I} \alpha_i f_i(\mu_i(B_i))$, which contradicts the fact that $(A_1, \dots, A_n)$ solves (P_α). Therefore, $(A_1, \dots, A_n)$ is weakly Pareto optimal.

(ii) Let f_i be concave for each $i \in I$. The proof of part (i) demonstrates that if a partition solves (P_α) for $\alpha \in \Delta^{n-1}$, it is weakly Pareto optimal. We shall prove the converse implication. Let $(A_1, \dots, A_n)$ be a weakly Pareto optimal partition. Define the *utility possibility set* by

$$U = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \exists\, (a_1, \dots, a_n) \in M : x_i \le f_i(a_i)\ \forall\, i \in I\}.$$

Note that U is closed by the compactness of M assured in Proposition 1 (i) and the continuity of f_i, and is convex by the concavity of f_i, and the utility vector $(f_1(\mu_1(A_1)), \dots, f_n(\mu_n(A_n)))$ is in the boundary of U by the weak Pareto optimality of $(A_1, \dots, A_n)$. There then exists a nonzero vector $(\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ such that $\sum_{i \in I} \beta_i x_i \le \sum_{i \in I} \beta_i f_i(\mu_i(A_i))$ for any $(x_1, \dots, x_n) \in U$ by virtue of the supporting hyperplane theorem. Since U is unbounded from below, we can derive $\beta_i \ge 0$ for each $i \in I$. Normalizing $\alpha_i = (\sum_{i \in I} \beta_i)^{-1} \beta_i$ for each $i \in I$ yields $\alpha = (\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$ and $\sum_{i \in I} \alpha_i x_i \le \sum_{i \in I} \alpha_i f_i(\mu_i(A_i))$ for any $(x_1, \dots, x_n) \in U$. Since $(f_1(\mu_1(B_1)), \dots, f_n(\mu_n(B_n))) \in U$ for any $(B_1, \dots, B_n) \in \mathcal{P}_n$, we obtain $\sum_{i \in I} \alpha_i f_i(\mu_i(B_i)) \le \sum_{i \in I} \alpha_i f_i(\mu_i(A_i))$ for any $(B_1, \dots, B_n) \in \mathcal{P}_n$. Therefore, $(A_1, \dots, A_n)$ solves (P_α).

(iii) The fact that Pareto optimality implies weak Pareto optimality is evident. We demonstrate the converse implication. Let $(A_1, \dots, A_n)$ be a weakly Pareto optimal partition. Suppose to the contrary that $(A_1, \dots, A_n)$ is not Pareto optimal.

Then there exists a partition $(B_1, \ldots, B_n)$ such that $f_i(\mu_i(A_i)) \le f_i(\mu_i(B_i))$ for each $i \in I$ and $f_i(\mu_i(A_i)) < f_i(\mu_i(B_i))$ for some $i \in I$. Hence, $(\mu_1(A_1), \ldots, \mu_n(A_n))$ does not belong to M^*. The weak Pareto optimality of $(A_1, \ldots, A_n)$ implies that $(\mu_1(A_1), \ldots, \mu_n(A_n))$ belongs to M^w, and hence it belongs to M^* by Proposition 1 (ii), a contradiction. Therefore, $(A_1, \ldots, A_n)$ is Pareto optimal. □

Remark 1. Theorem 1 contains a generalization of the results with the additivity hypothesis provided by the literature. Part (i) extends the result of Dubins and Spanier [3] and part (ii) extends the result of Barbanel and Zwicker [1]. In proving part (iii) the role of the mutual absolute continuity was recognized by Dubins and Spanier [3], and Barbanel and Zwicker [1], but it is impossible to find an available proof in the literature.

4. CORE PARTITION WITH NTU

A nonempty subset of I is called a *coalition*. We denote the collection of coalitions by $\mathcal{N}$. Let $(\Omega_1, \ldots, \Omega_n) \in \mathcal{P}_n$ be an *initial partition* in which individual $i \in I$ is endowed with a measurable subset Ω_i of Ω. A partition $(A_1, \ldots, A_n)$ is an *S-partition* if $\bigcup_{i\in S} A_i = \bigcup_{i\in S} \Omega_i$ for coalition S.

Definition 2. A coalition S *improves upon* a partition $(A_1, \ldots, A_n)$ with NTU if there exists some S-partition $(B_1, \ldots, B_n)$ such that $u_i(A_i) < u_i(B_i)$ for each $i \in S$. A partition with NTU that cannot be improved upon by any coalition is a *core partition* with NTU.

It is obvious from the definitions that a core partition with NTU is weakly Pareto optimal. Note that if f_i is continuous and strictly increasing for each $i \in I$ and μ_i is mutually absolutely continuous, then a core partition with NTU is also Pareto optimal by Theorem 1 (iii).

Theorem 2. Let $\mu_1, \ldots, \mu_n$ be nonatomic finite measures of a measurable space $(\Omega, \mathcal{F})$. If f_i is continuous and quasi-concave for each $i \in I$, then there exists a core partition with NTU.

Proof. Define the n-person game $V : \mathcal{N} \to 2^{\mathbb{R}^n}$ with NTU by

$$V(S) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \exists\, (a_1, \ldots, a_n) \in M : x_i \le f_i(a_i)\ \forall\, i \in S\}.$$

The core of the n-person game V, denoted by $\mathrm{Core}(V)$, is defined by

$$\mathrm{Core}(V) = \{(x_1, \ldots, x_n) \in V(I) \mid \nexists\, S \in \mathcal{N}\ \nexists\, y \in V(S) : x_i < y_i\ \ \forall\, i \in S\}.$$

By the compactness of M asserted in Proposition 1 (i) and the continuity of f_i, it follows that $V(S)$ is closed.

We demonstrate that V is a balanced game. To this end, let $\mathcal{B}$ be a balanced family with balanced weights $\{\lambda^S \geq 0 \mid S \in \mathcal{B}\}$. Let $\mathcal{B}_i = \{S \in \mathcal{B} \mid i \in S\}$. We then have $\sum_{S\in\mathcal{B}_i} \lambda^S = 1$ for each $i \in I$. Define

$$\chi_i^S = \begin{cases} 1 & \text{if } S \in \mathcal{B}_i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t^S = \frac{1}{n}\sum_{i\in I} \lambda^S \chi_i^S.$$

As a result we have:

$$\sum_{S\in\mathcal{B}} t^S = \frac{1}{n}\sum_{S\in\mathcal{B}}\sum_{i\in I}\lambda^S\chi_i^S = \frac{1}{n}\sum_{i\in I}\left(\sum_{S\in\mathcal{B}}\lambda^S\chi_i^S\right) = \frac{1}{n}\sum_{i\in I}\left(\sum_{S\in\mathcal{B}_i}\lambda^S\right) = 1.$$

Choose any $(x_1, \ldots, x_n) \in \bigcap_{S\in\mathcal{B}} V(S)$. Then for each $S \in \mathcal{B}$ there exists some $(a_1^S, \ldots, a_n^S) \in M$ such that $x_i \leq f_i(a_i^S)$ for each $i \in S$. Define $b_i = \sum_{S\in\mathcal{B}} t^S a_i^S$ for each $i \in I$. By the convexity of M asserted in Proposition 1 (i), we have $(b_1, \ldots, b_n) \in M$. By the quasi-concavity of f_i, we have $x_i \leq f_i(b_i)$ for each $i \in I$. We thus obtain $(x_1, \ldots, x_n) \in V(I)$. Therefore, $\bigcap_{S\in\mathcal{B}} V(S) \subset V(I)$, and consequently V is balanced.

Since the balanced game V obviously satisfies other sufficient conditions guaranteeing the nonemptiness of the core of V (see Scarf [7]), we can choose an element $(x_1, \ldots, x_n)$ in Core(V). Then, there exists some $(a_1, \ldots, a_n) \in M$ such that $x_i \leq f_i(a_i)$ for each $i \in I$, and hence there exists a partition $(A_1, \ldots, A_n)$ such that $x_i \leq f_i(\mu_i(A_i))$ for each $i \in I$. Suppose that $(A_1, \ldots, A_n)$ is not a core partition. Then, there exists some S-partition $(B_1, \ldots, B_n)$ such that $f_i(\mu_i(A_i)) < f_i(\mu_i(B_i))$ for each $i \in S$. We then have $(f_1(\mu_1(B_1)), \ldots, f_n(\mu_n(B_n))) \in V(S)$ and $x_i < f_i(\mu_i(B_i))$ for each $i \in S$, which contradicts the fact that $(x_1, \ldots, x_n)$ is in Core(V). $\square$

Remark 2. The nonemptiness of core partitions with NTU when the utility function of each individual is a quasi-concave continuous transformation of a nonatomic finite measure appears to be a new result. Sagara and Vlach [6] investigated more general convex continuous preferences to demonstrate the existence of core partitions with NTU.

5. CORE PARTITION WITH TU

The definition of core partitions is modified in the case of TU as follows:

Definition 3. A coalition S *improves upon* a partition $(A_1, \ldots, A_n)$ with TU if there exists some S-partition $(B_1, \ldots, B_n)$ such that $\sum_{i\in S} u_i(A_i) < \sum_{i\in S} u_i(B_i)$. A partition with TU that cannot be improved upon by any coalition is a *core partition* with TU.

It is obvious from the definitions that a core partition with TU is weakly Pareto optimal and that a core partition with TU is a core partition with NTU. Note that if f_i is continuous and strictly increasing for each $i \in I$ and μ_i is mutually absolutely continuous, then a core partition with TU is also Pareto optimal by Theorem 1 (iii).

Theorem 3. Let $\mu_1, \dots, \mu_n$ be nonatomic finite measures of a measurable space $(\Omega, \mathcal{F})$. If f_i is continuous and concave for each $i \in I$, then there exists a core partition with TU.

Proof. Let f_i be continuous and concave for each $i \in I$. Since f_i is bounded for each $i \in I$, there exists a constant $\bar{f}$ such that $f_i - \bar{f} \le 0$ for each $i \in I$. Thus, without loss of generality, we may assume that f_i has nonpositive real values for each $i \in I$ by replacing f_i with $f_i - \bar{f}$ in Definition 3, if necessary. Define the n-person game $v : \mathcal{N} \to \mathbb{R}$ with TU by

$$v(S) = \max \left\{ \sum_{i \in S} f_i(a_i) \mid (a_1, \dots, a_n) \in M \right\}.$$

By the compactness of M asserted in Proposition 1 (i) and the continuity of f_i, the maximum in the above is indeed reached. The core of the n-person game v, denoted by core(v), is defined by

$$\mathrm{core}(v) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i \in I} x_i \le v(I) \text{ and } v(S) \le \sum_{i \in S} x_i \ \forall\, S \in \mathcal{N} \right\}.$$

We demonstrate that v is balanced. Let $\mathcal{B}$ be a balanced family with balancing weights $\{\lambda^S \ge 0 \mid S \in \mathcal{B}\}$, and let $\mathcal{B}_i$ and t^S be defined as in the proof of Theorem 2. Then for each $S \in \mathcal{B}$ there exists some $(a_1^S, \dots, a_n^S) \in M$ such that $\sum_{i \in S} f_i(a_i^S) = v(S)$. Define $b_i = \sum_{S \in \mathcal{B}} t^S a_i^S$ for each $i \in I$. By the convexity of M asserted in Proposition 1 (i), we have $(b_1, \dots, b_n) \in M$. By the concavity of f_i, we have $\sum_{S \in \mathcal{B}} t^S f_i(a_i^S) \le f_i(b_i)$ for each $i \in I$. Summing this inequality over $i \in I$ yields:

$$\begin{aligned} \sum_{i \in I} f_i(b_i) &\ge \sum_{i \in I} \sum_{S \in \mathcal{B}} t^S f_i(a_i^S) = \frac{1}{n} \sum_{i \in I} \sum_{S \in \mathcal{B}} \lambda^S \chi_i^S f_i(a_i^S) \ge \frac{1}{n} \sum_{i \in I} \sum_{S \in \mathcal{B}} \lambda^S f_i(a_i^S) \\ &\ge \frac{1}{n} \sum_{i \in I} \sum_{S \in \mathcal{B}} \lambda^S v(S) = \sum_{S \in \mathcal{B}} \lambda^S v(S), \end{aligned}$$

where the third and fourth inequalities follow from the nonpositivity of f_i, and the fact that $0 \le \chi_i^S \le 1$ and $v(S) \le f_i(a_i^S)$ for each $i \in I$ and $S \in \mathcal{B}$. Therefore, $\sum_{S \in \mathcal{B}} \lambda^S v(S) \le \sum_{i \in I} f_i(b_i) \le v(I)$, and hence v is balanced.

Using the celebrated theorem of Bondareva–Shapley (see [2, 8]), there exists some $(x_1, \dots, x_n)$ in core(v). We then have $\sum_{i \in I} x_i = v(I)$. By the compactness of M asserted in Proposition 1 (i) and the continuity of f_i, there exists a partition $(A_1, \dots, A_n)$ such that $\sum_{i \in I} f_i(\mu_i(A_i)) = v(I)$. Suppose that $(A_1, \dots, A_n)$ is not a core partition. Then for some coalition S and some S-partition $(B_1, \dots, B_n)$, we have $\sum_{i \in S} f_i(\mu_i(A_i)) < \sum_{i \in S} f_i(\mu_i(B_i))$. Since $f_i(\mu_i(A_i)) \le v(i) \le x_i$ for each $i \in I$, we have $x_i = f_i(\mu_i(A_i))$ for each $i \in I$ in view of $\sum_{i \in I} x_i = \sum_{i \in I} f_i(\mu_i(A_i))$. Therefore, we have $\sum_{i \in S} x_i = \sum_{i \in S} f_i(\mu_i(A_i)) < \sum_{i \in S} f_i(\mu_i(B_i)) \le v(S)$, which contradicts the fact that $(x_1, \dots, x_n) \in \mathrm{core}(v)$. □

Remark 3. Theorem 3 is not a contribution, as Legut [4] proved the existence of core partitions with a TU game when the utility function of each individual is a concave continuous transformation of a nonatomic finite measure. However, the proof presented in this paper is somewhat different from that of Legut [4] in that Lebesgue integrals are not used explicitly in our proof. While the existence result in our proof crucially depends on the use of Proposition 1, its proof does not require the Lebesgue integrals (see [5]). For the existence of core partitions with a TU game with more general convex continuous preferences, see [6].

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