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PERFORMANCE OF HEDGING STRATEGIES IN INTERVAL MODELS

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For a proper assessment of risks associated with the trading of derivatives, the performance of hedging strategies should be evaluated not only in the context of the idealized model that has served as the basis of strategy development, but also in the context of other models. In this paper we consider the class of so-called interval models as a possible testing ground. In the context of such models the fair price of a derivative contract is not uniquely determined and we characterize the interval of fair prices for European-style options with convex payoff both in terms of strategies and in terms of martingale measures. We compare interval models to tree models as a basis for worst-case analysis. It turns out that the added flexibility of the interval model may have an important effect on the size of the worst-case loss.

Keywords: uncertain volatility, robustness, option pricing, delta hedging, binomial tree martingale measure

AMS Subject Classification: 62P05

1. INTRODUCTION

Since the publication of the Black–Scholes formula (Black and Scholes [6]), the theory of option pricing has gone through extensive developments both in theory and in applications. Today it is the basis of a multibillion dollar industry which covers not only stock options but also contracts written on interest rates, exchange rates, and so on. The theory has implications for the pricing of derivatives, but also for the way in which the risks associated with these contracts can be hedged by taking market positions in related assets. In fact the two sides of the theory are linked together inextricably, since the theoretical price of an option is usually based on model assumptions that imply that all risk can be eliminated by suitable hedging. In daily financial practice, hedging is a theme that is at least as important as pricing; indeed, probably greater losses have been caused by misconstrued hedging schemes than by incorrect pricing.

Given the size of the derivatives markets, it is imperative that the risks associated with derivative contracts are properly quantified. The idealized model assumptions that usually form the basis of hedging constructions are clearly not enough to create a reliable assessment of risk. Value-at-Risk (VaR) has been introduced by J. P. Morgan (Risk Magazine [15]) as a way of measuring the sensitivity of the value of portfolios to typical changes in asset prices. Although the VaR concept has been criticized on theoretical grounds (see for instance Artzner et al. [1]) it has become a standard that is used by regulatory authorities worldwide. For portfolios with a strong emphasis on derivative contracts, the normality assumptions underlying the VaR methodology may not be suitable and additional ways of measuring risk are called for to generate a more complete picture.

Often, *stress testing* is recommended, in particular by practitioners, as a method that should supplement other measures to create a full picture of portfolio risk (see for instance Basle Comittee [3], Laubsch [12], Greenspan [8]). The method evaluates the performance of given strategies under fairly extreme scenarios. In particular in situations where worst-case scenarios are not easily identified, stress testing on the basis of a limited number of selected scenarios may be somewhat arbitrary, however. It would be more systematic, although also more computationally demanding, to carry out a comprehensive worst-case search among all scenarios that satisfy certain limits.

Major concerns associated to worst-case analysis are firstly, as already mentioned, computational cost, and secondly, the dependence of the results on the restrictions placed on scenarios. The latter problem cannot be avoided in any worst-case setting; in the absence of restrictions on scenarios, the analysis will not lead to meaningful results. To some extent the second problem may be obviated (at the cost of increased computational complexity) by looking at the results as a function of the imposed constraints. Among an array of risk management tools that are likely to be used jointly in practice, worst-case analysis may be valued as a method that is easily understood also by non-experts. In this paper we consider a fairly simple framework for worst-case analysis. We derive some theoretical properties and investigate issues of computational complexity. The model that we use can be described as follows.

In the standard Black–Scholes model, there is one parameter that is not directly observable, namely the *volatility*. When the value of this parameter is inferred from actual option prices, quite a bit of variation is seen both through time and across various option types. It is therefore natural that uncertainty modeling in the context of option pricing and hedging has concentrated on the volatility parameter. In particular, the so-called *uncertain volatility model* has been considered by a number of authors (Avellaneda et al. [2], Lyons [13], Wilmott [16]). In this model, volatility is assumed to range between certain given bounds, and prices and hedges are computed corresponding to a worst-case scenario.

The uncertain volatility model as proposed in the cited references assumes continuous trading, which is of course an idealization. In this paper we consider a discrete-time version which we call an *interval model*. In such a model the relative price changes of basic assets from one point in time to the next are bounded below and above, but no further assumptions concerning price movements are made. The purpose of the present paper is threefold: (i) to add to the knowledge of this model by proving some theoretical properties; (ii) to investigate in a few computational experiments whether worst-case analysis in the setting of interval models adds substantially to what can be inferred from parametric analysis; (iii) to suggest a way of reducing computational complexity without much loss of impact of the analysis.

We shall consider the hedging of a European-style derivative (i. e., fixed time of expiry) on a single underlying asset; in parts of the development below we assume that the derivative has a convex payoff function, such as is the case for instance for a call option or a put option. Taking the total costs of hedging and final closure of the position as a measure of performance, we show that for each given hedge strategy there is an interval of associated costs corresponding to the set of paths in an interval model. We investigate first the best obtainable hedging results within a given interval model both for a short position and for a long position in the derivative. The intersection of the cost intervals associated to all strategies can also be interpreted as the set of option premiums that are consistent with arbitrage pricing, and therefore we refer to it as the *fair price interval.* We show that, when the payoff is convex, this interval can in fact be obtained as the intersection of the cost intervals corresponding to only two strategies that are based on a maximum-volatility assumption and a minimum-volatility assumption respectively.

The use of interval models as a tool for worst-case analysis is investigated in a number of test cases. We take standard binomial delta strategies as examples and compute worst-case costs in a number of cases with misspecified volatility parameters, comparing the results to the ones that would have been obtained from an analysis based on a binomial tree model. It turns out that, even when we consider hedging of a European product with convex payoff, the additional flexibility in asset prices offered by the interval model may have an important impact on computed worst-case costs. The effect appears to derive from the fact that the interval model provides more freedom for the price of the underlying to enter a worst-case regime. Our results suggest that the use of an interval specification is particularly crucial in the first simulation steps; this suggests a way of achieving flexibility at limited computational cost.

Interval models have been studied before by Kolokoltsov [11] who gave a characterization of the interval of fair prices for European options with a nondecreasing and convex payoff function in these models. His characterization is in terms of iterated Bellman operators and leads to an explicit expression for the upper bound of the fair price interval. Here we characterize the interval of fair prices in a different way, associating both bounds of the fair price interval to particular hedge strategies. Kolokoltsov also discusses options on several underlyings and the continuous-time limit of the interval model. These are topics that we do not consider here; we emphasize the role of interval models in worst-case analysis. Howe and Rustem [9] have used interval models as a basis for optimization of portfolio strategies, taking transaction costs into account. Their examples concern optimization over one or two time steps. The performance of the resulting strategies over the full lifetime of the option might be assessed in the way proposed in this paper; for simplicity however we have chosen to take simple delta strategies as examples.

Interval models have also been studied recently, and independently, by Bernhard [4,5], from the perspective of robust control and game theory. The same upper bound on fair prices is derived for options with convex payoff (like in Kolokoltsov [11]), together with the corresponding optimal hedge. In addition, continuous time limits are discussed, and, in the second paper, the effect of transaction costs is analysed in detail. Results that are obtained here and that are not covered by Bernhard's papers include minimum fair prices, the connection with martingale measures, and the worst-case analysis under non-optimal (delta-)hedging.

The paper is structured as follows. The nonprobabilistic framework that we use is discussed briefly in general terms in the next section. Section 3 introduces the interval model and presents some basic results concerning this model. Then in Section 4 we turn to the use of interval models in worst-case analysis. We consider a series of test cases in which we vary both strategy parameters and model parameters. Conclusions follow in Section 5.

2. FRAMEWORK

2.1. Nonprobabilistic asset price models

We work in a discrete-time setting; time points are indicated by t_j, $j = 0, 1, 2, \ldots$. We consider in this section a market with a single underlying asset. There are no conceptual difficulties, however, in extending the analysis to a situation with multiple assets. To simplify formulas we assume zero interest rates; this assumption is not essential.

Our basic framework is nonprobabilistic. An *asset-price path* is a sequence

$$\mathcal{S} = \{S_0, S_1, S_2, \ldots, S_N\} \tag{1}$$

where S_j denotes the price of the underlying asset at time t_j, and t_N represents the time horizon, which will be fixed in the discussion below. A *model* $\mathbb{M}$ is a collection of such sequences of real numbers,

$$\mathbb{M} \subset (\mathbb{R}^+)^{N+1}; \tag{2}$$

no probability structure is imposed at the outset. A *European derivative* maturing at time t_N is specified by a payoff function $F(\cdot)$; the value of the derivative at time t_N for a path $\{S_0, \ldots, S_N\}$ is $F(S_N)$. In this paper we shall consider models in which asset prices are always positive and so we can look at the payoff function as a function from $(0, \infty)$ to $\mathbb{R}$. We note that if such a function is convex, it is also continuous.

2.2. Hedging strategies

For the purposes of this paper we assume that a particular derivative has been given once and for all. We consider portfolios consisting of one option owed (short position) and a quantity γ of the underlying asset held (long position). Positions are closed at the expiry of the derivative. A *strategy* is a collection of *strategy functions*

$$\{g_0(S_0), g_1(S_0, S_1), \ldots, g_{N-1}(S_0, \ldots, S_{N-1})\}$$

which at each time j determine the amount of the underlying asset to be held.

Note that we take strategies to be non-anticipating by definition. Path independent strategies take only the current price of the underlying into account and can therefore be characterized by strategy functions $g_j(S_j)$.

For use later on, we mention some examples of strategies:

- the (left-continuous) *stop-loss* strategy: $g_j(S_j) = 0$ if $S_j \leq X$, and $g_j(S_j) = 1$ if $S_j > X$, where X is a given parameter;
- the *delta* strategy with parameters $F(\cdot)$, u and d, which is given by strategy functions Δ_j that are defined recursively by

$$\Delta_{N-1}(S_{N-1}) = \frac{F(uS_{N-1}) - F(dS_{N-1})}{(u-d)S_{N-1}} \tag{3}$$

$$\Delta_j(S_j) = \lambda\Delta_{j+1}(uS_j) + (1-\lambda)\Delta_{j+1}(dS_j) \tag{4}$$

where $\lambda := \frac{u(1-d)}{u-d}$.

The parameters that are used in the stop-loss strategy and in the delta strategy are *strategy parameters*; they need not coincide with parameters of the interval model. A strategy g is said to be *continuous* if the strategy functions are continuous functions of their arguments. The delta strategy is continuous; the stop-loss strategy is not.

2.3. Fair prices

To a given hedging strategy $g := \{g_0(S_0), \ldots, g_{N-1}(S_0, \ldots, S_{N-1})\}$ and a given price path $\mathcal{S} = \{S_0, \ldots, S_N\}$ we associate the *total cost of hedging and closure* defined by

$$Q^g(F, \mathcal{S}) := F(S_N) - \sum_{j=0}^{N-1} g_j(S_0, \ldots, S_j)(S_{j+1} - S_j). \tag{5}$$

The first term represents the cost of closure of a short position in the derivative at time of expiry, and the second term (appearing with a minus sign) represents the gains from trading in the underlying according to the hedging strategy. For a given model $\mathbb{M}$ and a given initial price S of the underlying asset, the *cost range* of a strategy g is defined as the set of all possible total costs for paths in the model that start at the given initial price:

$$I^g(\mathbb{M}, F, S) := \{Q^g(F, \mathcal{S}) \mid \mathcal{S} = (S_0, \ldots, S_N) \in \mathbb{M},\ \ S_0 = S\}. \tag{6}$$

Given some inital value S for the underlying asset, a price f for a European derivative with payoff function F is said to be *a fair price within the model* $\mathbb{M}$ if for all strategies g there are paths $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathbb{M}$ such that

$$Q^g(F, \mathcal{S}_1) \leq f \leq Q^g(F, \mathcal{S}_2). \tag{7}$$

For any given subset I of $\mathbb{R}$, let $\operatorname{co} I$ denote the smallest convex subset of $\mathbb{R}$ containing I. Then the above definition of a fair price may also be expressed as

$$f \in \cap_g \operatorname{co} I^g(\mathbb{M}, F, S) \tag{8}$$

where the intersection takes place over all strategies. The right hand side in (8) is an interval, which possibly may reduce to a single point. We shall refer to this set as the *fair price interval* FPI($\mathbb{M}$, F, S) corresponding to the model $\mathbb{M}$, the payoff function $F(\cdot)$, and the initial price S. From the definition it follows that

$$\text{if } \mathbb{M}_1 \subset \mathbb{M}_2 \text{ then } \mathrm{FPI}(\mathbb{M}_1, F, S) \subset \mathrm{FPI}(\mathbb{M}_2, F, S). \tag{9}$$

Intervals of fair prices are discussed by Pliska [14, § 1.5] in a single-period setting and also appear in a stochastic continuous-time context; see for instance El Karoui and Quenez [10].

Remark. In the definition above, a price f can be fair even if there exists a strategy that generates costs that are equal to f along some (but not all) paths and that are less than f along all other paths. It should be noted, though, that in our nonprobabilistic setting no positive statement is made concerning the probability that a path with costs less than f will occur. We believe that, among the various possible definitions of the notion of a 'fair price', the one proposed above has to be chosen if one wants to capture both the usual Cox–Ross–Rubinstein price in the binomial model and the monotonicity property (9).

We now introduce martingale measures. We consider price paths of a fixed length $N+1$ with a given initial value S_0 and so the measures that we shall consider can be thought of as probability measures on the vector space $\mathbb{R}^N$. Any such measure $\mathbf{Q}$ will be called a *martingale measure for the model* $\mathbb{M}$ *with initial value* S_0 if it assigns probability 1 to the paths in the model $\mathbb{M}$ with initial value S_0 and if the martingale property holds, that is, $E^{\mathbf{Q}}(S_{j+k} \mid S_j, S_{j-1}, \ldots, S_0) = S_j$ for all j and $k \geq 0$. The set of all martingale measures for a model $\mathbb{M}$ with initial condition S will be denoted by $\mathcal{Q}(\mathbb{M}, S)$. The most important property of martingale measures that we shall need is the fact that the expected gain from any trading strategy under a martingale measure is zero. From this it follows immediately (see (5) and (6)) that, for any hedging strategy g applied to a European derivative with payoff function F, we have

$$E_{\mathbf{Q}}F \in \operatorname{co} I^g(\mathbb{M}, F, S)$$

for any martingale measure $\mathbf{Q} \in \mathcal{Q}(\mathbb{M}, S)$. Consequently, we can write

$$\{E_{\mathbf{Q}}F \mid \mathbf{Q} \in \mathcal{Q}(\mathbb{M}, S)\} \subset \cap_g \operatorname{co} I^g(\mathbb{M}, F, S)$$

where the intersection is taken over all strategies.

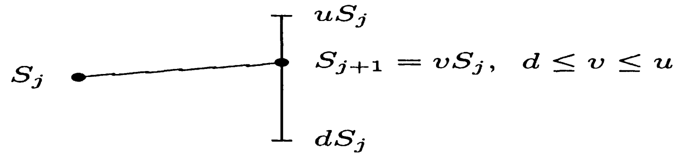
3. INTERVAL MODELS

3.1. Definition

The relations between the various sets that have been described above can be made more precise in the context of specific models. In this paper we shall be interested in particular in models of the following type. Recall that in our setting models are just collection of paths. An *interval model* is a model of the form

$$\mathbb{I}^{u,d} := \{S \mid S_{j+1} \in [dS_j, uS_j] \text{ for } j = 0, 1, 2, \ldots\} \tag{10}$$

where u and d are given parameters satisfying $d < 1 < u$. The figure below illustrates a typical step in a price path of an interval model.



The interval model may be compared to the standard binomial tree model with parameters u and d (Cox, Ross and Rubinstein [7])

$$\mathbb{B}^{u,d} := \{S \mid S_{j+1} \in \{dS_j, uS_j\} \text{ for } j = 0, 1, 2, \ldots\}. \tag{11}$$

Binomial models are motivated mainly because they can be used to approximate continuous-time models by letting the time step tend to zero. In contrast, the interval model may be taken seriously on its own, even for time steps that are not small.

3.2. Cost intervals

Our first result states that if asset prices behave according to an interval model, then the cost range of any strategy is an interval. The term 'interval' is understood here as 'convex subset of $\mathbb{R}$'; that is to say, intervals may be closed, open, or half-open, or may consist of a single point. In the proposition below we assume only continuity (rather than convexity) of the payoff function.

Proposition. Consider an interval model $\mathbb{I}^{u,d}$. For any strategy g with respect to a European derivative with continuous payoff function $F(\cdot)$ and for any initial price S, the cost range $I^g(\mathbb{I}^{u,d}, F, S)$ is an interval. If the strategy g is continuous, then the cost interval is closed.

Proof. The proof proceeds by induction with respect to the number of periods N. For $N = 1$ the cost of a strategy is given by $F(S_1) - \gamma_0(S_1 - S_0)$ for some real number $\gamma_0 = g_0(S_0)$, so it depends continuously on S_1. Since S_1 is restricted to an interval and since continuous functions map intervals to intervals, I^g must be an interval.

Next assume that the proposition is true for models with less than N steps, and consider the total cost range I^g in an N-step model for some fixed strategy g. First consider the costs of price paths $\{S_0, \ldots, S_N\}$ with $S_N = S_{N-1}$. It follows from the induction hypothesis that the cost range of the strategy g over these paths forms an interval, I' say. Take $p \in I^g$ and let $\{S_0, \ldots, S_{N-1}, S_N\}$ be the corresponding path. Consider the paths $\{S_0, \ldots, S_{N-1}, \alpha S_N + (1-\alpha)S_{N-1}\}$ for $0 \leq \alpha \leq 1$. Since the corresponding costs depend continuously on α, they form an interval that contains p and that also contains at least one point of I'. Therefore, the set I^g may be written as a union of intervals that all have at least one point in common with the interval I', and so I^g is itself an interval.

If a strategy is continuous, the cost function associated to it is continuous in the price paths. Because the set $\mathbb{I}^{u,d} \subset \mathbb{R}^{N+1}$ is compact, the cost function then achieves both its maximum and its minimum value on $\mathbb{I}^{u,d}$. □

An example of a cost interval that is not closed is provided by the stop-loss strategy as defined above in the case of a two-period model with $u = 1.1$, $d = 0.8$, and $S_0 = 100$, applied to a call option with exercise price $X = 80$. One readily computes that the cost interval is $[0, 36)$.

The computation of the cost interval amounts to determining the best and worst case costs over all price paths in a given interval model, and algorithms can be designed according to the principles of dynamic programming. We briefly sketch the standard idea.

Let θ_j denote a state variable at time t_j that summarizes all information over the strict past $t_0, \ldots, t_{j-1}$ that is relevant to a given strategy g. Replacing past prices by θ_j in the argument of the strategy functions g_j we obtain the state space system

$$\begin{aligned} \theta_{j+1} &= f_j(\theta_j, S_j), \quad \theta_0 \text{ fixed} \\ \gamma_j &= g_j(\theta_j, S_j) \end{aligned} \tag{12}$$

where f_j is a state evolution function and γ_j is the hedge position at t_j according to strategy g. Now determine, for every time instant, value functions $V^{\max}$ and $V^{\min}$ that assign to a state (θ_j, S_j) the worst-case and best-case costs respectively over all paths starting in S_j at t_j that satisfy the restrictions of the given interval model. Starting at expiry with boundary conditions

$$V^{\min}(N, S_N, \theta_N) = V^{\max}(N, S_N, \theta_N) := F(S_N) \tag{13}$$

we are led to a backward recursive optimization

$$V^{\min}(j, S_j, \theta_j) := \min_{v \in [d,u]} V^{\min}(j+1, vS_j, f_j(S_j, \theta_j)) - g_j(S_j, \theta_j)(v-1)S_j \tag{14}$$

(and for $V^{\max}$ 'min' replaced by 'max'). For discontinuous strategies the minima and maxima above need not exist; taking infima and suprema instead, we actually compute the closure of the cost interval.

The complexity of the algorithm depends on the number of state variables in the hedge strategy and the number of underlyings. The number of required operations is quadratic in NK where N is the number of time steps and K is the number of grid points in the state space of the θ and S variables. For regular grids, K depends exponentially on the dimension of θ and S. Variations of the above algorithm such as using a forward rather than a backward recursion do not fundamentally affect this complexity. In this paper we consider path-independent hedging strategies for options on a single underlying.

3.3. Characterizations of the fair price interval

In this section we characterize the fair price interval for European-style derivatives with convex payoffs, both in terms of strategies and in terms of martingale measures.

3.3.1. Characterization in terms of strategies

Below we shall be interested in particular in two strategies. The first is the standard binomial delta strategy for the binomial tree with the same parameters as the given interval model. We call this the *extreme delta strategy* because it corresponds to paths that at each time step exhibit the largest possible jump that is allowed by the interval model in either the upward or the downward direction. The second strategy that we shall consider is defined as follows. For a given convex function $F : (0, \infty) \mapsto \mathbb{R}$, the *subdifferential* $\partial F(x)$ of F at $x \in (0, \infty)$ is defined as the set of all scalars γ such that $F(y) \geq F(x) + \gamma(y - x)$ for all y. The fact that $\partial F(x)$ is non-empty for all x follows from the assumed convexity of F. The elements of $\partial F(x)$ are called the *subgradients* of F at x. We call a *subgradient strategy* for a European derivative with convex payoff F any strategy g such that $g(S_j) \in \partial F(S_j)$. For instance the stop-loss strategy defined above is a subgradient strategy for the European call option.

The special role played by the extreme delta and the subgradient strategies is indicated in the theorem below. In the theorem we place ourselves in the position of an institution that holds a short position in a certain derivative and that is looking for a hedging strategy. We shall identify strategies that minimize worst-case costs and strategies that maximize best-case costs. The first are of course simple to interpret; the latter strategies are more easily viewed as the opposites of strategies that maximize worst-case gain for a party holding a long position in the derivative. The theorem states that, in a situation described by an interval model, an institution holding a short position in a European option with a convex payoff can minimize its downward risk by hedging as if maximal volatility is going to occur. On the other hand, an institution holding a long position will minimize its downward risk by hedging as if minimal (actually zero) volatility will occur. Part 1 of the theorem below can also be found in Kolokoltsov [11].

Theorem 1. Consider a frictionless market in which the price paths of an underlying asset follow an interval model with parameters u and d, where $d < 1 < u$; the initial value S_0 of the underlying is given. Let $F(\cdot)$ be the payoff function of a European derivative, and assume that F is convex. We consider portfolios that consist of (i) a given short position in the option, and (ii) a position in the underlying asset that is determined at each time point by a trading strategy.

1. Lowest worst-case costs are generated by the extreme delta hedging strategy. The corresponding costs, which we denote by $f_{\max}$, are given by the Cox–Ross–Rubinstein price of the derivative in the binomial tree model with the same parameters as the interval model. Worst-case costs are achieved for paths in this tree model.

2. Highest best-case costs are generated by any subgradient strategy. The corresponding costs are equal to $f_{\min} := F(S_0)$ and are realized along the constant path.

3. The fair price interval for the derivative is $[f_{\min}, f_{\max}]$.

The proof requires the following two technical lemmas. The first result is given without proof by Kolokoltsov [11]; we provide a brief argument for completeness.

Lemma 1. Let u and d be such that $d < 1 < u$. If $h : (0, \infty) \mapsto \mathbb{R}$ is convex, the function $\tilde{h}(x)$ defined for $x > 0$ by

$$\tilde{h}(x) \;=\; \min_{\gamma \in \mathbb{R}} \max_{dx \leq y \leq ux} [h(y) - \gamma(y - x)] \tag{15}$$

is convex as well.

Proof. Since $h(y) - \gamma(y - x)$ is convex as a function of y, the maximum in (15) must be taken at the boundary of the interval $[dx, ux]$, so

$$\tilde{h}(x) \;=\; \min_{\gamma} \max[h(dx) + \gamma(1 - d)x, h(ux) - \gamma(u - 1)x].$$

Since the first argument in the 'max' operator is increasing in γ and the second is decreasing, the minimum is achieved when both are equal, that is to say, when γ is given by

$$\gamma \;=\; \frac{h(ux) - h(dx)}{(u - d)x}.$$

Therefore we have the following explicit expression for $\tilde{h}$ in terms of h:

$$\tilde{h}(x) \;=\; \frac{1 - d}{u - d} h(ux) + \frac{u - 1}{u - d} h(dx). \tag{16}$$

Since the property of convexity is preserved under scaling and under taking positive linear combinations, it is seen from the above that the function $\tilde{h}$ is convex. □

Lemma 2. Let $h(\cdot)$ be a convex function, and let u and d be such that $d < 1 < u$. Then we have

$$\max_{\gamma \in \mathbb{R}} \min_{dx \leq y \leq ux} [h(y) - \gamma(y - x)] \;=\; h(x). \tag{17}$$

Proof. We obviously have

$$\min_{dx \leq y \leq ux} [h(y) - \gamma(y - x)] \;\leq\; h(x)$$

for all γ, since the value at the right hand side is achieved at the left hand side for $y = x$. So to complete the proof it suffices to show that there exists γ such that

$$h(y) \;\geq\; h(x) + \gamma(y - x)$$

for all y. Clearly, any subgradient of h at x can serve as such. □

Proof of Theorem 1.

1. The value function for the problem of minimizing worst-case costs is given by

$$V(S,j) = \min \max_{S_j=S} \left[F(S_N) - \sum_{k=j}^{N-1} \gamma_k (S_{k+1} - S_k) \right] \tag{18}$$

where the minimum is taken over all strategies, and the maximum is taken over all paths in the given interval model that satisfy $S_j = S$. The value function satisfies the recursion

$$V(S,j-1) = \min_{\gamma} \max_{dS \le S' \le uS} [V(S',j) - \gamma(S' - S)] \tag{19}$$

and of course we have

$$V(S,N) = F(S). \tag{20}$$

It follows from Lemma 1 that the functions $V(\cdot,j)$ are convex for all j. Therefore the strategy that minimizes maximal costs is the same as the minmax strategy for the binomial tree model with parameters u and d, and the corresponding worst-case paths are the paths of this tree model.

2. The proof is *mutatis mutandis* the same as above; use Lemma 2 rather than Lemma 1.

3. This is by definition a consequence of 1. and 2. □

In the case of a call option, the stop-loss strategy is best in the worst-case sense for a party holding a long position, and the corresponding worst-case paths are those in which the strike level is not crossed. More generally, it can be easily verified that if we have a piecewise linear payoff function, then the worst-case paths for a party holding a long position in the derivative and following a subgradient hedge strategy are those in which the successive values of the underlying are confined to one of the regions where the payoff function behaves linearly.

3.3.2. Characterization in terms of martingale measures

It is clear that interval models allow many martingale measures. For instance, for an interval model with parameters u and d, all martingale measures associated to binomial tree models with parameters u' and d' satisfying $d \le d' < 1 < u' \le u$ are also martingale measures for the interval model. We have already shown that for interval models the fair price interval is closed. In the following theorem we show that all fair prices are generated by martingale measures, and we indicate the measures that generate the extreme points of the fair price interval.

Theorem 2. Let an interval model $\mathbb{I}^{u,d}$ and an initial asset value S_0 be given, and let $\mathcal{Q}$ denote the set of all martingale measures that can be placed on the collection of paths in $\mathbb{I}^{u,d}$ that start at S_0. Consider a European derivative with convex payoff $F(\cdot)$, and denote the fair price interval for the derivative by $[f_{\min}, f_{\max}]$.

1. We have
$$\{E_{\mathbf{Q}}[F(S_N)] \mid \mathbf{Q} \in \mathcal{Q}\} \;=\; [f_{\min}, f_{\max}].$$

2. The minimal option price $f_{\min}$ is the expected value of the derivative under the martingale measure that assigns probability one to the constant path $S_j = S_0$ for all j.

3. The maximal option price $f_{\max}$ is the expected value of the derivative under the martingale measure that assigns probability one to the collection of paths in the submodel $\mathbb{B}^{u,d}$ (the binomial tree model with parameters u and d).

Proof. Items 2. and 3. are clear from the previous theorem. One part of item 1. follows easily from the characterization of the consistent price interval as the intersection of all cost intervals. Indeed, if $\mathbf{Q}$ is a martingale measure, then $E_{\mathbf{Q}}F(S_N)$ is in the cost interval I^g for any strategy g, since the expected result from any trading strategy under the martingale measure is zero. So $E_{\mathbf{Q}}F(S_N)$ is in the intersection of all cost intervals. To show that every such premium can be obtained as an expected value under some martingale measure, let $\mathbf{Q}^\alpha$ denote the martingale measure associated to the binomial tree $\mathbb{B}^{u,d}$ with parameters $u_\alpha := 1+\alpha(u-1)$ and $d_\alpha := 1/u_\alpha$. For $0 \le \alpha \le 1$, the measure $\mathbf{Q}^\alpha$ is also a martingale measure on $\mathbb{I}^{u,d}$. The expected option value $f_\alpha := E_{\mathbf{Q}^\alpha}F(S_N)$ is continuous in α; moreover $f_\alpha = f_{\min}$ for $\alpha = 0$ and $f_\alpha = f_{\max}$ for $\alpha = 1$. Hence every price $f \in [f_{\min}, f_{\max}]$ occurs as an expected option value under some martingale measure. □

For general incomplete markets in a single-period setting, the relation between martingale measures and fair price intervals is given by Pliska [14, § 1.5]. There may be many martingale measures along with the one mentioned in item 2. of the theorem that generate the minimal price; for instance if the option is a call option, then any martingale measure under which there is zero probability of crossing the strike level will generate this price. On the other hand the maximal price is generated uniquely by the measure indicated in item 3., except in the (trivial) case in which the payoff function $F(\cdot)$ is linear; for instance, if the option is a call option, the measure is unique until the asset price in a path becomes too high or low for crossing the exercise level.

All intermediate prices are generated by many different martingale measures, and unlike the extreme prices, they obviously allow for 'equivalent martingale measures', in the sense that every set of paths in the interval model with positive (Lebesgue) measure is assigned a positive probability.

Example. For a simple illustration of the above results, consider a call option in a one-step interval model. In such a model the choice of a strategy comes down to the choice of a real number which indicates the position to be taken in the underlying at time 0. In Figure 1 we show the results of the extreme delta and the stop-loss strategies for a range of initial values S_0. For both strategies we indicate the worst and the best case under the interval model. Since we have shown that for the call option the boundary points of the consistent price interval are given by these two strategies, the consistent price interval can be read off for each value of S_0 as the intersection of the cost intervals of these two strategies.

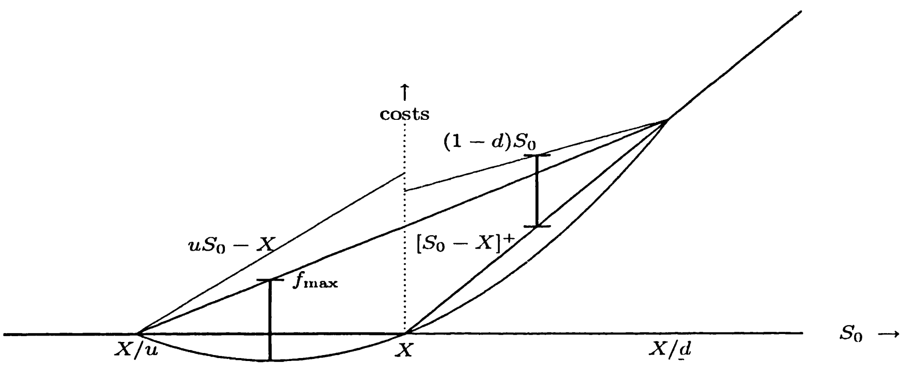


Fig. 1. Consistent price intervals in one-step models.
The thick lines correspond to the consistent price interval in the interval model $\mathbb{I}^{u,d}$, as a function of the initial price S_0. Specifically, the upper bound is $f_{\max} = \frac{1-d}{u-d}(uS_0 - X)$ and the lower bound is $f_{\min} = [S_0 - X]^+$. The thin lines with discontinuity in $S_0 = X$ denote the worst-case costs for the Stop-loss strategy; the curved line below denotes the best-case costs under the extreme delta strategy, which are given by $\frac{(uS_0-X)(S_0-X)}{(u-d)S_0}$. In addition, for both strategies a cost interval is shown: for the delta strategy one with an initial price below X, and for the stop-loss strategy one with an initial price above X.

4. WORST-CASE ANALYSIS

In this section we compare the results obtained from an interval model with those obtained from a simpler model (the standard binomial model) in a number of test cases. The derivative that we consider is a European call option. One may of course in principle envisage many hedging strategies, but we shall restrict ourselves to delta strategies derived from binomial tree models. Specifically we denote by Δ^σ the standard hedge for the binomial tree model with parameters u_σ and d_σ, where for each given number $\sigma > 0$ the parameters u_σ and d_σ are chosen such that $d_\sigma = 1/u_\sigma$ and the price of the option in the tree model with parameters u_σ and d_σ is equal to the price in the continuous-time Black–Scholes model with volatility parameter σ. In this way we have a one-parameter family of strategies that we shall test.

The tests will be carried out in an interval model. As always when one is carrying out worst-case analysis, one has to specify the range of situations that will be considered; for an interval model this comes down to the choice of the parameters u and d. The results of the test will depend on this choice; the choice is, however, to some extent arbitrary. One way out is to carry out tests for a range of parameter values. In view of the moderate computational demands associated to path-independent hedging of derivatives on single assets, we will in fact proceed in this manner. We shall consider interval models with parameters u_τ and d_τ which are determined by the single parameter τ in the same way as above. These interval models will be denoted by $\mathbb{I}^\tau$, and the tree models with the same parameters will be denoted by $\mathbb{B}^\tau$.

When looking for worst-case scenarios, one may be tempted to think of paths with extreme jumps. It turns out however that not always the paths with the largest possible jumps are the ones that generate the worst costs. This is demonstrated in the following simple example.

4.1. A non-extreme worst case

Consider an at-the-money European call option with exercise price $X = S_0 = 100$ in a two-period model. Let $u_\sigma = 1.20$ and $u_\tau = 1.25$; this means that the hedge strategy is based on $\sigma = 0.16$, whereas the actual volatility in the model is $\tau = 0.19$. The price of the option in the tree model $\mathbb{B}^\sigma$ is $f_{\max} = 9.09$; in the tree model $\mathbb{B}^\tau$ the price is $f_{\max} = 11.11$. The latter quantity also represents the maximal worst-case costs in $\mathbb{I}^\tau$ which are achieved by the extreme delta hedge Δ^τ. If however the strategy Δ^σ is applied in the model $\mathbb{I}^\tau$, then the worst-case costs are found to be $f_{\max} = 13.26$. The corresponding worst-case path is $\{S_0, S_1, S_2\} = \{100, 83.3, 104.2\}$. This is not an extreme path. If we limit paths to the tree $\mathbb{B}^\tau$ and we compute the worst-case costs for the strategy Δ^σ in this model, then we find the value $f_{\text{bin}} = 11.36$; there are two corresponding worst-case paths, namely $\{100, 125, 100\}$ and $\{100, 125, 156.25\}$.

The conclusions from this example may be summarized as follows. The worst-case costs for the strategy Δ^σ in the interval model $\mathbb{I}^\sigma$ are equal to 9.09. A worst-case analysis in the tree model $\mathbb{B}^\tau$ suggests that this figure may increase to 11.36 if the actual volatility turns out to be $\tau = 0.19$ rather than $\sigma = 0.16$. However, if the analysis is carried out in the interval model $\mathbb{I}^\tau$ rather than in the tree model $\mathbb{B}^\tau$ it turns out that actually costs may go up to 13.26. So if the option is sold for 9.09 corresponding to the implied volatility $\sigma = 0.16$, the potential loss in an interval model with volatility parameter $\tau = 0.19$ is almost twice as big as in the binomial tree model with the same parameter.

4.2. Worst cases in interval models vs. tree models

In a more extensive experiment, we consider the hedging of a European call option in a ten-period model for several combinations of hedging strategies and interval models. The following parameter values are used:

initial price	$S_0 = 100$
exercise price	$X = 100$
exercise time	$T = 1$
interest rate	$r = 0$
time step	$h = 0.1$; so $N = 10$.

As our main reference point we take $\sigma^* = \tau^* = 0.2$. We compute the worst-case costs of hedging strategies Δ^σ in the models $\mathbb{I}^\tau$, with σ and τ ranging from 0.1 to 0.3 in steps of 0.05. Worst cases are determined as indicated in Section 3.2, where the one-dimensional optimizations are implemented on a grid for the logarithms of prices. The results are shown in Figure 2.

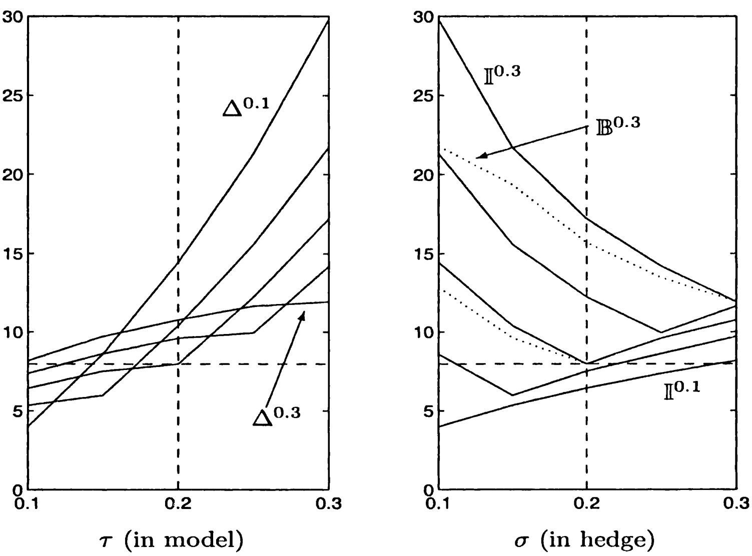


Fig. 2. Worst-case costs for Δ^σ in $\mathbb{I}^\tau$.
In the left plot each line corresponds to worst-case costs under a fixed strategy Δ^σ for a range of interval models, in the right plot every line denotes the worst-case costs in a fixed interval model $\mathbb{I}^\tau$ for a range of hedging strategies. The dotted lines denote worst-case costs in the binomial trees $\mathbb{B}^{0.2}$ and $\mathbb{B}^{0.3}$; on the left these are not shown.

Because all paths of an interval model with a given volatility parameter are also contained in interval models with a larger volatility parameter, the worst-case costs corresponding to a fixed strategy must be nondecreasing as a function of τ; this is seen in the left-hand plot. Both plots also show the optimality in a worst case sense of Δ^σ within the model $\mathbb{I}^\sigma$; for $\sigma^* = 0.2$ this is indicated by dashed lines. There is a striking asymmetry between over-hedging and under-hedging: the loss due to under-hedging according to $\Delta^{0.1}$ in the interval model $\mathbb{I}^{0.3}$ is much larger than the loss due to over-hedging according to $\Delta^{0.3}$ in $\mathbb{I}^{0.1}$. The dotted lines in the right-hand plot again show that worst-case analysis in a binomial tree setting may produce results that are quite a bit more optimistic than the results that are obtained from an interval model, especially when the hedge strategy is based on a value of the volatility that is considerably too low. So the risk associated with a too low specification of volatility is higher when volatility is non-constant than in the case in which volatility is higher than expected but constant.

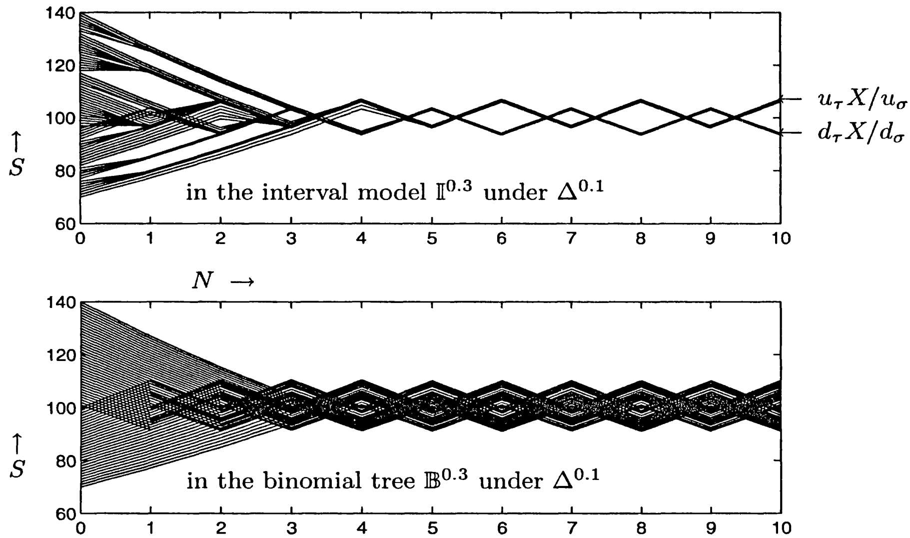


Fig. 3. Worst-case paths for various initial prices.
The upper plot contains worst-case paths in $\mathbb{I}^\tau$ under Δ^σ for $\tau = 0.3$ and $\sigma = 0.1$, with initial price ranging from 70 to 140. The lower plot shows the worst case paths in $\mathbb{B}^\tau$.

The question may arise how non-extreme price fluctuations contribute to extreme costs. In order to discover a pattern we consider several worst-case paths, for a range of initial prices S_0 and all other parameters kept constant. These are compared with worst-case paths in the binomial tree model in Figure 3. The graphs indicate that in both models, costs are maximal for paths that cross the exercise level as often as possible with extreme jumps. They differ however in the levels of the peaks in the end regime. In the binomial model (with $u = 1/d$) all prices are of the form $u^j S_0$, where j may be positive or negative, and hence the peak levels are at $u^j S_0$ where j is the smallest integer such that $u^j S_0 > X$. Non-extreme jumps allow a change in the level of peaks, and this extra freedom in interval models may increase the cost substantially. The simulations suggest that worst-case costs are achieved for upward peaks at X/u_σ or downward peaks at X/d_σ; a formal statement in this direction remains to be proven, however. The graphs clearly suggest that most of the freedom allowed by interval models is used in the first few time steps. This in turn suggests that a reduction in computational load of a worst-case search may be achieved by using an interval model for the first few time steps (or even just for the first one) and a binomial model thereafter.

A similar effect is apparent when the exercise level X is varied with initial prices kept fixed. This is illustrated in Figure 4, in which worst-case costs in interval models and binomial trees are compared for various exercise prices X. There is considerable variation in the size of the underestimation of worst-case costs by binomial models as compared to interval models. Again the irregular pattern for the binomial tree is explained by the fact that worst-case paths are restrained to fixed grid points $u^j S_0$.

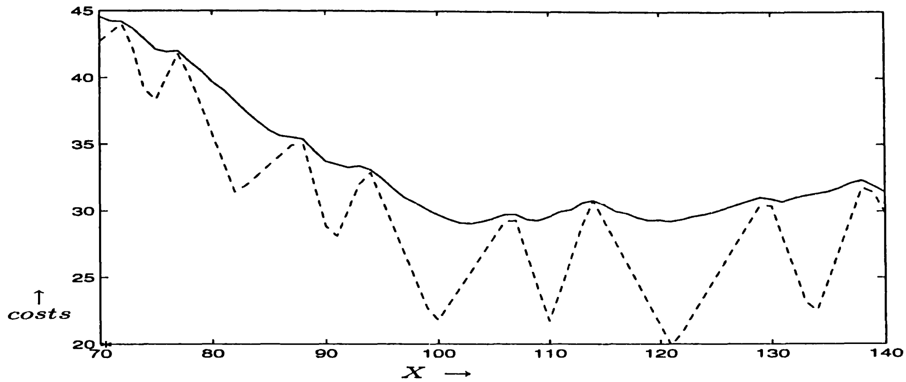


Fig. 4. Worst-case costs for various exercise levels.
The solid line corresponds to worst-case costs in $\mathbb{I}^{0.3}$ under $\Delta^{0.1}$, with X ranging from 70 to 140 (by unit steps). The dashed line represents worst-case costs in $\mathbb{B}^{0.3}$.

5. CONCLUSIONS

Like the well known binomial tree models, interval models have a certain didactical value in that they allow certain concepts to be explained in a fairly simple context. In particular interval models allow a discussion of incomplete markets and consistent price intervals. They provide an easily understood context in which one can do worst-case analysis and so they may play a role in risk management in addition to the standard tools based on sensitivity analysis and VaR computations.

In this paper we have studied hedging strategies for European derivatives with convex payoff functions in the context of interval models. It turns out that even for such rather simple derivatives the interval analysis has added value with respect to an analysis based on constant-volatility models; indeed, worst-case paths may show a mixture of moderate and extreme price changes. In the case of delta hedging strategies for standard call options, consideration of a few test cases has suggested that most of the value added by the interval model derives from the first time step, so that a good approximation to worst-case costs in an interval model would already be obtained, at considerably reduced computational cost, by following the interval model in the first step and the tree model for all following steps. Further analysis is needed to see whether similar simplifying rules may also be formulated for other derivatives.

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