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ON THE DOMINANCE RELATION BETWEEN ORDINAL SUMS OF CONJUNCTORS

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This contribution deals with the dominance relation on the class of conjunctors, containing as particular cases the subclasses of quasi-copulas, copulas and t-norms. The main results pertain to the summand-wise nature of the dominance relation, when applied to ordinal sum conjunctors, and to the relationship between the idempotent elements of two conjunctors involved in a dominance relationship. The results are illustrated on some well-known parametric families of t-norms and copulas.

Keywords: conjunctor, copula, dominance, ordinal sum, quasi-copula, t-norm

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1. INTRODUCTION

The dominance relation was introduced in the framework of probabilistic metric spaces as a binary relation on the class of all triangle functions [25], and was soon generalized to operations on a partially ordered set [24]. It plays an important role in the construction of Cartesian products of probabilistic metric spaces (see, e. g. [24, 25]), but also in the preservation of several properties, most of them expressed by some inequality, during (dis-)aggregation processes [3, 4, 7, 9, 22, 23]. Therefore, the dominance property was also introduced in the framework of aggregation operators where it enjoyed further development [19, 22, 23].

In this paper, we restrict ourselves to a broad class of aggregation operators, namely those with neutral element 1. They are known as conjunctors and encompass all quasi-copulas, copulas and t-norms. Our emphasis lies on the dominance relation between ordinal sums of conjunctors.

In Section 2, we review the various classes of conjunctors considered in this work and extend the ordinal sum construction and the dominance relation to conjunctors. In the following section, we briefly discuss the dominance relation between ordinally irreducible conjunctors. In Section 4, we lay bare the summand-wise nature of the dominance relation. Finally, we identify interesting properties of the sets of idempotent elements of two conjunctors connected through the dominance relation and illustrate the results on some parametric families of t-norms/copulas.

2. THE DOMINANCE RELATION ON THE CLASS OF CONJUNCTORS

2.1. Conjunctors

In recent years, various classes of binary operators on the unit interval have gained interest in fuzzy set theory and probability theory. Triangular norms, originally introduced in the field of probabilistic metric spaces, now live a second life as models for the pointwise intersection of fuzzy sets or as models for the many-valued conjunction in fuzzy logic. Copulas, and in particular 2-copulas as considered here, connect the marginal distributions of a random vector into the joint distribution. Weaker operators, such a quasi-copulas, are appearing frequently in probability theory, as well as in fuzzy set theory. All of the operators mentioned have two properties in common: neutral element 1 and monotonicity. We now state the formal definitions.

Definition 1. ([6, 13]) A binary operation $C : [0,1]^2 \to [0,1]$ is called a *conjunctor* if it satisfies:

(i) *Neutral element* 1: for any $x \in [0,1]$ it holds that $C(x,1) = C(1,x) = x$.

(ii) *Monotonicity:* C is increasing in each variable.

Note that any conjunctor C coincides on $\{0,1\}^2$ with the Boolean conjunction and satisfies:

(i') *Absorbing element* 0: for any $x \in [0,1]$ it holds that $C(x,0) = C(0,x) = 0$.

The comparison of two conjunctors C_1 and C_2 is done pointwisely, i. e. if for all $x, y \in [0,1]$ it holds that $C_1(x,y) \leq C_2(x,y)$, then we say that C_1 is *weaker* than C_2, or that C_2 is *stronger* than C_1, and denote it by $C_1 \leq C_2$. For any conjunctor C it holds that $T_{\mathbf{D}} \leq C \leq T_{\mathbf{M}}$, with

$$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y), & \text{otherwise,} \end{cases}$$

known as the *drastic product*, and $T_{\mathbf{M}}(x,y) = \min(x,y)$.

For a conjunctor C and an order isomorphism $\varphi \colon [0,1] \to [0,1]$, i. e. an increasing bijection, its isomorphic transform is the conjunctor $C_\varphi \colon [0,1]^2 \to [0,1]$ defined by $C_\varphi(x,y) = \varphi^{-1}(C(\varphi(x), \varphi(y)))$. The conjunctors C and C_φ are then referred to as *isomorphic operations*, or also as being isomorphic to each other.

In this paper, we are mainly interested in three particular classes of conjunctors: the class of triangular norms (t-norms), the class of copulas and the class of quasi-copulas. Where t-norms have the additional properties of associativity and commutativity, copulas have the property of moderate growth, while quasi-copulas have the 1-Lipschitz property. Note that conjunctors are also known as semi-copulas [11].

Definition 2. ([12]) A conjunctor $C : [0,1]^2 \to [0,1]$ is called a *quasi-copula* if it satisfies:

(iii) *1-Lipschitz property:* for any $x_1, x_2, y_1, y_2 \in [0,1]$ it holds that:

$$|C(x_1,y_1) - C(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|.$$

Definition 3. ([20]) A conjunctor $C : [0,1]^2 \to [0,1]$ is called a 2-*copula* (*copula* for short) if it satisfies:

(iv) *Moderate growth:* for any $x_1, x_2, y_1, y_2 \in [0,1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$ it holds that:

$$C(x_1, y_2) + C(x_2, y_1) \leq C(x_1, y_1) + C(x_2, y_2)\,.$$

As implied by the terminology used, any copula is a quasi-copula, and therefore has the 1-Lipschitz property; the opposite is, of course, not true.

Definition 4. ([15, 24]) A conjunctor $C : [0,1]^2 \to [0,1]$ is called a *t-norm* if it satisfies:

(v) *Commutativity:* for any $x, y \in [0,1]$ it holds that:

$$C(x, y) = C(y, x)\,.$$

(vi) *Associativity:* for any $x, y, z \in [0,1]$ it holds that:

$$C(x, C(y, z)) = C(C(x, y), z)\,.$$

It is well known that a copula is a t-norm if and only if it is associative; conversely, a t-norm is a copula if and only if it is 1-Lipschitz (see, e. g. [15, 20]). The three main continuous t-norms are the minimum operator $T_{\mathbf{M}}$, the algebraic product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ (defined by $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$); they are at the same time associative and commutative copulas. For any quasi-copula C it holds that $T_{\mathbf{L}} \leq C \leq T_{\mathbf{M}}$ (see, e. g. [12]).

2.2. The ordinal sum construction

The ordinal sum construction appears quite frequently, e. g. in the framework of partially ordered sets [2] and in the context of algebraic operations and structures (ordinal sums of semigroups [5], in particular t-norms [14, 16, 21], as well as copulas [20], and aggregation operators [8]). The aim is always the same, namely the preservation of properties of the summand operations into the resulting ordinal sum. Here, we follow a particular approach known as the *id-lower ordinal sum* [8].

Definition 5. Let $(]a_i, b_i[)_{i \in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$ and let $(C_i)_{i \in I}$ be a family of conjunctors. Then the *ordinal sum* $C = (\langle a_i, b_i, C_i \rangle)_{i \in I} : [0,1]^2 \to [0,1]$ is the conjunctor defined by

$$C(x, y) = \begin{cases} a_i + (b_i - a_i)\, C_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right), & \text{if } (x, y) \in [a_i, b_i]^2\,, \\ \min(x, y), & \text{otherwise}\,. \end{cases}$$

Note that each conjunctor C_i is squeezed into the corresponding square $[a_i, b_i]^2$ by a linear transformation. The triplets $\langle a_i, b_i, C_i \rangle$ are called the *summands* of the ordinal sum. The intervals $[a_i, b_i]$ are called the *summand carriers*, the conjunctors C_i the *summand operations*. A conjunctor C that has no ordinal sum representation different from $(\langle 0, 1, C\rangle)$ is called *ordinally irreducible*. Obviously, $T_{\mathbf{M}}$ is not ordinally irreducible.

The ordinal sum construction is powerful as it preserves a lot of properties, such as commutativity, (1-Lipschitz) continuity, etc. For instance, an ordinal sum is continuous if and only if all its summand operations are continuous. Combining various properties, it holds that the classes of quasi-copulas, copulas and triangular norms are all closed under the ordinal sum construction. The ordinal sum construction even allows for the full characterization of continuous t-norms [17].

Proposition 1. A binary operation $T\colon [0, 1]^2 \to [0, 1]$ is a continuous t-norm if and only if it is uniquely representable as an ordinal sum of t-norms that are either isomorphic to the Łukasiewicz t-norm $T_{\mathbf{L}}$ or to the product $T_{\mathbf{P}}$.

2.3. The dominance relation

The dominance relation was introduced in the framework of probabilistic metric spaces as a relation between triangle functions which ensures that the Cartesian product of two probabilistic metric spaces is again a probabilistic metric space of the same type ([24, 25]). It was generalized to operations on a partially ordered set [24] and introduced into the framework of t-norms (see also [15]). The dominance relation is indispensable when refining fuzzy partitions and when building Cartesian products of fuzzy equivalence and fuzzy order relations [3, 7]. Moreover, it plays an important role in the preservation of T-transitivity of fuzzy relations involved in a (dis-)aggregation process [9, 23], giving way to its generalization to aggregation operators [23].

Definition 6. Consider two conjunctors C_1 and C_2. We say that C_1 *dominates* C_2, denoted by $C_1 \gg C_2$, if for all $x, y, u, v \in [0, 1]$ it holds that

$$C_1(C_2(x, y), C_2(u, v)) \geq C_2(C_1(x, u), C_1(y, v)). \tag{1}$$

For any two conjunctors C_1 and C_2 and any order isomorphism $\varphi\colon [0, 1] \to [0, 1]$, it holds that $C_1 \gg C_2$ if and only if $(C_1)_\varphi \gg (C_2)_\varphi$ (see also [22, 23]). We will refer to this relationship as the *isomorphism property of dominance*.

Due to the fact that 1 is the common neutral element of all conjunctors, dominance of one conjunctor by another conjunctor implies their comparability: $C_1 \gg C_2$ implies $C_1 \geq C_2$ (see also [22]). Obviously, the converse does not hold. Consequently, the dominance relation is antisymmetric on the class of all conjunctors. A conjunctor C for which $C \gg C$ is said to be *self-dominant*. Self-dominance is evidently equivalent with the bisymmetry property [1]

$$C(C(x, y), C(u, v)) = C(C(x, u), C(y, v)).$$

Commutativity and associativity clearly imply bisymmetry. Moreover, bisymmetry together with 1 being the neutral element imply commutativity and associativity. Hence any t-norm is self-dominant and on the class of all t-norms the dominance relation is not only antisymmetric, but also reflexive. This is, however, not the case for the class of copulas.

Example 1. Consider the family of copulas $(C_\theta)_{\theta\in[0,1]}$ defined by

$$C_\theta(x,y) = \begin{cases} \min(x, y-\theta), & \text{if } (x,y) \in [0,1-\theta] \times [\theta,1]\,, \\ \min(x+\theta-1, y), & \text{if } (x,y) \in [1-\theta,1] \times [0,\theta]\,, \\ T_{\mathbf{L}}(x,y), & \text{otherwise.} \end{cases}$$

The copula $C_{0.5}$ is the only commutative member of this family (see also [20]). As it is not associative, it is also not bisymmetric, and does therefore not dominate itself (choose, e. g. $x = 0.5$, $y = 1$, $u = v = 0.75$).

Before turning to ordinal sums of conjunctors let us recall some basic results about dominance between (ordinally irreducible) conjunctors, in particular involving the extreme elements of various subclasses of conjunctors.

3. DOMINANCE BETWEEN (ORDINALLY IRREDUCIBLE) CONJUNCTORS

3.1. Conjunctors

Due to their monotonicity, it is immediately clear that any conjunctor C is dominated by $T_{\mathbf{M}}$. Conversely, since dominance implies comparability, $T_{\mathbf{M}}$ is the only conjunctor dominating $T_{\mathbf{M}}$. On the other hand, it is easily verified that any conjunctor C dominates the weakest conjunctor $T_{\mathbf{D}}$.

In [23], several methods for constructing dominating aggregation operators from given ones have been proposed. As a consequence, we can immediately pose the following lemma.

Lemma 1. Consider conjunctors C_1, C_2, C_3 and C. If $C_i \gg C$, for any $i \in \{1,2,3\}$, then also the binary operation $C^*\colon [0,1]^2 \to [0,1]$ defined by

$$C^*(x,y) = C_3(C_1(x,y), C_2(x,y))$$

dominates C. Moreover, C^* is a conjunctor if and only if $C_3 = T_{\mathbf{M}}$.

3.2. Quasi-copulas and copulas

The strongest (quasi-)copula $T_{\mathbf{M}}$ dominates all other conjunctors, in particular all (quasi-)copulas. However, not all (quasi-)copulas dominate the weakest (quasi-) copula $T_{\mathbf{L}}$, as the following example demonstrates.

Example 2. Consider the copula $C\colon [0,1]^2 \to [0,1]$ defined by

$$C(x,y) = \begin{cases} \frac{1}{2}T_{\mathbf{L}}(2x,2y), & \text{if } (x,y) \in \left[0,\frac{1}{2}\right]^2, \\ T_{\mathbf{M}}(x,y), & \text{otherwise.} \end{cases}$$

Putting $x = y = u = v = \frac{5}{8}$ yields

$$0 = C\left(\tfrac{1}{4},\tfrac{1}{4}\right) = C\left(T_{\mathbf{L}}\left(\tfrac{5}{8},\tfrac{5}{8}\right), T_{\mathbf{L}}\left(\tfrac{5}{8},\tfrac{5}{8}\right)\right) < T_{\mathbf{L}}\left(C\left(\tfrac{5}{8},\tfrac{5}{8}\right), C\left(\tfrac{5}{8},\tfrac{5}{8}\right)\right) = T_{\mathbf{L}}\left(\tfrac{5}{8},\tfrac{5}{8}\right) = \tfrac{1}{4}$$

and therefore C does not dominate $T_{\mathbf{L}}$. Note that C is an ordinal sum copula and a member of the Mayor–Torrens family as discussed also later in Section 5.2.2.

However, the 1-Lipschitz property is a necessary condition for a conjunctor to dominate $T_{\mathbf{L}}$ (see also [9, 19]).

Proposition 2. If a conjunctor C dominates $T_{\mathbf{L}}$, then it is a quasi-copula.

Proof. Suppose that a conjunctor C dominates $T_{\mathbf{L}}$, i.e. for all $x, y, u, v \in [0,1]$ it holds that

$$C(T_{\mathbf{L}}(x,y), T_{\mathbf{L}}(u,v)) \geq T_{\mathbf{L}}(C(x,u), C(y,v))\,. \tag{2}$$

In order to show that C fulfills the 1-Lipschitz property, it suffices, due to its increasingness, to prove that

$$C(a,b) - C(a-\varepsilon, b-\delta) \leq \varepsilon + \delta$$

whenever $0 \leq \varepsilon \leq a$, $0 \leq \delta \leq b$ for arbitrary $a, b \in [0,1]$. We first choose $x = a$, $y = 1$, $u = b$, $v = 1 - \delta$ for some $0 \leq \delta \leq b$ with arbitrary but fixed $a, b \in [0,1]$. Then $T_{\mathbf{L}}(u,v) = \max(u+v-1, 0) = \max(b-\delta, 0) = b - \delta$ and hence it follows using Eq. (2) that

$$\begin{aligned} C(a, b-\delta) &= C(T_{\mathbf{L}}(a,1), T_{\mathbf{L}}(b, 1-\delta)) \\ &\geq T_{\mathbf{L}}(C(a,b), C(1, 1-\delta)) \\ &= T_{\mathbf{L}}(C(a,b), 1-\delta) = \max(C(a,b) - \delta, 0) \\ &\geq C(a,b) - \delta. \end{aligned}$$

Analogously, by putting $x = a$, $y = 1 - \varepsilon$, $u = b$, $v = 1$ with $0 \leq \varepsilon \leq a$, we can conclude that $C(a-\varepsilon, b) \geq C(a,b) - \varepsilon$. As a consequence

$$C(a-\varepsilon, b-\delta) \geq C(a-\varepsilon, b) - \delta \geq C(a,b) - \varepsilon - \delta\,.$$

Therefore, C is 1-Lipschitz, and thus a quasi-copula. □

3.3. Triangular norms

The class of ordinally irreducible continuous t-norms consists of all continuous Archimedean t-norms, i.e. those t-norms that are either isomorphic to the product $T_{\mathbf{P}}$ (called *strict* t-norms) or to the Łukasiewicz t-norm $T_{\mathbf{L}}$ (called *nilpotent* t-norms). The following observations are important, as they imply that it suffices to consider the t-norms $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ in order to understand dominance of a continuous Archimedean t-norm T by a conjunctor C:

(i) If T is strict, there exists an order isomorphism $\varphi\colon [0,1] \to [0,1]$ such that $T = (T_{\mathbf{P}})_\varphi$, leading to the equivalence $\quad C \gg T \Leftrightarrow C_{\varphi^{-1}} \gg T_{\mathbf{P}}$.

(ii) If T is nilpotent, there exists an order isomorphism $\varphi\colon [0,1] \to [0,1]$ such that $T = (T_{\mathbf{L}})_\varphi$, leading to the equivalence $\quad C \gg T \Leftrightarrow C_{\varphi^{-1}} \gg T_{\mathbf{L}}$.

We have already seen in Proposition 2 that being a quasi-copula is a necessary condition for a conjunctor to dominate $T_{\mathbf{L}}$. It is remarkable that the same condition applies for a conjunctor to dominate $T_{\mathbf{P}}$.

Proposition 3. If a conjunctor C dominates $T_{\mathbf{P}}$, then it is a quasi-copula.

Proof. Suppose that a conjunctor C dominates $T_{\mathbf{P}}$, i. e. for all $x, y, u, v \in [0,1]$ it holds that

$$C(xy, uv) \geq C(x,u)C(y,v)\,. \tag{3}$$

Again it suffices, due to the increasingness of C, to show that

$$C(a,b) - C(a-\varepsilon, b-\delta) \leq \varepsilon + \delta$$

whenever $0 \leq \varepsilon \leq a$, $0 \leq \delta \leq b$ for arbitrary $a, b \in [0,1]$. In case that $a = 0$ (resp. $b = 0$), it holds that $\varepsilon = 0$ (resp. $\delta = 0$), and the inequality is trivially fulfilled. Therefore, it remains to prove that it holds for arbitrary $a, b \in]0,1]$. We first choose $x = a$, $y = 1 - \frac{\varepsilon}{a}$, $u = b$, $v = 1$ with $0 \leq \varepsilon \leq a$. Then it follows from Eq. (3) that

$$C(a-\varepsilon, b) \geq C(a,b)C(1-\frac{\varepsilon}{a}, 1) = C(a,b)(1-\frac{\varepsilon}{a})\,.$$

Since $C \leq T_{\mathbf{M}}$ it then holds for all $0 < a \leq 1$, $0 \leq b \leq 1$ and $0 \leq \varepsilon \leq a$ that

$$C(a,b) - C(a-\varepsilon, b) \leq C(a,b)(1-(1-\frac{\varepsilon}{a})) = \frac{\varepsilon}{a}C(a,b) \leq \varepsilon\,.$$

Similarly, we can conclude from Eq. (3), by choosing $x = a$, $y = 1$, $u = b$, $v = 1 - \frac{\delta}{b}$ with $0 \leq \delta \leq b$, that for all $0 \leq a \leq 1$, $0 < b \leq 1$ with $0 \leq \delta \leq b$ also $C(a,b) - C(a, b-\delta) \leq \delta$. Hence,

$$\begin{aligned} C(a,b) - C(a-\varepsilon, b-\delta) &= C(a,b) - C(a,b-\delta) + C(a,b-\delta) - C(a-\varepsilon, b-\delta) \\ &\leq \varepsilon + \delta \end{aligned}$$

whenever $0 \leq \varepsilon \leq a$, $0 \leq \delta \leq b$ for arbitrary $a, b \in]0,1]$. Therefore, C is 1-Lipschitz, and thus a quasi-copula. □

4. DOMINANCE BETWEEN ORDINAL SUM CONJUNCTORS

4.1. Summand-wise dominance

As the ordinal sum construction is generally applicable, it is important to investigate dominance between two ordinal sum conjunctors in order to gain a deeper understanding of the dominance relation. In a first proposition we show that if both ordinal sum conjunctors are based on the same summand carriers, dominance between these conjunctors is based on the dominance between the corresponding summand operations.

Proposition 4. Consider two ordinal sum conjunctors $C_1 = (\langle a_i, b_i, C_{1,i}\rangle)_{i\in I}$ and $C_2 = (\langle a_i, b_i, C_{2,i}\rangle)_{i\in I}$. Then C_1 dominates C_2 if and only if $C_{1,i}$ dominates $C_{2,i}$ for all $i \in I$.

Proof. Suppose that $C_1 \gg C_2$, i.e. for all $x, y, u, v \in [0, 1]$ it holds that

$$C_1(C_2(x, y), C_2(u, v)) \geq C_2(C_1(x, u), C_1(y, v)) \,. \tag{4}$$

We want to show that for all $i \in I$ it holds that $C_{1,i} \gg C_{2,i}$. Choose arbitrary $x, y, u, v \in [0, 1]$ and some $i \in I$. Since $\varphi_i : [a_i, b_i] \to [0, 1]$, $x \mapsto \frac{x-a_i}{b_i-a_i}$ is an increasing bijection, there exist unique $x', y', u', v' \in [a_i, b_i]$ such that $\varphi_i(x') = x$, $\varphi_i(y') = y$, $\varphi_i(u') = u$ and $\varphi_i(v') = v$. Since Eq. (4) is fulfilled for all $x, y, u, v \in [0, 1]$ and in particular for $x', y', u', v' \in [a_i, b_i]$, it can be equivalently expressed as

$$\begin{array}{ll} & \varphi_i^{-1} \circ C_{1,i}(C_{2,i}(\varphi_i(x'), \varphi_i(y')), C_{2,i}(\varphi_i(u'), \varphi_i(v'))) \\ \geq & \varphi_i^{-1} \circ C_{2,i}(C_{1,i}(\varphi_i(x'), \varphi_i(u')), C_{1,i}(\varphi_i(y'), \varphi_i(v'))) \,, \end{array}$$

taking into account the ordinal sum structure of C_1 and C_2. The previous inequality is in turn equivalent to

$$\varphi_i^{-1} \circ C_{1,i}(C_{2,i}(x, y), C_{2,i}(u, v)) \geq \varphi_i^{-1} \circ C_{2,i}(C_{1,i}(x, u), C_{1,i}(y, v)) \,.$$

Applying φ_i to both sides of the above inequality yields $C_{1,i} \gg C_{2,i}$.

Conversely, suppose that for all $i \in I$ it holds that $C_{1,i} \gg C_{2,i}$, then Eq. (4) is fulfilled for all $x, y, u, v \in [a_i, b_i]$ due to the isomorphism property. Next, we will make use of the following observation: for any $p, q \in [0, 1]$ such that $\min(p, q) \in [a_i, b_i]$ for some $i \in I$, it holds that

$$C_1(p, q) = C_1(\min(p, b_i), \min(q, b_i)) \,.$$

Now consider arbitrary $x, y, u, v \in [0, 1]$ and suppose w.l.o.g. that $x = \min(x, y, u, v)$, then we can distinguish the following cases.

Case 1. Suppose $x \in [a_i, b_i]$ for some $i \in I$. Let $y^* = \min(y, b_i)$, $u^* = \min(u, b_i)$ and $v^* = \min(v, b_i)$. Note that $C_1(x, u) = C_1(x, u^*)$. Moreover, if $\min(y, v) \in [a_i, b_i]$, then also $C_1(y, v) = C_1(y^*, v^*)$. As x, y^*, u^*, v^* all belong to $[a_i, b_i]$, the assumption $C_{1,i} \gg C_{2,i}$ and the increasingness of C_1 and C_2 imply that

$$\begin{array}{rcl} C_2(C_1(x, u), C_1(y, v)) & = & C_2(C_1(x, u^*), C_1(y^*, v^*)) \\ & \leq & C_1(C_2(x, y^*), C_2(u^*, v^*)) \\ & \leq & C_1(C_2(x, y), C_2(u, v)) \,. \end{array}$$

On the other hand, if $\min(y, v) \notin [a_i, b_i]$, we know that $C_1(y, v) \geq b_i$. Since $C_1(x, u^*) \leq b_i$ it follows that

$$\begin{array}{rcl} C_2(C_1(x, u), C_1(y, v)) & = & C_2(C_1(x, u^*), C_1(y, v)) \\ & = & \min(C_1(x, u^*), C_1(y, v)) = C_1(x, u^*) \,. \end{array}$$

Due to the increasingness of C_1 it holds that

$$\begin{aligned} C_1(x,u^*) &= \min(C_1(x,u^*),C_1(x,v),C_1(y,u^*),C_1(y,v)) \\ &= C_1(\min(x,y),\min(u^*,v)) \\ &= C_1(C_2(x,y),C_2(u^*,v)) \\ &\le C_1(C_2(x,y),C_2(u,v))\,. \end{aligned}$$

Case 2. If $x \notin [a_i,b_i]$ for all $i \in I$, then $C_1(x,\cdot) = C_2(x,\cdot) = T_{\mathbf{M}}(x,\cdot)$. One easily verifies that $C_1(y,v) \ge x$ and $C_2(u,v) \ge x$. This leads to

$$\begin{aligned} C_2(C_1(x,u),C_1(y,v)) &= C_2(x,C_1(y,v)) \\ &= \min(x,C_1(y,v)) = x = \min(x,C_2(u,v)) \\ &= C_1(x,C_2(u,v)) = C_1(C_2(x,y),C_2(u,v))\,. \end{aligned}$$

This completes the proof that C_1 dominates C_2. □

4.2. Ordinal sums with different summand carriers

In case the structure of both ordinal sum conjunctors is not the same, we are able to provide some necessary conditions which lead to a characterization of dominance between ordinal sum conjunctors in general. Assume that the ordinal sum conjunctors under consideration are based on two at least partially different families of summand carriers, i. e. $C_1 = (\langle a_{1,i},b_{1,i},C_{1,i}\rangle)_{i\in I}$ and $C_2 = (\langle a_{2,j},b_{2,j},C_{2,j}\rangle)_{j\in J}$. W.l.o.g. we can assume that these representations are the finest possible, i. e. that each summand operation is ordinally irreducible.

Since any conjunctor is bounded from above by $T_{\mathbf{M}}$ and dominance implies comparability, the following proposition follows immediately.

Proposition 5. If a conjunctor C_1 dominates a conjunctor C_2, then $C_1(x,y) = T_{\mathbf{M}}(x,y)$ whenever $C_2(x,y) = T_{\mathbf{M}}(x,y)$.

Geometrically speaking, if an ordinal sum conjunctor C_1 dominates an ordinal sum conjunctor C_2, then it must necessarily consist of more regions where it acts as $T_{\mathbf{M}}$ than does C_2. Two such cases are displayed in Figure 1 (a) and (c). Note that no dominance relationship between C_1 and C_2 is possible in a case like illustrated in Figure 1 (b). Therefore, we can immediately state the following corollary.

Corollary 1. Consider two ordinal sum conjunctors $C_1 = (\langle a_{1,i},b_{1,i},C_{1,i}\rangle)_{i\in I}$ and $C_2 = (\langle a_{2,j},b_{2,j},C_{2,j}\rangle)_{j\in J}$ with ordinally irreducible summand operations only. If C_1 dominates C_2 then

$$(\forall i \in I)(\exists j \in J)([a_{1,i},b_{1,i}] \subseteq [a_{2,j},b_{2,j}])\,. \tag{5}$$

Note that each $[a_{2,j},b_{2,j}]$ can contain several or even none of the summand carriers $[a_{1,i},b_{1,i}]$ (see also Figure 1 (a) and (c)). Hence, for each $j \in J$ we can consider the

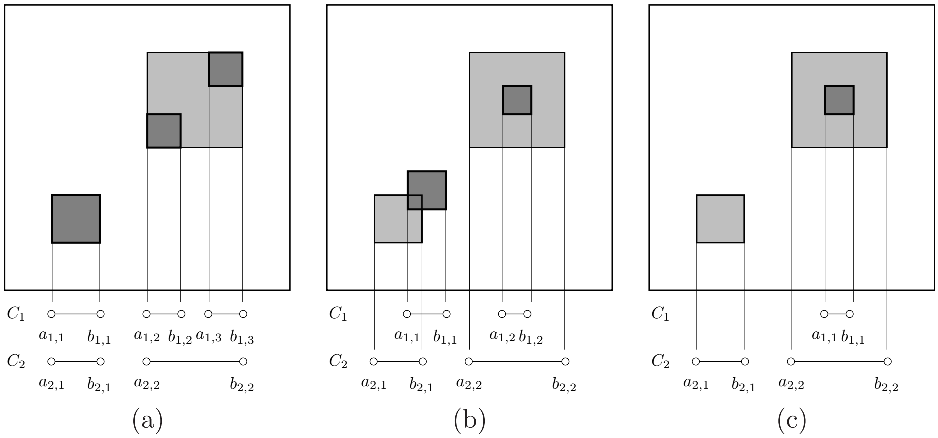


Fig. 1. Examples of two ordinal sum conjunctors C_1 and C_2 differing in their summand carriers.

following subset of I:

$$I_j = \{i \in I \mid [a_{1,i}, b_{1,i}] \subseteq [a_{2,j}, b_{2,j}]\}\,. \tag{6}$$

Based on these notions and due to Proposition 4, dominance between two ordinal sum conjunctors can be reformulated in the following way.

Proposition 6. Consider two ordinal sum conjunctors $C_1 = (\langle a_{1,i}, b_{1,i}, C_{1,i}\rangle)_{i\in I}$ and $C_2 = (\langle a_{2,j}, b_{2,j}, C_{2,j}\rangle)_{j\in J}$ with ordinally irreducible summand operations only. Then C_1 dominates C_2 if and only if

(i) $\cup_{j\in J} I_j = I$,

(ii) $C_1^j \gg C_{2,j}$ for all $j \in J$ with

$$C_1^j = (\langle \varphi_j(a_{1,i}), \varphi_j(b_{1,i}), C_{1,i}\rangle)_{i\in I_j} \tag{7}$$

and $\varphi_j \colon [a_{2,j}, b_{2,j}] \to [0,1]$, $\varphi_j(x) = \frac{x - a_{2,j}}{b_{2,j} - a_{2,j}}$.

Proof. Under condition (i) it is easily verified that C_1 can be equivalently expressed as an ordinal sum based on the summand carriers of C_2 in the following way

$$C_1 = (\langle a_{2,j}, b_{2,j}, C_1^j\rangle)_{j\in J}$$

with C_1^j defined by Eq. (7). With Corollary 1 and Proposition 4, the proposition now follows immediately. □

Note that due to Proposition 6, the study of dominance between ordinal sum conjunctors can be reduced to the study of the dominance of a single ordinally irreducible conjunctor by some ordinal sum conjunctor.

5. THE ROLE OF IDEMPOTENT ELEMENTS

5.1. A basic result

Before turning to particular families of ordinal sum conjunctors, we will next discuss the influence of idempotent elements to the property of dominance. We will denote the set of idempotent elements of some conjunctor C by $\mathcal{I}(C)$, i.e.

$$\mathcal{I}(C) = \{x \in [0,1] \mid C(x,x) = x\}\,.$$

Due to the construction of an ordinal sum conjunctor C, the endpoints of its summand carriers belong to its set of idempotent elements.

Proposition 7. If a conjunctor C_1 dominates a conjunctor C_2, then the following hold:

(i) $\mathcal{I}(C_2) \subseteq \mathcal{I}(C_1)$,

(ii) $\mathcal{I}(C_1)$ is closed under C_2.

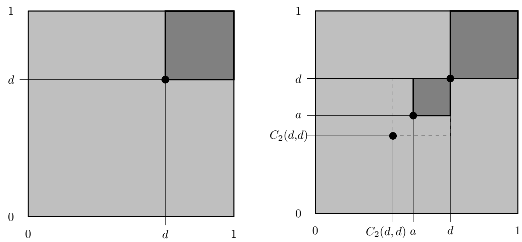
Proof. The inclusion follows immediately from Proposition 5. Next, suppose that $d_1, d_2 \in \mathcal{I}(C_1)$, then

$$\begin{aligned} C_2(d_1,d_2) &= C_2(C_1(d_1,d_1), C_1(d_2,d_2)) \\ &\leq C_1(C_2(d_1,d_2), C_2(d_1,d_2)) \\ &\leq T_{\mathbf{M}}(C_2(d_1,d_2), C_2(d_1,d_2)) = C_2(d_1,d_2), \end{aligned}$$

showing that $C_1(C_2(d_1,d_2), C_2(d_1,d_2)) = C_2(d_1,d_2)$ and therefore $C_2(d_1,d_2) \in \mathcal{I}(C_1)$. □

This proposition has some interesting consequences for the boundary elements of the summand carriers. Firstly, all idempotent elements of C_2 are idempotent elements of C_1, i.e. either boundary elements themselves, elements of some domain where C_1 acts as $T_{\mathbf{M}}$, or isomorphic transformations of idempotent elements of some summand operation. Secondly, for any two idempotent elements d_1 and d_2 of C_1 also $C_2(d_1,d_2)$ is an idempotent element of C_1. Consequently, if C_1 is some ordinal sum that dominates $C_2 = T_{\mathbf{P}}$, resp. $C_2 = T_{\mathbf{L}}$, and $d \in \mathcal{I}(C_1)$ then also $d^n \in \mathcal{I}(C_1)$, resp. $\max(nd - n + 1, 0) \in \mathcal{I}(C_1)$, for all $n \in \mathbb{N}$.

Example 3. Consider a conjunctor C with trivial idempotent elements only, i.e. $\mathcal{I}(C) = \{0,1\}$. We are now interested in constructing ordinal sums C_1 with summands based on C which fulfill the necessary conditions for dominating $C_2 = C$ as expressed by Proposition 7. Clearly, $C_1 = (\langle d,1,C\rangle)$ is a first possibility (see Figure 2 (a)). Adding one further summand to C_1, i.e. building $C_1' = (\langle a,d,C\rangle, \langle d,1,C\rangle)$, demands that $a \geq C_2(d,d)$, since otherwise $C_2(d,d) \notin \mathcal{I}(C_1')$ (see also Figure 2 (b)).



(a) (b)

Fig. 2. Illustrations to Example 3.

5.2. Applications to some parametric families

To conclude, we consider two families consisting of conjunctors with only one summand but varying boundary elements. All members of these families are t-norms as well as copulas. We have opted for these families as they involve $T_{\mathbf{P}}$, resp. $T_{\mathbf{L}}$, only as summand operation.

5.2.1. A family involving $T_{\mathbf{P}}$

The members of the family of Dubois–Prade t-norms [10] are given by $T^{\mathbf{DP}}_{\lambda} = (\langle 0, \lambda, T_{\mathbf{P}} \rangle)$ for $\lambda \in [0,1]$. Obviously, they are ordinal sums with the product as single summand operation. The case $\lambda = 0$ corresponds to $T_{\mathbf{M}}$, the case $\lambda = 1$ to $T_{\mathbf{P}}$. If $\lambda_1 \leq \lambda_2$, then $T^{\mathbf{DP}}_{\lambda_1} \geq T^{\mathbf{DP}}_{\lambda_2}$. Therefore, if $T^{\mathbf{DP}}_{\lambda_1} \gg T^{\mathbf{DP}}_{\lambda_2}$ then $\lambda_1 \leq \lambda_2$.

If $\lambda_1 = 0$ or $\lambda_1 = \lambda_2$, then the dominance property is trivially fulfilled. Therefore, suppose that $0 < \lambda_1 < \lambda_2$. For better readability we denote $T^{\mathbf{DP}}_{\lambda_1}$, resp. $T^{\mathbf{DP}}_{\lambda_2}$, by T_1, resp. T_2. Suppose that T_1 dominates T_2. For each T_i, $i \in \{1,2\}$, its set of idempotent elements is given by

$$\mathcal{I}(T_i) = \{0\} \cup [\lambda_i, 1] \,.$$

Due to Proposition 7, it holds that $T_2(\lambda_1, \lambda_1) \in \mathcal{I}(T_1)$. However,

$$0 \neq T_2(\lambda_1, \lambda_1) = \lambda_2 \cdot T_{\mathbf{P}}(\tfrac{\lambda_1}{\lambda_2}, \tfrac{\lambda_1}{\lambda_2}) = \tfrac{\lambda_1}{\lambda_2} \cdot \lambda_1 < \lambda_1$$

due to the strict monotonicity of $T_{\mathbf{P}}$. This leads to a contradiction.

Consequently, the only dominance relationships in the family of Dubois–Prade t-norms are $T_{\mathbf{M}}$ dominating all other members and self-dominance. Hence, there exists no triplet of pairwisely different t-norms $T^{\mathbf{DP}}_{\lambda_1}$, $T^{\mathbf{DP}}_{\lambda_2}$ and $T^{\mathbf{DP}}_{\lambda_3}$ fulfilling $T^{\mathbf{DP}}_{\lambda_1} \gg T^{\mathbf{DP}}_{\lambda_2}$ and $T^{\mathbf{DP}}_{\lambda_2} \gg T^{\mathbf{DP}}_{\lambda_3}$, implying that the dominance relation is (trivially) transitive, and therefore a partial order, on this family.

5.2.2. A family involving $T_{\mathbf{L}}$

Similarly, the members of the family of Mayor–Torrens t-norms [18] are given by $T^{\mathbf{DP}}_{\lambda} = (\langle 0, \lambda, T_{\mathbf{L}} \rangle)$ for $\lambda \in [0,1]$. Obviously, they are ordinal sums with $T_{\mathbf{L}}$ as single summand operation. The case $\lambda = 0$ corresponds to $T_{\mathbf{M}}$, the case $\lambda = 1$ to $T_{\mathbf{L}}$. Again, $T^{\mathbf{MT}}_{\lambda_1} \gg T^{\mathbf{MT}}_{\lambda_2}$ implies $\lambda_1 \leq \lambda_2$.

If $\lambda_1 = 0$ or $\lambda_1 = \lambda_2$, then the dominance property is trivially fulfilled. Therefore, suppose that $0 < \lambda_1 < \lambda_2$. We denote $T^{\mathbf{MT}}_{\lambda_1}$, resp. $T^{\mathbf{MT}}_{\lambda_2}$, by T_1, resp. T_2. The sets of idempotent elements are of the following form

$$\mathcal{I}(T_i) = \{0\} \cup [\lambda_i, 1] \ .$$

Due to Proposition 7, it holds that $T_2(\lambda_1, \lambda_1) \in \mathcal{I}(T_1)$. Since $T_2(\lambda_1, \lambda_1) \leq \lambda_1$, either $T_2(\lambda_1, \lambda_1) = 0$ or $T_2(\lambda_1, \lambda_1) = \lambda_1$. The latter implies that $\lambda_1 \in \mathcal{I}(T_2)$, a contradiction. Hence, $T_2(\lambda_1, \lambda_1) = 0$ or equivalently $\lambda_1 \leq \frac{\lambda_2}{2}$. Now choose x such that

$$\frac{\lambda_2}{2} < x < \frac{\lambda_2}{2} + \frac{\lambda_1}{4}$$

and put $u = v = y = x$, then $T_1(T_2(x, y), T_2(u, v)) = 0$ and $T_2(T_1(x, u), T_1(y, v)) = 2x - \lambda_2 > 0$, a final contradiction.

Therefore, also in the Mayor–Torrens family, there exist no other dominance relationships than $T_{\mathbf{M}}$ dominating all other members and self-dominance.

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