Kybernetika

Fabrizio Durante; José Quesada-Molina; Carlo Sempi

Semicopulas: characterizations and applicability

Kybernetika, Vol. 42 (2006), No. 3, 287--302

Persistent URL: http://dml.cz/dmlcz/135715

KYBERNETIKA – VOLUME *42* (2006), NUMBER 3, PAGES 287 – 302

SEMICOPULAS: CHARACTERIZATIONS AND APPLICABILITY

Fabrizio Durante, José Juan Quesada–Molina and Carlo Sempi

We characterize some bivariate semicopulas and, among them, the semicopulas satisfying a Lipschitz condition. In particular, the characterization of harmonic semicopulas allows us to introduce a new concept of depedence between two random variables. The notion of multivariate semicopula is given and two applications in the theory of fuzzy measures and stochastic processes are given.

Keywords: semicopula, quasi-copula, Lipschitz condition, aggregation operator

AMS Subject Classification: 60E05, 60E15, 26B35, 03E72

1. INTRODUCTION

A (bivariate) *semicopula* is a function from the unit square $[0,1]^2$ into the unit interval $[0,1]$ that is increasing in each argument and which has neutral element 1. For the first time, this term was used by B. Bassan and F. Spizzichino ([8]) in their investigations on the bivariate notion of aging. Instead, the study of semicopulas in the context of aggregation operators can be found in [15] and [11].

Note that, in the context of fuzzy theory, semicopulas are often used under the name of *conjunctors*, an extension of a boolean conjunction from $\{0,1\}$ to the whole interval $[0,1]$ (see, e. g., [4]). Recently, in fuzzy preference modelling, conjunctors are used to define a general notion of transitivity for fuzzy relations and, in particular, some special classes of 1-Lipschitz conjunctors are considered ([5, 6]). A semicopula is also a generalization of the concepts of copula and quasi-copula, which are largely used in statistics (see [24]).

These two sources, fuzzy theory and statistics, motivate the problems here presented. Sections 2 and 3 are devoted, respectively, to the characterization of some basic semicopulas, which are also t-norms ([18]), and of the class of k-Lipschitz semicopulas; in particular, a new class of 1-Lipschitz semicopulas (i. e. quasi-copulas) is constructed, generalizing a family already used in the study of cycle-transitivity of fuzzy relations ([7]). The characterization of harmonic semicopulas will also induce new results in the study of dependence between random variables (subsection 2.1).

Finally, (Section 5), we speculate about the definition of multivariate semicopulas. In particular, we present two interesting connections between semicopulas and, respectively, the theory of stochastic processes and that of fuzzy measures.

2. BASIC PROPERTIES AND CHARACTERIZATIONS

A function $S : [0,1]^2 \to [0,1]$ is a *semicopula* if, and only if, it satisfies the two following conditions:

$$S(x,1) = S(1,x) = x \quad \text{for all } x \text{ in } [0,1]; \tag{1}$$

$$S(x,y) \le S(x',y') \quad \text{for all } x \le x' \text{ and } y \le y'. \tag{2}$$

In other words, a semicopula is a binary aggregation operator with neutral element 1 and, consequently, annihilator 0. Important examples of semicopulas are the functions M, Π and W, given by

$$M(x,y) = \min\{x,y\}, \quad \Pi(x,y) = xy, \quad W(x,y) = \max\{x+y-1,0\}.$$

A semicopula S is 1-*Lipschitz* if, and only if, for all x, x', y, y' in $[0,1]$,

$$|S(x,y) - S(x',y')| \le |x-x'| + |y-y'|. \tag{3}$$

A 1-Lipschitz semicopula is called a *quasi-copula*, a concept introduced by C. Alsina, R. B. Nelsen and B. Schweizer (see [2]) in order to characterize binary operations on distribution functions (= d. f.'s) that can be derived from operations on random variables defined on the same probability space. Quasi-copulas were characterized in [16] and, now, they are used both in finding pointwise best-possible bounds in the set of d. f.'s with given marginals ([23]) and in the study of cycle-transitivity between two random variables (see [5, 6]). Notice that M, Π and W are quasi-copulas and that, for every quasi-copula Q, one has

$$W(x,y) \le Q(x,y) \le M(x,y) \quad \text{for every } (x,y) \in [0,1]^2. \tag{4}$$

A semicopula S is called *copula* if it satisfies the *moderate growth property*, namely for all points x, x', y and y' in $[0,1]$ with $x \le x'$ and $y \le y'$,

$$S(x',y') + S(x,y) \ge S(x,y') + S(x',y).$$

In particular, one proves that every copula is also a quasi-copula. The great importance of copulas in statistics stems from *Sklar's Theorem*: given two continuous random variables (=r. v.'s) X and Y with joint d.f. H and marginal d.f.'s F_X and F_Y, there exists a unique copula C such that $H(x,y) = C(F_X(x), F_Y(y))$ for every x and y in $\overline{\mathbb{R}}$ (see [22] for more details).

At a first glance, the definition of semicopula might appear somewhat more general than actually is. In this sense, it will be shown in this section that condition (1) is quite restrictive and that it allows to characterize some basic semicopulas.

Proposition 1. Let S be a semicopula. The following statements are equivalent:

(a) S is concave;

(b) S is super-homogeneous, viz. $S(\lambda x, \lambda y) \geq \lambda\, S(x,y)$ for all x, y and λ in $[0,1]$;

(c) S is idempotent, viz. $S(x,x) = x$ for every $x \in [0,1]$;

(d) $S = M$.

Proof. If S is concave, then $S(\lambda x, \lambda y) = S(\lambda(x,y) + (1-\lambda)(0,0)) \geq \lambda S(x,y)$, and (b) holds. If S is super-homogeneous, then $S(x,x) \geq xS(1,1) = x$, which together with $S(x,x) \leq S(x,1) = x$, leads to (c). If S is idempotent, then $M(x,y) = S(M(x,y), M(x,y)) \leq S(x,y)$ and, since $S(x,y) \leq \min\{S(x,1), S(1,y)\} = M(x,y)$, this implies $S = M$. Finally, it is clear that M is concave. □

Proposition 2. Let S be a semicopula. The following statements are equivalent:

(a) S is convex and 1-Lipschitz;

(b) S is a function of the sum of its arguments, i. e. $S(x,y) = F(x+y)$ for some function F from $[0,2]$ into $[0,1]$;

(c) $S = W$.

Proof. (a) $\Longrightarrow$ (b) Suppose that S is convex and 1-Lipschitz. If $x+y \leq 1$, define $\lambda := y/(x+y)$, which is in $[0,1]$; then $(x,y) = \lambda(0, x+y) + (1-\lambda)(x+y, 0)$. Now, since S is convex,

$$0 \leq S(x,y) \leq \lambda S(0, x+y) + (1-\lambda) S(x+y, 0) = 0;$$

therefore, $S(x,y) = 0$. If $x+y \geq 1$, define $\lambda := (1-y)/[2-(x+y)]$, which is in $[0,1]$, in order to obtain $(x,y) = \lambda(1, x+y-1) + (1-\lambda)(x+y-1, 1)$. Again, since S is convex,

$$S(x,y) \leq \lambda S(1, x+y-1) + (1-\lambda) S(x+y-1, 1) = x+y-1,$$

and, since it is 1-Lipschitz,

$$S(1,1) - S(x,y) \leq 1 - x + 1 - y.$$

Therefore $S(x,y) = x+y-1$, and (b) holds.

(b) $\Longrightarrow$ (c) Suppose that there exists a function F from $[0,2]$ into $[0,1]$ such that $S(x,y) = F(x+y)$. If t is in $[0,1]$, then $F(t) = S(0,t) = 0$, and if t is in $[1,2]$, then $F(t) = S(1, t-1) = t-1$. Therefore, $F(t) = \max\{0, t-1\}$, and $S(x,y) = F(x+y) = \max\{x+y-1, 0\} = W(x,y)$.

Finally, it is clear that W is convex and 1-Lipschitz. □

Proposition 3. The following properties are equivalent for a semicopula S:

(a) S is positively homogeneous with respect to one variable, viz. for every x, y, λ in $[0,1]$, either $S(x,\lambda y) = \lambda S(x,y)$ or $S(\lambda x, y) = \lambda S(x,y)$;

(b) S has separate variables, viz. there exist two functions F_1 and F_2 defined from $[0,1]$ into $[0,1]$ such that $S(x,y) = F_1(x) \cdot F_2(y)$;

(c) S has linear sections in both the variables;

(d) $S = \Pi$.

Proof. Without loss of generality assume that S is homogeneous with respect to the first variable; then

$$S(x,y) = x\,S(1,y) = xy;$$

therefore (a) implies (b).

Let S be a semicopula with separate variables. Notice that one has both $F_1(1) \neq 0$ and $F_2(1) \neq 0$ since $F_1(1)F_2(1) = S(1,1) = 1$. Then, from (1), for all $x \in [0,1]$

$$S(1,x) = x = F_1(1)F_2(x) \quad \text{and} \quad S(x,1) = x = F_1(x)F_2(1).$$

Therefore

$$S(x,y) = F_1(x) \cdot F_2(y) = \frac{xy}{F_1(1)F_2(1)} = xy.$$

Now, suppose that (b) holds and let $S(x,y) = F_1(x) \cdot F_2(y)$ be a semicopula. From $S(1,1) = 1$, it follows that $F_1(1) = F_2(1) = 1$ and, hence, $S(x,1) = F_1(x) = x$. Therefore, for every $a \in [0,1]$, one has $S(x,a) = F_1(x) \cdot F_2(a) = F_2(a) \cdot x$, viz. the horizontal section of S at the point a is linear. The same result holds for the vertical section of S.

If S has linear sections in both the variables, then $S(x,a) = ax$ for every a and x in $[0,1]$, viz. $S = \Pi$. Finally, Π obviously satisfies (a). □

2.1. Superharmonic and subharmonic semicopulas

Let Ω be an open subset of $\mathbb{R}^2$. A twice continuously differentiable function $F : \Omega \to \mathbb{R}$ is said to be *harmonic* if

$$\Delta F(x,y) := \frac{\partial^2 F(x,y)}{\partial x^2} + \frac{\partial^2 F(x,y)}{\partial y^2} = 0 \quad \text{for all } (x,y) \in \Omega.$$

Moreover, such a function F is said to be *superharmonic* (resp. *subharmonic*) if $\Delta F \leq 0$ (resp. $\Delta F \geq 0$). For more details on harmonic function theory, see [3]. Here two important results for harmonic functions are recalled.

Theorem 1. (Maximum-minimum principle for harmonic functions)
Let Ω be a connected open subset of $\mathbb{R}^2$ and F a harmonic function on Ω. If F has either a maximum or a minimum on Ω, then F is constant on Ω.

Theorem 2. Let Ω be a connected open subset of $\mathbb{R}^2$ and F a superharmonic (respectively, subharmonic) function on Ω. If F has a minimum (respectively, a maximum) on Ω, then it is constant on Ω.

Proposition 4. The only harmonic semicopula is Π.

P r o o f. It is easily shown that Π is harmonic. Suppose that there exists another harmonic semicopula F and let (x_0, y_0) be a point in $]0,1[^2$ such that $\Pi(x_0, y_0) \neq F(x_0, y_0)$. Now, $G := F - \Pi$ is a harmonic function that vanishes on the boundary of $[0,1]^2$. Therefore, G has either a maximum or a minimum on $]0,1[^2$, and, in view of the maximum-minimum principle for harmonic functions, G is constant, and this constant is equal to zero, viz. $F = \Pi$. □

Corollary 1. If S is a superharmonic (resp. subharmonic) semicopula, then, for every $(x, y) \in [0,1]^2$, $S(x, y) \geq \Pi(x, y)$ (resp. $S(x, y) \leq \Pi(x, y)$).

P r o o f. If S is a superharmonic semicopula, then $G := S - \Pi$ is also superharmonic; moreover, it vanishes on the boundary of $[0,1]^2$. Therefore, $S(x, y) - \Pi(x, y) \geq 0$, for every (x, y) in $[0,1]^2$, because, otherwise, Theorem 2 would imply $S = \Pi$. A similar argument holds for subharmonic semicopulas. □

This simple remark turns out to be important when one restricts one's attention to *copulas*

$$S(x', y') + S(x, y) \geq S(x, y') + S(x', y).$$

In view of Sklar's Theorem, a copula can be uniquely associated to a continuous random pair (X, Y) and describes its dependence properties (see [22] for more details), as asserted below.

Proposition 5. Let X and Y be continuous r. v.'s with copula C. Then the following statements are equivalent:

(a) X and Y are *positively quadrant dependent* (shortly, PQD), viz. for every (x, y) in $\mathbb{R}^2$, $P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y)$;

(b) $C(x, y) \geq \Pi(x, y)$ for every $(x, y) \in [0,1]^2$.

Proposition 6. Let X and Y be continuous r. v.'s with copula C. Then the following statements are equivalent:

(a) Y is *stochastically increasing* in X (shortly $SI(Y|X)$), viz. the mapping $x \mapsto P(Y > y | X = x)$ is an increasing function for all y;

(b) the mapping $x \mapsto C(x, y)$ is concave for every $y \in [0,1]$.

Proposition 7. Let X and Y be continuous r. v.'s with copula C. Then the following statements are equivalent:

(a) X is *stochastically increasing* in Y (shortly $SI(X|Y)$), viz. the mapping $y \mapsto P(X > x|Y = y)$ is an increasing function for all x;

(b) the mapping $y \mapsto C(x, y)$ is concave for every $x \in [0, 1]$.

By obvious modifications of the above inequalities, analogous results can also be given for the symmetric concepts of *negative quadrant dependence* (shortly, NQD) and *stochastical decreasingness* (briefly, SD).

Proposition 8. Let (X, Y) be a continuous random pair with copula C. If C is superharmonic, then (X, Y) is positively quadrant dependent. Analogously, if C is subharmonic, then (X, Y) is negatively quadrant dependent.

Proposition 9. Let (X, Y) be a continuous random pair with a twice-differentiable copula C.

(a) If Y is stochastically increasing in X (briefly, $SI(Y|X)$) and if X is stochastically increasing in Y (briefly, $SI(X|Y)$), then C is superharmonic.

(b) If Y is stochastically decreasing in X (briefly, $SD(Y|X)$) and if X is stochastically decreasing in Y (briefly, $SD(X|Y)$), then C is subharmonic.

Proof. In view of Propositions 6 and 7, the property $SI(Y|X)$ is equivalent to the concavity of the function $x \mapsto C(x, y)$ for every $y \in [0, 1]$, and $SI(X|Y)$ is equivalent to the concavity of the function $y \mapsto C(x, y)$ for every $x \in [0, 1]$. Because C is twice differentiable, it follows that $\partial^2_{xx} C(x, y) \le 0$ and $\partial^2_{yy} C(x, y) \le 0$, from which $\Delta C(x, y) \le 0$. The proof of part (b) is analogous. □

This fact allows to introduce the concepts of super- and sub-harmonicity in the scheme of bivariate dependence concepts.

$$\text{SI(Y|X) \& SI(X|Y)} \quad \Longrightarrow \quad \text{Superharmonicity} \quad \Longrightarrow \quad \text{PQD(X,Y)}$$

$$\text{SD(Y|X) \& SD(X|Y)} \quad \Longrightarrow \quad \text{Subharmonicity} \quad \Longrightarrow \quad \text{NQD(X,Y)}$$

The converse implications in the above schemes are, in general, false.

Example 1. Consider the class of copulas given by $C_{fg}(x, y) = xy + \lambda f(x)\, g(y)$, where f and g are suitable functions and $\lambda > 0$ (see [25]). One has

$$\Delta C_{fg}(x, y) = \lambda(f''(x)\, g(y) + f(x)\, g''(y)).$$

If $f(t) = t(1-t)^2$ and $g(t) = t(1-t)$, then C_{fg} is a PQD copula, but

$$\Delta C_{fg}(x,y) = \lambda \left[(6x-4)y(1-y) - 2x(1-x)^2\right]$$

is (strictly) positive on the set $\{(x,y) \in [0,1]^2 : x = 1\}$ and it is (strictly) negative on the set $\{(x,y) \in [0,1]^2 : 0 \le x < 2/3\}$; thus C_{fg} is neither superharmonic nor subharmonic.

Analogously, one can find two functions f and g such that C_{fg} is superharmonic, but f and g are not both concave and, thus, C_{fg} is not $SI(Y|X)$ and $SI(X|Y)$.

3. k–LIPSCHITZ SEMICOPULAS

A semicopula S is said to be *k-Lipschitz* if there exists a constant k such that, for all x, x', y and y' in $[0,1]$,

$$|S(x,y) - S(x',y')| \le k\,(|x-x'| + |y-y'|)\,. \tag{5}$$

Notice that one necessarily has $k \ge 1$, as can easily be seen from the inequality $S(1,1) - S(1,x) = 1-x \le k(1-x)$ for all $x \in [0,1]$. The k-Lipschitz property ensures the stability of an aggregation operator, in the sense that, roughly speaking, small input errors give small output errors. This property has recently been studied, in the case of triangular norms, by several authors inspired by an open problem posed in [1] (see also [21] and [17]).

For every k-Lipschitz semicopula S, the following bounds hold (see [20]):

$$\max\{0, \min\{x,y\} - k(1 - \max\{x,y\})\} \le S(x,y) \le M(x,y). \tag{6}$$

Obviously, for $k = 1$, i. e. for quasi-copulas, one obtains inequalities (4).

In general, several results about quasi-copulas can be extended, with slight modifications, to the class $\mathcal{S}_k$ of k-Lipschitz semicopulas. In particular:

(a) $\mathcal{S}_k$ is convex;

(b) $\mathcal{S}_k$ is compact under the topology of uniform convergence;

(c) $\mathcal{S}_k$ is a complete lattice with respect to the pointwise ordering.

Below a characterization of k-Lipschitz semicopulas is given.

Theorem 3. Let S be a semicopula. If the mappings $x \mapsto S(x,y)$ and $y \mapsto S(x,y)$ are differentiable on $[0,1]$ except at countably many points, then the following statements are equivalent:

(a) S is a k-Lipschitz semicopula;

(b) S satisfies the following two conditions:

  (b1) S is continuous;

(b2) for every (x, y) in $[0,1]^2$ in which S admits first-order partial derivatives

$$0 \le \partial_x S(x,y) \le k \quad \text{and} \quad 0 \le \partial_y S(x,y) \le k.$$

By using Theorem 3, when $k = 1$, another characterization of quasi-copulas is obtained.

The following two lemmas will be needed for the proof of Theorem 3 (see, respectively, page 333 and 337 of [28]).

Lemma 1. Let $f : [a,b] \to \mathbb{R}$ be given. If f is continuous on $[a,b]$ and differentiable except at countably many points of $[a,b]$, and f' is Lebesgue integrable on $[a,b]$, then f is absolutely continuous on $[a,b]$.

Lemma 2. Let $f : [a,b] \to \mathbb{R}$ be given. The following statements are equivalent:

(a) for some $k \in \mathbb{R}$, $|f(x) - f(y)| \le k|x - y|$ for all $x, y \in [a,b]$;

(b) f is absolutely continuous on $[a,b]$ and $|f'(t)| \le k$ on $[a,b]$ for some $k \in \mathbb{R}$.

P r o o f o f T h e o r e m 3. The implication (a) $\Longrightarrow$ (b) is trivial.

In order to prove (b) $\Longrightarrow$ (a), let $S_y(t) := S(t, y)$ be the horizontal section of S at $y \in [0,1]$ and $S_x(t) := S(x,t)$ be the vertical section of S at $x \in [0,1]$. The functions S_x and S_y are continuous, differentiable on $[0,1]$ except at countably many points and their derivatives are bounded; therefore, by Lemma 1, it follows that they are absolutely continuous. But, now, if S_x and S_y are absolutely continuous and their derivatives are bounded from above by k, Lemma 2 ensures that S_x and S_y are Lipschitz with constant k. Therefore, for every (x,y) and (x',y') in $[0,1]^2$, one has

$$\begin{aligned}|S(x,y) - S(x',y')| &\le |S(x,y) - S(x',y)| + |S(x',y) - S(x',y')| \\ &\le |S_y(x) - S_y(x')| + |S_{x'}(y) - S_{x'}(y')| \\ &\le k\left(|x - x'| + |y - y'|\right),\end{aligned}$$

which is the desired assertion. □

By simple differentiations, Theorem 3 provides also the characterizations of two construction methods for k-Lipschitz semicopulas.

Proposition 10. Let S be a k-Lipschitz semicopula and let $h : [0,1] \to [0,1]$ be an increasing bijection. The *transform* of S by h is the function S_h defined on $[0,1]^2$ by

$$S_h(x,y) := h^{-1}\left(S(h(x), h(y))\right). \tag{7}$$

Suppose that S_h admits first derivatives except at countably many points. Then S_h is a k-Lipschitz semicopula if, and only if, a. e. on $[0,1]^2$

$$\begin{aligned}h'(x) \cdot \partial_x S(h(x), h(y)) &\le k \cdot h'\left(S_h(x,y)\right), \\ h'(y) \cdot \partial_y S(h(x), h(y)) &\le k \cdot h'\left(S_h(x,y)\right).\end{aligned}$$

Proposition 11. Let A and B be k-Lipschitz semicopulas. Let H be an idempotent binary aggregation operator and let F be the composition of A and B through H defined by

$$F(x,y) := H(A(x,y), B(x,y)) \qquad \text{for all } (x,y) \in [0,1]^2. \tag{8}$$

Suppose that F admits first derivatives except at countably many points. Then F is k-Lipschitz if, and only if, a. e. on $[0,1]^2$

$$0 \le \partial_x H(A(x,y), B(x,y)) \cdot \partial_x A(x,y) + \partial_y H(A(x,y), B(x,y)) \cdot \partial_x B(x,y) \le k,$$
$$0 \le \partial_x H(A(x,y), B(x,y)) \cdot \partial_y A(x,y) + \partial_y H(A(x,y), B(x,y)) \cdot \partial_y B(x,y) \le k.$$

4. A NEW CLASS OF QUASI–COPULAS

In this section, we obtain a characterization of a large class of quasi-copulas, depending on two univariate functions, that includes a subclass of Archimedean t-norms (see [18]).

The construction of new families of aggregation operators with desirable properties is often useful in real applications, where it is useful to have at disposal a large class of aggregation operators, depending on some parameters that can be fitted to real data. In particular, this new class generalizes a family used in [7] for the study of fuzzy preferences.

Denote by Φ the class of all functions $\varphi : [0,1] \to [0,+\infty]$ that are continuous and strictly decreasing, and by Ψ the class of all functions $\psi : [0,1] \to [0,+\infty]$ that are continuous, decreasing and such that $\psi(1) = 0$. Moreover, set $\Phi_0 = \Phi \cap \Psi$.

The *pseudo-inverse* of a function φ of Φ is defined by $\varphi^{[-1]}(t) := \varphi^{-1}(t)$ if $t < \varphi(0)$, and by $\varphi^{[-1]}(t) := 0$ otherwise. It can be proved that, for all t in $[0,1]$, $\varphi^{[-1]}(\varphi(t)) = t$, and, for all $t \ge 0$, $\varphi\left(\varphi^{[-1]}(t)\right) = \min\{t, \varphi(0)\}$. If $\varphi(0) = +\infty$, then the pseudo-inverse of φ coincides with its inverse, viz. $\varphi^{[-1]} = \varphi^{-1}$.

For all $(\varphi, \psi) \in \Phi \times \Psi$, one introduces the function $Q_{\varphi,\psi} : [0,1]^2 \to [0,1]$ defined by

$$Q_{\varphi,\psi}(x,y) := \varphi^{[-1]}\left(\varphi(x \wedge y) + \psi(x \vee y)\right), \tag{9}$$

where $x \wedge y := \min\{x,y\}$ and $x \vee y := \max\{x,y\}$.

It is easily proved that, for all $x \in [0,1]$,

$$Q_{\varphi,\psi}(x,1) = \varphi^{[-1]}\left(\varphi(x)\right) = x = Q_{\varphi,\psi}(1,x)$$

and $Q_{\varphi,\psi}$ is increasing in each place, namely it is always a semicopula. Below we shall investigate under which conditions on φ and ψ, $Q_{\varphi,\psi}$ is 1-Lipschitz, and, therefore, a quasi-copula.

Theorem 4. Let φ and ψ belong to Φ and to Ψ, respectively, and let $Q_{\varphi,\psi}$ be the function defined by (9). Then $Q_{\varphi,\psi}$ is a quasi-copula if, and only if, the two following statements hold:

(a) for every $r \le s$ and $t \in \left[0, \left(\psi \circ \varphi^{[-1]}\right)(r)\right]$, one has

$$\varphi^{[-1]}(r+t) - \varphi^{[-1]}(s+t) \le \varphi^{[-1]}(r) - \varphi^{[-1]}(s);$$

(b) for every $r \le s$ and $t \ge \left(\varphi \circ \psi^{[-1]}\right)(r)$, one has

$$\varphi^{[-1]}(r+t) - \varphi^{[-1]}(s+t) \le \psi^{[-1]}(r) - \psi^{[-1]}(s).$$

Proof. It suffices to show that conditions (a) and (b) are equivalent to the Lipschitz condition for $Q := Q_{\varphi,\psi}$.

Assume, first, that $x_1 < x_2 \le y$; then the inequality

$$Q(x_2, y) - Q(x_1, y) \le x_2 - x_1 \tag{10}$$

is equivalent to

$$\begin{aligned}\varphi^{[-1]}\left(\varphi(x_2) + \psi(y)\right) - \varphi^{[-1]}\left(\varphi(x_1) + \psi(y)\right) &\le x_2 - x_1 \\ &= \varphi^{[-1]}(\varphi(x_2)) - \varphi^{[-1]}(\varphi(x_1)).\end{aligned}$$

By setting $r := \varphi(x_2)$, $s := \varphi(x_1)$, $t := \psi(y)$, one has $t \in \left[0, \left(\psi \circ \varphi^{[-1]}\right)(r)\right]$ and $r \le s$; moreover, the last inequality is equivalent to (a).

Next assume $y \le x_1 < x_2$; then inequality (10) is equivalent to

$$\begin{aligned}\varphi^{[-1]}\left(\varphi(y) + \psi(x_2)\right) - \varphi^{[-1]}\left(\varphi(y) + \psi(x_1)\right) &\le x_2 - x_1 \\ &= \psi^{[-1]}(\psi(x_2)) - \psi^{[-1]}(\psi(x_1)).\end{aligned}$$

By setting $r := \psi(x_2)$, $s := \psi(x_1)$, $t := \varphi(y)$, one has $t \ge \left(\varphi \circ \psi^{[-1]}\right)(s)$ and $r \le s$. Because of the arbitrariness of $s \ge r$, it follows that $t \ge \left(\varphi \circ \psi^{[-1]}\right)(r)$ and the last inequality is equivalent to condition (b).

The final case, $x_1 \le y \le x_2$, follows from the two previous cases, since

$$\begin{aligned}Q(x_2, y) - Q(x_1, y) &= Q(x_2, y) - Q(y, y) + Q(y, y) - Q(x_1, y) \\ &\le x_2 - y + y - x_1 = x_2 - x_1,\end{aligned}$$

which concludes the proof. □

Although Theorem 4 characterizes quasi-copulas of type (9), conditions (a) and (b) might be somewhat impractical. However, these conditions are equivalent to the convexity of φ, when $\varphi = \psi$, as is shown by the following result.

Corollary 2. Let φ belong to Φ_0 and let $Q_{\varphi,\varphi}$ be a function of type (9). Then $Q_{\varphi,\varphi}$ is a quasi-copula if, and only if, φ is convex.

Proof. By Theorem 4, $Q_{\varphi,\varphi}$ is a quasi-copula if, and only if, for every $r \le s$ and for every $t \ge 0$, one has

$$\varphi^{[-1]}(r+t) - \varphi^{[-1]}(s+t) \le \varphi^{[-1]}(r) - \varphi^{[-1]}(s),$$

which can be written in the form

$$\varphi^{[-1]}(r+t)+\varphi^{[-1]}(s)\le\varphi^{[-1]}(s+t)+\varphi^{[-1]}(r). \tag{11}$$

Now, set $\alpha := t/(s+t-r)$, so that

$$r+t=\alpha\,(s+t)+(1-\alpha)\,r \quad \text{and} \quad s=(1-\alpha)\,(s+t)+\alpha\,r.$$

If φ is convex, so is $\varphi^{[-1]}$, and therefore

$$\begin{aligned}&\varphi^{[-1]}(r+t)+\varphi^{[-1]}(s)\\ &=\varphi^{[-1]}(\alpha(s+t)+(1-\alpha)\,r)+\varphi^{[-1]}((1-\alpha)(s+t)+\alpha r)\\ &\le\alpha\varphi^{[-1]}(s+t)+(1-\alpha)\varphi^{[-1]}(r)+(1-\alpha)\varphi^{[-1]}(s+t)+\alpha\varphi^{[-1]}(r)\\ &=\varphi^{[-1]}(s+t)+\varphi^{[-1]}(r).\end{aligned}$$

Conversely, if (11) holds, then putting, for all $a,b\ge 0$,

$$r:=a, \qquad t:=\frac{b-a}{2} \qquad s:=\frac{a+b}{2}$$

yields

$$2\varphi^{[-1]}\left(\frac{a+b}{2}\right)\le\varphi^{[-1]}(a)+\varphi^{[-1]}(b),$$

viz. $\varphi^{[-1]}$ is Jensen-convex; thus, because $\varphi^{[-1]}$ is continuous, it follows that $\varphi^{[-1]}$ is convex, and hence so is φ. □

In this way, one obtains the well-known fact that an Archimedean quasi-copula is necessarily a copula (see [22]). Moreover, note that some results on a similar class of copulas were given in [14].

A "more tractable" sufficient condition that ensures that $Q_{\varphi,\psi}$ is a quasi-copula is given here.

Proposition 12. Let φ and ψ belong to Φ and to Ψ, respectively. If φ is convex, then, for the function $Q_{\varphi,\psi}$ defined by (9), the following statements are equivalent

(a) $Q_{\varphi,\psi}$ is a quasi-copula;

(b) for every $\lambda\in[\varphi(1),\varphi(0)]$ the function $\rho_\lambda : \left[\varphi^{[-1]}(\lambda),1\right]\to\mathbb{R}$ defined by $\rho_\lambda(t):=\varphi^{[-1]}\left(\lambda+\psi(t)\right)-t$ is decreasing.

P r o o f. In order to show that $Q := Q_{\varphi,\psi}$ is 1-Lipschitz, assume, first, $x_1\le x_2\le y$. Then the inequality

$$Q(x_2,y)-Q(x_1,y)\le x_2-x_1 \tag{12}$$

is equivalent to

$$\varphi^{[-1]}\left(\varphi(x_2)+\psi(y)\right)+\varphi^{[-1]}(\varphi(x_1)) \leq \varphi^{[-1]}\left(\varphi(x_1)+\psi(y)\right)+\varphi^{[-1]}(\varphi(x_2)).$$

Set

$$s_1 := \varphi(x_2)+\psi(y),\ s_2 := \varphi(x_1),\ t_1 := \varphi(x_1)+\psi(y),\ t_2 := \varphi(x_2),$$

and $\alpha := (t_2 - s_1)/(t_2 - t_1)$ in order to obtain

$$s_1 = \alpha t_1 + (1-\alpha)t_2 \quad \text{and} \quad s_2 = (1-\alpha)t_1 + \alpha t_2;$$

therefore, since $\varphi^{[-1]}$ is convex, (12) is satisfied. In this case, the Lipschitz condition is a consequence of the convexity of φ alone.

Next assume $y \leq x_1 \leq x_2$; then inequality (12) is equivalent to

$$\varphi^{[-1]}\left(\varphi(y)+\psi(x_2)\right)-\varphi^{[-1]}\left(\varphi(y)+\psi(x_1)\right) \leq x_2 - x_1;$$

viz. condition (b).

The final case, $x_1 \leq y \leq x_2$, follows from the two previous cases. □

Example 2. Take $\varphi(t) = 1-t$ and $\psi(t) = \alpha(1-t)$ with $\alpha \in [0,1]$. For every $\lambda \in [0,1]$, the function ρ_λ of the previous proposition, given, for every $t \in [1-\lambda, 1]$, by

$$\rho_\lambda(t) = \max\{0, 1-\lambda-\alpha(1-t)\} - t,$$

is decreasing. Then one obtains the interesting class of quasi-copulas (see [7])

$$Q_{\varphi,\psi}(x,y) = \max\left\{0, x \wedge y - \alpha(1 - x \vee y)\right\}.$$

Notice that the lower bound in (6) belongs to this class.

Example 3. Let $\delta : [0,1] \to [0,1]$ be an increasing function such that $\delta(1) = 1$, $\delta(t) \leq t$ for all $t \in [0,1]$ and $|\delta(t) - \delta(s)| \leq 2|t-s|$ for all $t, s \in [0,1]$. Moreover, suppose that the function $t \mapsto t - \delta(t)$ is decreasing, take $\varphi(t) = 1-t$ and $\psi(t) = t - \delta(t)$. For every $\lambda \in [0,1]$, the function ρ_λ of the previous proposition is given, for every $t \in [1-\lambda, 1]$, by

$$\rho_\lambda(t) = \max\{0, 1-\lambda-t+\delta(t)\} - t;$$

by assumption, it is decreasing. The corresponding class of quasi-copulas

$$Q_{\varphi,\psi}(x,y) = \max\{0, \delta(x \vee y) - x \vee y + x \wedge y\}$$

coincides with the family of MT-quasi-copulas (that are also copulas), characterized and studied in [13].

Example 4. Take the functions

$$\varphi(t) := -\ln t \quad \text{and} \quad \psi(t) := -\ln\left(t + t^2 - t^3\right).$$

For every $\lambda \in [0, +\infty]$ the function $\rho_\lambda : [\exp(-\lambda), 1] \to \mathbb{R}$ is given by

$$\rho_\lambda(t) := \exp(-\lambda)\left(t + t^2 - t^3\right) - t.$$

Now, (φ, ψ) is in $\Phi \times \Psi$ and ρ_λ is decreasing, therefore Theorem 4 ensures that the function $Q_{\varphi,\psi}$, given by (9), is a quasi-copula. Notice that $Q_{\varphi,\psi}$ is not a copula, as shown in [12], where $Q_{\varphi,\psi}$ was expressed in a slightly different form.

5. MULTIVARIATE SEMICOPULAS: APPLICATIONS

A function $S : [0,1]^n \to [0,1]$ is said to be an *n-semicopula* ($n \geq 3$) if it satisfies the following two conditions:

(a) $S(x_1, x_2, \ldots, x_n) = x_i$ for x_i in $[0,1]$ ($i = 1, 2, \ldots, n$), and $x_j = 1$ for all $j \neq i$;

(b) $S(x_1, x_2, \ldots, x_n)$ is increasing in each place.

This definition is a direct generalization of the concept of n-copulas and has also a curious application in fuzzy measures.

For every $n \geq 2$, let $\mathcal{B}(\overline{\mathbb{R}}^n)$ be the class of Borel sets in $\overline{\mathbb{R}}^n$. A mapping $\nu : \mathcal{B}(\overline{\mathbb{R}}^n) \to [0,1]$ is called a *fuzzy measure* if it is *monotone*, viz. $\nu(A) \leq \nu(B)$ for all Borel sets $A \subseteq B$, and *normalized*, namely $\nu(\emptyset) = 0$ and $\nu(\overline{\mathbb{R}}^n) = 1$. A fuzzy measure ν is called *supermodular* if, for all Borel sets A and B

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B).$$

Given a fuzzy measure ν, the *distribution function* associated with ν is the function $F_\nu : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ given by

$$F_\nu(x_1, \ldots, x_n) = \nu\left([-\infty, x_1] \times \cdots \times [-\infty, x_n]\right).$$

Moreover, denote by F_{ν_i} the marginal d.f. associated to ν_i, where ν_i is the ith projection of ν ($i = 1, 2, \ldots, n$). Notice that, due to the lack of additivity, a fuzzy measure is not completely characterized by its associated d. f.. Reformulating a result of M. Scarsini (see [26]) through the concept of multivariate semicopula yields the following result.

Theorem 5. Let ν be a supermodular fuzzy measure on the space $(\overline{\mathbb{R}}^n, \mathcal{B}(\overline{\mathbb{R}}^n))$, F_ν its associated d. f., and F_{ν_i}, ($i = 1, 2, \ldots, n$), the marginal d. f.'s associated to the projections $\nu_1, \nu_2, \ldots, \nu_n$ of ν. Then there exists a semicopula $S_\nu : [0,1]^n \to [0,1]$ such that, for every $(x_1, \ldots, x_n) \in \overline{\mathbb{R}}^n$

$$F_\nu(x_1, \ldots, x_n) = S_\nu\left(F_{\nu_1}(x_1), \ldots, F_{\nu_n}(x_n)\right).$$

In particular, if ν is a probability measure, then S is a copula.

Thus, in some sense, a semicopula links the d. f. of a fuzzy measure to its marginal d. f.'s.

Now, we discuss n-semicopulas in the context of aggregation operators. It is known that a *global aggregation operator* may be represented by a family of n-ary aggregation operators $\{A_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ where, in general, A_n and A_m $(n \neq m)$ need not be related (see [9]). In this way, a family of semicopulas $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is, obviously, a global aggregation operator, but it need not have the neutral element property, because $S_n(x_1,\dots,x_{n-1},1)$ need not be equal to $S_{n-1}(x_1,\dots,x_{n-1})$. Here a different definition of *global semicopula* is proposed.

Let $n \in \mathbb{N}$, $n \geq 2$. Let $\{i_1, i_2, \dots, i_k\}$ be a nonempty set of k indices of $\{1,2,\dots,n\}$ $(1 \leq k \leq n)$ and let S_n be an n-semicopula. The *k-marginals* of S_n $(1 \leq k < n)$ are the $\binom{n}{k}$ semicopulas $S^n_{i_1,\dots,i_k} : [0,1]^k \to [0,1]$ defined, for every $(y_1,\dots,y_k) \in [0,1]^k$ by

$$S^n_{i_1,\dots,i_k}(y_1,\dots,y_k) = S_n(x_1,\dots,x_n),$$

where $x_1,\dots,x_n$ in $[0,1]$ are defined by

$$x_j = \begin{cases} y_j, & \text{if } j \in \{i_1,\dots,i_k\}; \\ 1, & \text{if } j \notin \{i_1,\dots,i_k\}. \end{cases}$$

Clearly, the n 1-marginals of S_n are all equal to the identity on $[0,1]$. In general, for $k \geq 2$, the k-marginals of S_n need not be equal to each other, but they are when S_n is commutative.

Definition 1. A family of commutative semicopulas $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is called a *global semicopula* if $S_1 = \mathrm{id}_{[0,1]}$ and, for every $n \geq 2$, S_{n-1} is the $(n-1)$-marginal semicopula of S_n.

Analogous definitions can be given for the new concepts of *global quasi-copula* and *global copula*.

In this way, a global semicopula is a global aggregation operator with neutral element 1 and annihilator 0.

In practice, it is not difficult to construct a global semicopula. It suffices to take a commutative 2-semicopula S and construct the family $\{S_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ in such a way that $S_1 = \mathrm{id}_{[0,1]}$, and, for every $n \geq 2$,

$$S_n(x_1,\dots,x_n) := S(S_{n-1}(x_1,\dots,x_{n-1}), x_n).$$

This method can be also used for quasi-copulas, but not for copulas, because it is not simple to construct a copula starting from the marginals (see [27] for more details).

The concept of a global copula has an interesting use in a probabilistic context. Consider a stochastic process $\{X_n\}_{n\in\mathbb{N}}$ in which all the random variables are continuous. In view of Sklar's Theorem, a (unique) k-dimensional copula C_k can be associated to any choice of k r. v.'s $X_{i_1},\dots,X_{i_k}$. In particular, if the r. v.'s of the process are *exchangeable*, then the copula C_k is commutative and it does not depend on the choice of the r. v.'s. Moreover, C_{k-1} is the $(k-1)$-marginal copula of C_k.

Conversely, if $\{C_n : [0,1]^n \to [0,1]\}_{n\in\mathbb{N}}$ is a global copula, in view of the Kolmogorov compatibility Theorem (see [19]), we can construct an exchangeable stochastic process $\{X_n\}_{n\in\mathbb{N}}$ (where each r. v. X_n is uniformly distributed on $[0,1]$) such that, for every $n \in \mathbb{N}$, C_n is the copula associated to any choice of n r. v.'s of the process.

Thus a one-to-one correspondence between global copulas and exchangeable stochastic processes is established.

ACKNOWLEDGEMENTS

The research of the first and the third authors was supported by M.I.U.R. through the project "Metodi stocastici in finanza matematica". The research of the second author was supported by the Junta de Andalucía and the Ministerio de Ciencia y Tecnología (Spain) under research project BFM2003-06522.

(Received October 21, 2005)

REFERENCES

[1] C. Alsina, M. J. Frank, and B. Schweizer: Problems on associative functions. Aequationes Math. *66* (2003), 128–140.

[2] C. Alsina, R. B. Nelsen, and B. Schweizer: On the characterization of a class of binary operations on distribution functions. Statist. Probab. Lett. *17* (1993), 85–89.

[3] S. Axler, P. Bourdon, and W. Ramey: Harmonic Function Theory. (Graduate Texts in Mathematics 137.) Springer–Verlag, New York 2001.

[4] B. De Baets: Analytical solution methods for fuzzy relational equations. In: Fundamentals of Fuzzy Sets, The Handbooks of Fuzzy Sets Series (D. Dubois and H. Prade, eds.), Chapter 6, Vol. 1, Kluwer Academic Publishers, Dordrecht 2000, pp. 291–340.

[5] B. De Baets and H. De Meyer: Transitivity frameworks for reciprocal relations: cycle-transitivity versus FG-transitivity. Fuzzy Sets and Systems *152* (2005), 249–270.

[6] B. De Baets, H. De Meyer, B. De Schuymer, and S. Jenei: Cycle evaluation of transitivity of reciprocal relations. Soc. Choice Welfare. To appear.

[7] B. De Schuymer, H. De Meyer, and B. De Baets: On some forms of cycle-transitivity and their relation to commutative copulas. In: Proc. EUSFLAT–LFA Conference, Barcelona 2005, pp. 178–182.

[8] B. Bassan and F. Spizzichino: Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. J. Multivariate Anal. *93* (2005), 313–339.

[9] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar: Aggregation operators: properties, classes and construction methods. In: Aggregation Operators. New Trends and Applications (T. Calvo, R. Mesiar, and G. Mayor, eds.), Physica–Verlag, Heidelberg 2002, pp. 3–106.

[10] D. Denneberg: Non-additive Measure and Integral. Kluwer Academic Publishers, Dordrecht 1994.

[11] F. Durante: What is a semicopula? In: Proc. AGOP – Summer School on Aggregation Operators, Lugano 2005, pp. 51–56.

[12] F. Durante: A new class of symmetric bivariate copulas. Preprint n. 19, Dipartimento di Matematica E. De Giorgi, Lecce, 2005.

[13] F. Durante, R. Mesiar, and C. Sempi: On a family of copulas constructed from the diagonal section. Soft Computing *10* (2006), 490–494.

[14] F. Durante, J. J. Quesada-Molina, and C. Sempi: A generalization of the Archimedean class of bivariate copulas. Ann. Inst. Statist. Math. (2006), to appear.

[15] F. Durante and C. Sempi: Semicopulæ. Kybernetika *41* (2005), 315–328.
[16] C. Genest, J. J. Quesada Molina, J. A. Rodríguez Lallena, and C. Sempi: A characterization of quasi-copulas. J. Multivariate Anal. *69* (1999), 193–205.
[17] R. Ghiselli Ricci and R. Mesiar: k-Lipschitz strict triangular norms. In: Proc. EUSFLAT–LFA Conference, Barcelona 2005, pp. 1307–1312.
[18] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.
[19] A. N. Kolmogorov: Grundbegriffe der Wahrscheinlichkeitsrechnung. Springer–Verlag, Berlin 1933. Reprinted in: Foundations of the Theory of Probability. Chelsea, Bronxm NY 1950.
[20] A. Mesiarová: k-Lipschitz aggregation operators. In: Proc. AGOP – Summer School on Aggregation Operators, Lugano 2005, pp. 89–92.
[21] A. Mesiarová: Triangular norms and k-Lipschitz property. In: Proc. EUSFLAT–LFA Conference, Barcelona 2005, pp. 922–926.
[22] R. B. Nelsen: An Introduction to Copulas. (Lecture Notes in Statistics 139.) Springer–Verlag, New York 1999.
[23] R. B. Nelsen, J. J. Quesada-Molina, J. A. Rodríguez-Lallena, and M. Úbeda-Flores: Best-possible bounds on sets of bivariate distribution functions. J. Multivariate Anal. *90* (2004), 348–358.
[24] R. B. Nelsen: Copulas and quasi-copulas: an introduction to their properties and applications. In: Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms (E. P. Klement and R. Mesiar, eds.), Elsevier, Amsterdam 2005, pp. 391–413.
[25] J. A. Rodríguez-Lallena and M. Úbeda-Flores: A new class of bivariate copulas. Statist. Probab. Lett. *66* (2004), 315–325.
[26] M. Scarsini: Copulæ of capacities on product spaces. In: Distribution Functions with Fixed Marginals and Related Topics (L. Rüschendorf, B. Schweizer, and M. D. Taylor, eds.), Institute of Mathematical Statistics (Lecture Notes – Monograph Series Volume 28), Hayward 1996, pp. 307–318.
[27] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. North Holland, New York 1983. 2nd edition: Dover Publications, Mineola, New York 2005.
[28] K. R. Stromberg: An Introduction to Classical Real Analysis. Chapman & Hall, London 1981.

Fabrizio Durante and Carlo Sempi, Dipartimento di Matematica "Ennio De Giorgi", Università di Lecce, 73100 Lecce. Italy.
e-mails: fabrizio.durante@unile.it, carlo.sempi@unile.it

José Juan Quesada-Molina, Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada. Spain.
e-mail: jquesada@ugr.es