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FUZZY DISTANCES

JOSEF BEDNÁŘ

In the paper, three different ways of constructing distances between vaguely described objects are shown: a generalization of the classic distance between subsets of a metric space, distance between membership functions of fuzzy sets and a fuzzy metric introduced by generalizing a metric space to fuzzy-metric one. Fuzzy metric spaces defined by Zadeh's extension principle, particularly to \mathbb{R}^n are dealt with in detail.

Keywords: fuzzy metric, fuzzy distance, fuzzy metric space, fuzzy contraction

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1. INTRODUCTION

When programming a fuzzy database with search [1] storing both exact and relatively vague information, we encountered a problem of identifying the item that has the least distance from a required item. In this paper, we will not be concerned with searching but, rather, with the definition itself of a fuzzy (or crisp) distance between fuzzy sets, which will certainly be useful also in other parts of fuzzy mathematics such as fuzzy clustering or fuzzy optimization.

The definitions of fuzzy distances published by different authors [3, 4, 7, 9] can be divided into three distinct groups:

1. generalization of the classic distance between subsets of a metric space (Section 3),
2. distance between membership functions of fuzzy sets (Section 4),
3. fuzzy metric introduced by generalizing a metric space to fuzzy-metric one (Section 5).

As none of the fuzzy distance definitions published fitted our purpose [2], we defined a fuzzy metric space on a set of fuzzy points (Definition 2.3) using Zadeh's extension principle. The properties of this fuzzy metric space are described in the following chapters and related to the fuzzy distances of other authors.

2. BASIC NOTIONS AND DENOTATIONS

Definition 2.1. Let $X \neq \emptyset$ be a set and $\mu_A : X \rightarrow [0, 1]$ a mapping. By a *fuzzy set* A on X we understand the set of all ordered pairs $\{(x, \mu_A(x)); x \in X; \mu_A(x) \in [0, 1]\}$.

We will also denote it by $A = (X, \mu_A)$. X will be called the *universe* and the function μ_A will be called the *membership function of A*. The number $\mu_A(x)$ will be called the *membership degree to the fuzzy set A of x*.

Next, we will work with generally known notions whose definitions are listed in [5, 6].

Definition 2.2. The *support of a fuzzy set* $A = (X, \mu_A)$ is defined as the ordinary set $\text{supp } A = \{x \in X; \mu_A(x) > 0\}$. The *kernel of a fuzzy set* $A = (X, \mu_A)$ is defined as the ordinary set $\text{ker } A = \{x \in X; \mu_A(x) = 1\}$. The α -*cut of a fuzzy set* $A = (X, \mu_A)$ is defined as the ordinary set $A_\alpha \subseteq X$ such that, $A_\alpha = \{x \in X; \mu_A(x) \geq \alpha\}$ for $\forall \alpha \in [0, 1]$. A fuzzy set $A = (X, \mu_A)$ is called *normal*, if $\text{ker } A \neq \emptyset$. A fuzzy set $A = (X, \mu_A)$ where X is a linear space is called *convex*, if all its α -cuts are convex. The *height of a fuzzy set* $A = (X, \mu_A)$ is defined as a number $h(A) = \sup_X \mu_A(x)$.

Denotation 2.1. *Cartesian product of fuzzy sets* A_1, A_2, \dots, A_n will be denoted by $A_1 \times A_2 \times \dots \times A_n$.

Definition 2.3. A fuzzy set $A = (X, \mu_A)$ where X is a linear space is called a *fuzzy point*, if the fuzzy set A is convex and normal. A fuzzy point is called a *fuzzy point continuous from above*, if all its α -cuts are closed sets. A fuzzy point $A = (R, \mu_A)$ is called a *fuzzy number*. A fuzzy number A is *non-negative*, if $\mu_A(x) = 0$, for $\forall x < 0$.

Denotation 2.2. The set of all fuzzy sets on a universe X will be denoted by X_{FS} . The set of all fuzzy numbers will be denoted by \mathbb{R}_{fn} and, similarly, the set of all fuzzy points in a linear space X will be denoted by X_{fp} . The set of all non-negative fuzzy numbers will be denoted by \mathbb{R}_{fn}^+ , and the set of all normal fuzzy sets on \mathbb{R}^+ will be denoted by ${}_n\mathbb{R}_{FS}^+$.

Using Zadeh's extension principle, the following lemma published in [4, 5] can be proved.

Lemma 2.1. If, a unary function $f : X \rightarrow Y$ has an inverse $f^{-1} : Y \rightarrow X$, then

$$\mu_{f(A)}(y) = \mu_A(f^{-1}(y)).$$

If $\circ \in \{+, -, \cdot, /\}$ is a binary operation on \mathbb{R} and $A, B \in \mathbb{R}_{fn}$ are non-interactive fuzzy numbers, then for the extension of the binary operation $A \circ B = (\mathbb{R}, \mu_{A \circ B}) \in \mathbb{R}_{fn}$ and we have:

$$\begin{aligned} \mu_{A \oplus B}(y) &= \sup_{x \in \mathbb{R}} \min \{ \mu_A(x), \mu_B(y - x) \}, y \in \mathbb{R}, \\ \mu_{A \ominus B}(y) &= \sup_{x \in \mathbb{R}} \min \{ \mu_A(x), \mu_B(x - y) \}, y \in \mathbb{R}, \\ \mu_{A \odot B}(y) &= \sup_{\substack{x \in \mathbb{R} \\ x \neq 0}} \min \{ \mu_A(x), \mu_B(y/x) \}, y \in \mathbb{R}, \\ \mu_{A \oslash B}(y) &= \sup_{\substack{x \in \mathbb{R} \\ x \neq 0}} \min \{ \mu_A(yx), \mu_B(x) \}, y \in \mathbb{R}. \end{aligned}$$

The fuzzy absolute value $|A| = (\mathbb{R}, \mu_{|A|}) \in \mathbb{R}_{fn}^+$ of the fuzzy number $A \in \mathbb{R}_{fn}$ has the membership function

$$\mu_{|A|}(x) = \begin{cases} \max \{ \mu_A(x), \mu_A(-x) \} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Definition 2.4. Let M be a non-empty set and, for every pair of elements $A, B \in M$ a function $\rho(A, B) : M \times M \rightarrow \mathbb{R}^+$ is defined satisfying the following conditions:

- $\rho(A, B) \geq 0$ with $\rho(A, B) = 0$ iff $A = B$,
- $\rho(A, B) = \rho(B, A)$,
- $\rho(A, B) \leq \rho(A, C) + \rho(C, B)$ for $\forall C \in M$,

then the function ρ is called a *metric* and the ordered pair (M, ρ) is called a *metric space*.

3. DISTANCE BETWEEN FUZZY SETS

In this chapter, we define fuzzy distances of two fuzzy sets as a generalization of the distance of two non-empty subsets of a metric space. To do this we have used the distances of the matching α -cuts of both the fuzzy sets.

Definition 3.1. Let (X, ρ) be a metric space. By the *distance of two non-empty sets* M and N of the metric space X we mean the non-negative real number

$$\rho(M, N) = \inf_{x \in M, y \in N} \rho(x, y).$$

Definition 3.2. Let (X, ρ) be a metric space. The *fuzzy distance of two non-empty fuzzy sets* M and N of the metric space X will be defined as the fuzzy set $D_f(M, N) = (\mathbb{R}^+, \mu_{D_f(M, N)})$ with the membership function

$$\mu_{D_f(M, N)}(y) = \begin{cases} \sup \{ \alpha \in [0, 1] \mid M_\alpha, N_\alpha \neq \emptyset; \rho(M_\alpha, N_\alpha) \leq y \} & \text{for } y \leq \lim_{\beta \rightarrow \min\{h(M), h(N)\}_-} \rho(M_\beta, N_\beta), \\ 0 & \text{otherwise.} \end{cases}$$

Gerla and Volpe [3] have introduced a (crisp) distance of two fuzzy sets as follows.

Definition 3.3. Let (X, ρ) be a metric space. By the *distance of two fuzzy sets* M and N of the metric space X we mean a non-negative real number

$$D(M, N) = \int_0^1 \rho(M_\alpha, N_\alpha) d\alpha, \quad \text{where } \rho(P, \emptyset) = 0 \text{ for } \forall P \in X.$$

Note 3.1. The distance $D(M, N) : X_{FS} \times X_{FS} \rightarrow \mathbb{R}^+$ is not a metric on X_{FS} .

Theorem 3.1. If (X, ρ) is a metric space and M, N two non-empty (crisp) sets on X , then

- a) $\rho(M, N) = D(M, N)$,
- b) $\mu_{D_f(M, N)}(y) = \begin{cases} 1 & \text{for } y = \rho(M, N), \\ 0 & \text{for } y \neq \rho(M, N). \end{cases}$

Proof. Let M, N be two non-empty (crisp) sets on X , then

a) $D(M, N) = \int_0^1 \rho(M_\alpha, N_\alpha) d\alpha = \int_0^1 \rho(M, N) d\alpha = [\alpha \rho(M, N)]_0^1 = \rho(M, N)$,

b) $\min \{h(M), h(N)\} = 1 \Rightarrow \lim_{\beta \rightarrow \min\{h(M), h(N)\}_-} \rho(M_\beta, N_\beta) = \rho(M, N)$

$$\Rightarrow \mu_{D_f(M, N)}(y) = \begin{cases} \sup_{\substack{\alpha \in [0, 1] \\ \rho(M_\alpha, N_\alpha) \leq y}} \alpha & \text{for } y \leq \rho(M, N) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{for } y < \rho(M, N), \\ 1 & \text{for } y = \rho(M, N), \\ 0 & \text{for } y > \rho(M, N). \end{cases}$$

□

Note 3.2. By the previous theorem, Definitions 3.2 and 3.3 naturally generalize the classic definition of the distance of two non-empty subsets of a metric space.

Theorem 3.2. If (X, ρ) is a metric space and M, N two non-empty fuzzy sets on X , then the fuzzy distance $D_f(M, N)$ is a convex set and

$$\int_0^{h(D_f(M, N))} \inf(D_f(M, N)_\alpha) d\alpha = D(M, N),$$

where $h(D)$ is height of a fuzzy set D .

Proof. $h(D_f(M, N)) = \min \{h(M), h(N)\}$. If we denote $h(D_f(M, N)) = h$, then $\mu_{D_f(M, N)}(y)$ is a non-decreasing function on $(-\infty, \lim_{\beta \rightarrow h_-} \rho(M_\beta, N_\beta)]$ and $\mu_{D_f(M, N)}(y) = 0$ otherwise so that the fuzzy distance $D_f(M, N)$ is a convex fuzzy set.

Next $D(M, N) = \int_0^1 \rho(M_\alpha, N_\alpha) d\alpha = \int_0^h \rho(M_\alpha, N_\alpha) d\alpha + \int_h^1 \rho(M_\alpha, N_\alpha) d\alpha = \int_0^h \rho(M_\alpha, N_\alpha) d\alpha + 0$.

Further $\inf(D_f(M, N)_\alpha) = \lim_{\beta \rightarrow \alpha_-} \rho(M_\beta, N_\beta)$ is a non-decreasing function for $\forall \alpha \in (0, h) \Rightarrow$ for $\forall \alpha \in (0, h)$ we have

$$\lim_{\beta \rightarrow \alpha_-} \rho(M_\beta, N_\beta) = \sup_{\beta \in (0, \alpha)} \rho(M_\beta, N_\beta) \leq \sup_{\beta \in (0, \alpha)} \rho(M_\beta, N_\beta) = \rho(M_\alpha, N_\alpha).$$

If $\lim_{\beta \rightarrow \alpha_-} \rho(M_\beta, N_\beta) < \rho(M_\alpha, N_\alpha)$, then $\rho(M_\alpha, N_\alpha)$ is discontinuous on the left at α . Moreover, the number of points at which the function $\rho(M_\alpha, N_\alpha)$ is discontinuous

is countable on $(0, h)$ since $\rho(M_\alpha, N_\alpha)$ is a non-decreasing function for $\forall \alpha \in (0, h)$. Then we have

$$\int_0^h \rho(M_\alpha, N_\alpha) d\alpha = \int_0^h \lim_{\beta \rightarrow \alpha^-} \rho(M_\beta, N_\beta) d\alpha = \int_0^{h(D_f(M,N))} \inf(D_f(M, N)_\alpha) d\alpha.$$

□

Example 3.1. The fuzzy distance $D_f(M, N)$ of the two fuzzy sets $M, N \in \mathbb{R}_{FS}$ is shown in Figure 3.2. The distance $D(M, N)$ is the measure of the grey area (Figure 3.1).

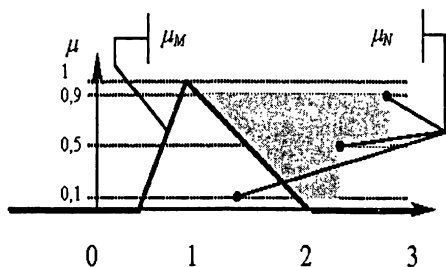


Fig. 3.1. Fuzzy sets M, N .

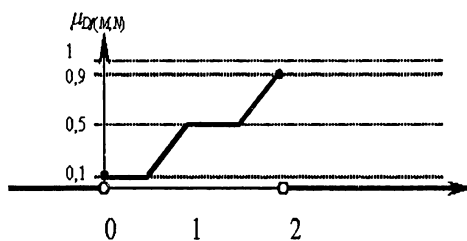


Fig. 3.2. Fuzzy sets $D_f(M, N)$.

Definition 3.4. Let (X, ρ) be a metric space. The maximal distance of two non-empty sets M and N of a metric space X will be defined as the non-negative real number

$$\rho_{\max}(M, N) = \sup_{x \in M, y \in N} \rho(x, y).$$

Definition 3.5. Let (ρ, X) be a metric space. The maximal fuzzy distance of two non-empty fuzzy sets M and N of a metric space X will be defined as a fuzzy set $D_{f \max}(M, N) = (\mathbb{R}, \mu_{D_{f \max}(M, N)})$ with the membership function

$$\mu_{D_{f \max}(M, N)}(y) = \begin{cases} \sup \{ \alpha \in [0, 1] \mid M_\alpha, N_\alpha \neq \emptyset; \rho_{\max}(M_\alpha, N_\alpha) \geq y \} & \text{for } y \geq \lim_{\beta \rightarrow \min\{h(M), h(N)\}^-} \rho_{\max}(M_\beta, N_\beta), \\ 0 & \text{otherwise.} \end{cases}$$

4. DISTANCE BETWEEN MEMBERSHIP FUNCTIONS

In this chapter, the distance of fuzzy sets is defined as the distance between their membership functions. The properties of membership function distances are listed in [8].

Definition 4.1. Let $X \neq \emptyset$ be a Lebesgue-measurable set, m a Lebesgue measure on X . The distance $d_p : X_{FS} \times X_{FS} \rightarrow \mathbb{R}^+$ of two fuzzy sets on X is a function assigning to each pair $M, N \in X_{FS}$ the number

$$d_p(M, N) = \begin{cases} (\int_X |\mu_M - \mu_N|^p dm)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \text{essential sup}_{x \in X} |\mu_M(x) - \mu_N(x)| & \text{for } p = \infty. \end{cases}$$

The ordered pair (X_{FS}, d_p) is called a *fuzzy metric space*.

Note 4.1. This fuzzy metric space complies with all the requirements for a metric space (Definition 2.11) with a single exception: if the distance of two fuzzy sets is zero, the membership functions of both fuzzy sets are the same almost everywhere thus

$$d_p(M, N) = 0 \Rightarrow \mu_M(x) = \mu_N(x) \text{ for } \forall x \in X - E, \text{ where } m(E) = 0.$$

Special cases of this definition are Definition 4.2 introduced by Abu Osman in [7] and Definition 4.3 introduced by E. Szmidt and J. Kacprzyk in [9].

Definition 4.2. The function $d_\infty : X_{FS} \times X_{FS} \rightarrow [0, 1]$, which, to every pair $M, N \in X_{FS}$, assigns the number $d_\infty(M, N) = \sup_{x \in X} |\mu_M(x) - \mu_N(x)|$ is called a *fuzzy metric* and the ordered pair (X_{FS}, d_∞) is called a *fuzzy metric space*.

Note 4.2. Abu Osman's fuzzy metric space is defined in an elegant way, but for technical applications, this definition is not suitable because, as long as at least one $x \in X$ exists such that $\mu_A(x) = 0$ and $\mu_B(x) = 1$, $d_\infty(A, B) = 1$, which is the maximum distance. Thus the fuzzy metric $d_\infty(A, B)$ discretizes X_{FS} too much. For example, for all intervals A, B on \mathbb{R} , where $A \neq B$, we have $d_\infty(A, B) = 1$.

Definition 4.3. The functions d_1 and $d_2 : X_{FS} \times X_{FS} \rightarrow \mathbb{R}^+$ where $X = \{x_1, x_2, \dots, x_n\}$ that, to every pair $M, N \in X_{FS}$, assign the number

$$d_1(M, N) = \sum_{i=1}^n |\mu_M(x_i) - \mu_N(x_i)|, \quad d_2(M, N) = \sqrt{\sum_{i=1}^n (\mu_M(x_i) - \mu_N(x_i))^2}$$

are called *fuzzy metrics*.

Note 4.3. Sometimes these metrics are written in the standardized form

$$n_1(M, N) = \frac{1}{n} \sum_{i=1}^n |\mu_M(x_i) - \mu_N(x_i)|, \quad n_2(M, N) = \sqrt{\frac{1}{n} \sum_{i=1}^n (\mu_M(x_i) - \mu_N(x_i))^2}.$$

5. FUZZY METRIC SPACE

In this chapter, we will generalize a metric space to a fuzzy metric one.

A very general definition (Definition 5.1) of fuzzy metric space has been shown by Kaleva and Seikkala [4]. They have defined the distance as a fuzzy function that, to any two elements of a set, assigns a non-negative fuzzy number continuous above and complies with some requirements, they have defined a fuzzy convergence and proved a fixed-point theorem. Due to the generality of the definition, no rules are shown enabling the construction of particular fuzzy metric spaces for particular point sets.

Denotation 5.1. Every $x \in X$ can be seen as a fuzzy number $\{x\} = (X, \mu_{\{x}\})$ with the membership function $\mu_{\{x\}}(y) = 1$ for $y = x$, $\mu_{\{x\}}(y) = 0$ for $y \neq x$. The set of all $\{x\}$ will be denoted ${}_{\text{crisp}}X_{fp}$ (obviously ${}_{\text{crisp}}X_{fp} \subset X_{fp}$).

Definition 5.1. Let $X \neq \emptyset$, $d : X \times X \rightarrow G$ where G is the set of all non-negative fuzzy numbers continuous from above and mappings $R, L : [0, 1] \times [0, 1] \rightarrow [0, 1]$ are symmetric, non-decreasing in both arguments with $L(0, 0) = 0$ and $R(1, 1) = 1$.

The ordered quadruple (X, d, L, R) is called a *fuzzy metric space* and the function d is called a *fuzzy distance*, if, for every $x, y, z \in X$, we have:

- 1) $d(x, y) = \{0\}$ iff $x = y$,
- 2) $d(x, y) = d(y, x)$ for every $x, y \in X$,
- 3a) $\mu_{d(x,y)}(s+t) \geq L(\mu_{d(x,z)}(s), \mu_{d(z,y)}(t))$, if $s \leq \inf \ker d(x, z)$, $t \leq \inf \ker d(z, y)$ and $s + t \leq \inf \ker d(x, y)$,
- 3b) $\mu_{d(x,y)}(s+t) \leq R(\mu_{d(x,z)}(s), \mu_{d(z,y)}(t))$, if $s \geq \inf \ker d(x, z)$, $t \geq \inf \ker d(z, y)$ and $s + t \geq \inf \ker d(x, y)$.

Note 5.1. Since the metric space need not be defined on a general set $X \neq \emptyset$, but rather on a set of fuzzy points, we define the fuzzy distance using Zadeh's extension principle. This means that the membership degree $\mu_{\rho_f(A,B)}(y)$ may be interpreted as a measure of the likelihood that the distance between $A \in X_{fp}$ and $B \in X_{fp}$ is y .

Definition 5.2. If (X, ρ) is a linear metric space and $A, B \in X_{fp}$ arbitrary fuzzy points, we define the *fuzzy distance of the fuzzy points* A and B as a fuzzy set $\rho_f(A, B) = (\mathbb{R}, \mu_{\rho_f(A,B)})$ with the membership function

$$\mu_{\rho_f(A,B)}(y) = \sup_{\substack{(x_1, x_2) \in X^2 \\ \rho(x_1, x_2) = y}} \min \{ \mu_A(x_1), \mu_B(x_2) \}.$$

If no $(x_1, x_2) \in X^2$ exists such that $\rho(x_1, x_2) = y$, we put $\mu_{\rho_f(A,B)}(y) = 0$. The ordered pair (X_{fp}, ρ_f) is called a *fuzzy metric space*.

Theorem 5.1. The fuzzy distance ρ_f as defined under Definition 5.2 is a mapping of X_{fp}^2 into the set of all normal non-negative fuzzy sets on \mathbb{R} , that is, $\rho_f : X_{fp}^2 \rightarrow {}_n\mathbb{R}_{FS}^+$.

Proof. For every $A, B \in X_{fp}$ there is $(x_1, x_2) \in X^2$ such that $x_1 \in \ker(A)$, $x_2 \in \ker(B)$. If $\rho(x_1, x_2) = y$, then $y \in \ker(\rho_f(A, B))$, which means that the fuzzy distance $\rho_f(A, B)$ is normal. The non-negativity of ρ_f follows from the assertion b) of Theorem 5.2. □

Note 5.2. In engineering practice, mostly linear normed spaces (X, g) are encountered where $g : X \rightarrow \mathbb{R}^+$ is a norm. If we define the metric $\rho(x_1, x_2) = g(x_2 - x_1)$ on such spaces, then the fuzzy distance ρ_f defined by Zadeh's extension principle in Definition 5.2 is a mapping of X_{fp}^2 into the set of all non-negative fuzzy numbers on \mathbb{R} , thus $\rho_f : X_{fp}^2 \rightarrow \mathbb{R}_{fn}^+$. An example of such a fuzzy metric space is the fuzzy Euclidean space $(\mathbb{R}_{fp}^n, \rho_{fk})$, or the fuzzy metric spaces $(\mathbb{R}_{fp}^n, \rho_{fk})$ where $1 \leq k < \infty$ as defined in Section 7.

Note 5.3. Now we have to verify whether the fuzzy metric space complies with the conditions listed in Definition 2.4 and thus deserves to be called metric. To do this, first an ordering needs to be defined on \mathbb{R}_{fn} and then generalized to ${}_n\mathbb{R}_{FS}$.

Definition 5.3. If $A, B \in \mathbb{R}_{fn}$, then a *partial ordering* \preceq on \mathbb{R}_{fn} is defined as a relation $A \preceq B$ if and only if $\inf A_\alpha \leq \inf B_\alpha$ and $\sup A_\alpha \leq \sup B_\alpha$ for all $\alpha \in (0, 1]$.

Note 5.4. The relation $A \preceq B$ signifies that the membership function $\mu_A(x)$ is "to the left" of the membership function $\mu_B(x)$. The set of all normal fuzzy sets on \mathbb{R} is much more complex ($\mathbb{R}_{fn} \subset {}_n\mathbb{R}_{FS}$) and therefore, rather than investigating the inner structure of individual fuzzy sets, we will be matching their least convex hulls.

Definition 5.4. If $A, B \in {}_n\mathbb{R}_{FS}$, then a *partial ordering* \preceq on ${}_n\mathbb{R}_{FS}$ is defined as a relation $A \preceq B$ if and only if $\inf A_\alpha \leq \inf B_\alpha$ and $\sup A_\alpha \leq \sup B_\alpha$ for all $\alpha \in (0, 1]$.

Denotation 5.2. Let $O \subset {}_n\mathbb{R}_{FS}^+$ a $\mu_O(0) = 1$, then the fuzzy set O will be called a fuzzy zero and the set of all the fuzzy zeros will be denoted O_{FS} .

Theorem 5.2. For every triple of elements $A, B, C \in X_{fp}$ we have:

- a) The fuzzy metric space (X_{fp}, ρ_f) is symmetric, thus $\rho_f(A, B) = \rho_f(B, A)$.
- b) The fuzzy metric space (X_{fp}, ρ_f) is positively semi-definite, thus $\{0\} \preceq \rho_f(A, B)$.
- c) In the fuzzy metric space (X_{fp}, ρ_f) , the triangle inequality holds, thus

$$\rho_f(A, B) \preceq \rho_f(A, C) \oplus \rho_f(C, B).$$

Proof. a) The original (crisp) metric is symmetric so we have

$$\mu_{\rho_f(A,B)}(y) = \sup_{\substack{(x_1, x_2) \in X^2 \\ \rho(x_2, x_1) = y}} \min \{ \mu_A(x_1), \mu_B(x_2) \} = \mu_{\rho_f(B,A)}(y)$$

and, if no $(x_1, x_2) \in X^2$ exists such that $\rho(x_1, x_2) = y$, then no $(x_2, x_1) \in X^2$ exists such that $\rho(x_2, x_1) = y$.

b) If $y \in \mathbb{R} - \mathbb{R}^+$, then no $(x_1, x_2) \in X^2$ exists such that $\rho(x_1, x_2) = y$. This means that $\mu_{\rho_f(A,B)}(y) = 0$, for $\forall y \in \mathbb{R} - \mathbb{R}^+$ and this implies that $\{0\} \preceq \rho_f(A, B)$.

c) The fuzzy sets $\rho_f(A, C)$ and $\rho_f(C, B)$ are interactive due to the fuzzy point C therefore it is not possible to calculate both fuzzy distances and add them up, the expression $\rho_f(A, C) \oplus \rho_f(C, B)$ needs to be handled as a fuzzy function dependent on the fuzzy points $A, B, C \in X_{fp}$.

For $\forall \alpha \in (0, 1]$ we have:

$$\begin{aligned} \lim_{\beta \rightarrow \alpha^-} \inf_{x_1 \in A_\beta, x_2 \in B_\beta} \rho(x_1, x_2) &\leq \lim_{\beta \rightarrow \alpha^-} \inf_{x_1 \in A_\beta} \rho(x_1, x_3) + \lim_{\beta \rightarrow \alpha^-} \inf_{x_2 \in B_\beta} \rho(x_3, x_2), \forall x_3 \in X, \\ \lim_{\beta \rightarrow \alpha^-} \sup_{x_1 \in A_\beta, x_2 \in B_\beta} \rho(x_1, x_2) &\leq \lim_{\beta \rightarrow \alpha^-} \sup_{x_1 \in A_\beta} \rho(x_1, x_3) + \lim_{\beta \rightarrow \alpha^-} \sup_{x_2 \in B_\beta} \rho(x_3, x_2), \forall x_3 \in X. \end{aligned}$$

If both above inequalities hold for $\forall x_3 \in X$, they also hold for $\forall x_3 \in \bigcap_{\beta < \alpha} C_\beta \subset X$ and thus

$$\rho_f(A, B) \preceq \rho_f(A, C) \oplus \rho_f(C, B). \quad \square$$

Theorem 5.3. For every pair of elements $A, B \in X_{fp}$ we have:

- a) If $\rho_f(A, B) = \{0\}$, then $A = B$ and $A, B \in_{\text{crisp}} X_{fp}$. However, it is not true that, for $A = B$, we have $\rho_f(A, B) = \{0\}$.
- b) If $\rho_f(A, B) \in_{\text{crisp}} \mathbb{R}_{fn}$, then $A, B \in_{\text{crisp}} X_{fp}$.
- c) If $A = B$, then $\rho_f(A, B) \in O_{FS}$. However, it is not true that, for $\rho_f(A, B) \in O_{FS}$, we have $A = B$.

Proof. The assertions are obvious. □

Note 5.5. If $A \in X_{fp} -_{\text{crisp}} X_{fp}$, then $\rho_f(A, A) \neq \{0\}$. Therefore, if the identity axiom is needed, a new metric may be defined

$$\rho_{nf}(A, B) = \begin{cases} \rho_f(A, B) & \text{for } A \neq B, \\ \{0\} & \text{for } A = B. \end{cases}$$

The fuzzy metric space (X_f, ρ_{nf}) defined in this way, however, is no longer obtained using Zadeh's extension principle. For this reason, it is often good to disregard the identity axiom and call as metric (as we have done) simply the space as defined under Definition 5.2.

6. FUZZY CONTRACTIVE MAPPING

Definition 6.1. Let (X, ρ) be a metric space. A mapping $C : X \rightarrow X$ is called *contractive* if a number $0 < q < 1$ exists such that, for any two points $x, y \in X$, we have

$$\rho(Cx, Cy) \leq q\rho(x, y).$$

Theorem 6.1. (Banach) Every contractive mapping has exactly one fixed point in a complete metric space (X, ρ) .

Definition 6.2. Let (X_{fp}, ρ_f) be fuzzy metric space. Mapping $C_f : X_{fp} \rightarrow X_{fp}$ is called *fuzzy contractive*, if a number $0 < q < 1$ exists such that, for any two fuzzy points $A, B \in X_{fp}$, we have $\rho_f(C_f A, C_f B) \preceq \{q\} \odot \rho_f(A, B)$.

Theorem 6.2. Let (X, ρ) be a complete metric space, $C : X \rightarrow X$ be a contractive mapping with $p \in X$ as a fixed point and mapping $C_f : X_{fp} \rightarrow X_{fp}$ be Zadeh's extension of the contractive mapping C , then C_f is a fuzzy contractive mapping and has exactly one fixed point

$$\{p\} \in_{\text{crisp}} X_{fp}.$$

Proof. Since, in the proof, intervals will be used of which it cannot be determined whether they are open or closed and this fact is irrelevant for the proof, we will denote such intervals $[a, b]$.

For $\forall \alpha \in (0, 1]$ we have:

$$\begin{aligned} \rho_f(A, B)_\alpha &= \lim_{\beta \rightarrow \alpha-} \left[\inf_{x \in A_\beta, y \in B_\beta} \rho(x, y), \sup_{x \in A_\beta, y \in B_\beta} \rho(x, y) \right], \\ q \odot \rho_f(A, B)_\alpha &= \lim_{\beta \rightarrow \alpha-} \left[q \inf_{x \in A_\beta, y \in B_\beta} \rho(x, y), q \sup_{x \in A_\beta, y \in B_\beta} \rho(x, y) \right], \\ \rho_f(C_f A, C_f B)_\alpha &= \lim_{\beta \rightarrow \alpha-} \left[\inf_{x \in C_f A_\beta, y \in C_f B_\beta} \rho(x, y), \sup_{x \in C_f A_\beta, y \in C_f B_\beta} \rho(x, y) \right] \end{aligned}$$

$$\begin{aligned} \lim_{\beta \rightarrow \alpha-} \inf_{x \in C_f A_\beta, y \in C_f B_\beta} \rho(x, y) &\leq \lim_{\beta \rightarrow \alpha-} \inf_{x \in A_\beta, y \in B_\beta} \rho(Cx, Cy) \leq \lim_{\beta \rightarrow \alpha-} q \inf_{x \in A_\beta, y \in B_\beta} \rho(x, y), \\ \lim_{\beta \rightarrow \alpha-} \sup_{x \in C_f A_\beta, y \in C_f B_\beta} \rho(x, y) &\leq \lim_{\beta \rightarrow \alpha-} \sup_{x \in A_\beta, y \in B_\beta} \rho(Cx, Cy) \leq \lim_{\beta \rightarrow \alpha-} q \sup_{x \in A_\beta, y \in B_\beta} \rho(x, y) \end{aligned}$$

and so $\rho_f(C_f A, C_f B) \preceq \{q\} \odot \rho_f(A, B) \Rightarrow C_f X_{fp} \rightarrow X_{fp}$ is a fuzzy contractive mapping.

For $\forall x \in X$ we have $\lim_{n \rightarrow \infty} C^n x = p$. By repeatedly applying Zadeh's extension principle we will obtain $\lim_{n \rightarrow \infty} C_f^n x = \{p\}$ for $\forall A \in X_{fp}$. The existence and uniqueness of a fixed point for mapping $C_f : X_{fp} \rightarrow X_{fp}$ follows from the existence and uniqueness of a fixed point for mapping $C : X \rightarrow X$.

7. FUZZY METRIC SPACE IN R^N

If $A, B \in \mathbb{R}_{fp}^n$ is a fuzzy point, then, using Definition 5.2, any metric space (\mathbb{R}^n, ρ) may be extended to a metric space $(\mathbb{R}_{fp}^n, \rho_f)$. In \mathbb{R}^n , mostly the metrics $\rho_0(A, B) = 0$ for $A = B$, $\rho_0(A, B) = 1$ for $A \neq B$, and $\rho_k(A, B) = \left(\sum_{i=1}^n |b_i - a_i|^k\right)^{1/k}$ are used where $1 \leq k < \infty$, for $k = 2$, the Euclidean metric $\rho_2(A, B) = \sqrt{\sum_{i=1}^n (b_i - a_i)^2}$ is obtained.

Example 7.1. Fuzzy distances of fuzzy points A and B in \mathbb{R}^2 (Figure 7.1) are shown in Figure 7.2.

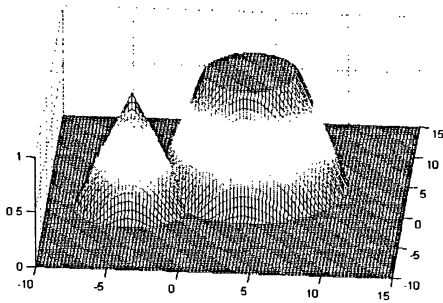


Fig. 7.1. Fuzzy points in \mathbb{R}^2 .

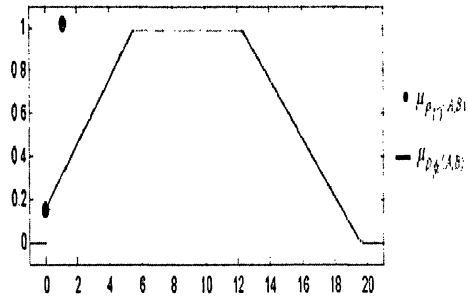


Fig. 7.2. Fuzzy distances $\rho_{fk}(A, B)$.

Theorem 7.1. The membership function of the fuzzy distance $\rho_{f0}(A, B) = (\mathbb{R}^+, \mu_{\rho_{f0}(A, B)}(y))$ is

$$\begin{aligned} \mu_{\rho_{f0}(A, B)}(0) &= h(A \cap B) \\ \mu_{\rho_{f0}(A, B)}(1) &= h((A \cup B) - \{x_0\}), \text{ where } x_0 \in \bigcap_{\beta < h(A \cap B)} (A \cup B)_\beta. \end{aligned}$$

Proof.

$$\begin{aligned} \mu_{\rho_{f0}(A, B)}(0) &= \\ &= \sup_{\substack{((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) \in \mathbb{R}^{2n} \\ (x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)}} \min \{ \mu_A(x_1, x_2, \dots, x_n), \mu_B(y_1, y_2, \dots, y_n) \} \\ &= \sup(A \cap B) = h(A \cap B) \end{aligned}$$

$$\begin{aligned} \mu_{\rho_{f0}(A, B)}(1) &= \\ &= \sup_{\substack{((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) \in \mathbb{R}^{2n} \\ (x_1, x_2, \dots, x_n) \neq (y_1, y_2, \dots, y_n)}} \min \{ \mu_A(x_1, x_2, \dots, x_n), \mu_B(y_1, y_2, \dots, y_n) \} \\ &= h((A \cup B) - \{x_0\}), \text{ where } x_0 \in \bigcap_{\beta < h(A \cap B)} (A \cup B)_\beta. \end{aligned}$$

If we define a fuzzy point $A = (\mathbb{R}^n, \mu_A(x_1, x_2, \dots, x_n))$ as the Cartesian product of the fuzzy numbers $A_i = (\mathbb{R}, \mu_{A_i}(x))$, $i = 1, \dots, n$, in each dimension, the expression for the fuzzy distance ρ_{fk} can be simplified using distances in each dimension.

Example 7.2. Fuzzy distances $\rho_f(A, B)$ of fuzzy points A and B in \mathbb{R}^2 (Figure 7.3) are shown in Figure 7.4.

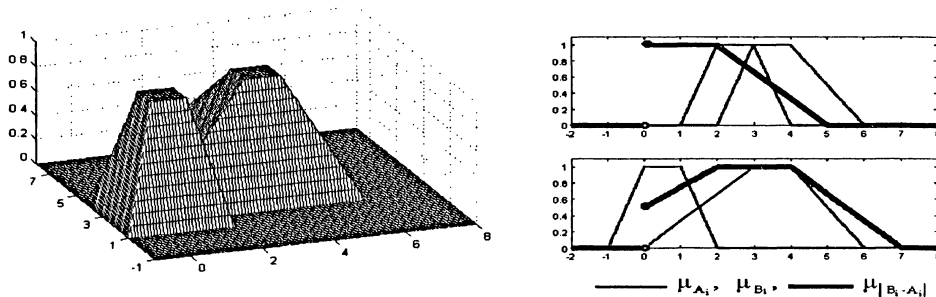


Fig. 7.3. Fuzzy points A and B in \mathbb{R}^2 and their distances in each dimension.

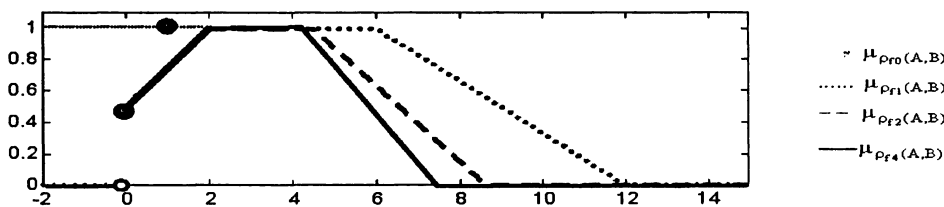


Fig. 7.4. Fuzzy distances $\rho_f(A, B)$ calculated using distances in each dimension.

8. RELATIONSHIP BETWEEN $D_F(A, B)$ AND $\rho_F(A, B)$

Definition 8.1. The *minimal convex hull* of a fuzzy set $A = (\mathbb{R}, \mu_A)$ is defined as a convex fuzzy set $\llbracket A \rrbracket = (\mathbb{R}, \mu_{\llbracket A \rrbracket})$ continuous from above where $\inf \llbracket A \rrbracket_\alpha = \lim_{\beta \rightarrow \alpha_-} \inf A_\beta$ and $\sup \llbracket A \rrbracket_\alpha = \lim_{\beta \rightarrow \alpha_-} \sup A_\beta$ for all $\alpha(0, h(A))$.

Theorem 8.1. If (X_{fp}, ρ_f) is a fuzzy metric space as defined by Definition 5.2, $A, B \in X_{fp}$, $D_f(A, B)$ is a fuzzy distance as defined by Definition 3.2 and $D_{f \max}(A, B)$ is a maximal fuzzy distance as defined by Definition 3.5, then

$$\llbracket D_f(A, B) \cup D_{f \max}(A, B) \rrbracket = \llbracket \rho_f(A, B) \rrbracket.$$

Proof. $\llbracket D_f(A, B) \cup D_{f \max}(A, B) \rrbracket$, $\llbracket \rho_f(A, B) \rrbracket$ are convex and, for $\forall \alpha \in (0, 1)$, we have:

$$\llbracket \rho_f(A, B) \rrbracket_\alpha =$$

$$\begin{aligned}
 &= \lim_{\beta \rightarrow \alpha^-} \left\langle \inf_{x \in A_\beta, y \in B_\beta} \rho(x, y), \sup_{x \in A_\beta, y \in B_\beta} \rho(x, y) \right\rangle, \\
 &\quad \llbracket D_f(A, B) \cup D_{f \max}(A, B) \rrbracket_\alpha = \\
 &= \lim_{\beta \rightarrow \alpha^-} \langle \rho(A_\beta, B_\beta), \rho_{\max}(A_\beta, B_\beta) \rangle = \lim_{\beta \rightarrow \alpha^-} \left\langle \inf_{x \in A_\beta, y \in B_\beta} \rho(x, y), \sup_{x \in A_\beta, y \in B_\beta} \rho(x, y) \right\rangle \\
 &\Rightarrow \llbracket D_f(A, B) \cup D_{f \max}(A, B) \rrbracket = \llbracket \rho_f(A, B) \rrbracket.
 \end{aligned}$$

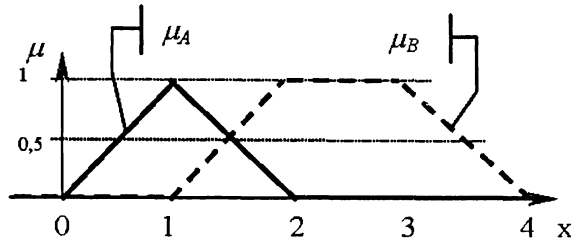


Fig. 8.1. Fuzzy points $A, B \in \mathbb{R}_{fn}$.

Example 8.1. Fuzzy distances $\rho_f(A, B)$, $D_f(A, B)$ and $D_{f \max}(A, B)$ between fuzzy points $A, B \in \mathbb{R}_{fn}$ (Figure 8.1) where

$$\rho(x, y) = \begin{cases} 0 & \text{for } x = y, \\ 0,5 & \text{for } |x - y| \in (0, 1), \\ |x - y| & \text{for } |x - y| \geq 1, \end{cases}$$

are shown in Figures 8.2 and 8.3.

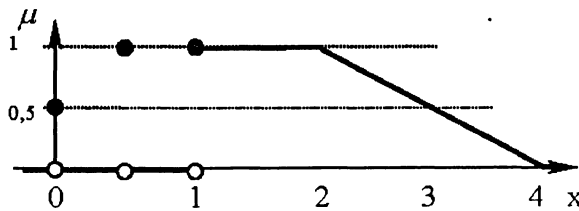


Fig. 8.2. Fuzzy distance $\rho_f(A, B)$.

9. CONCLUSION

In this paper, we have systematically classified the distances published previously by various authors [3, 4, 7, 9] into three groups by the construction method. We

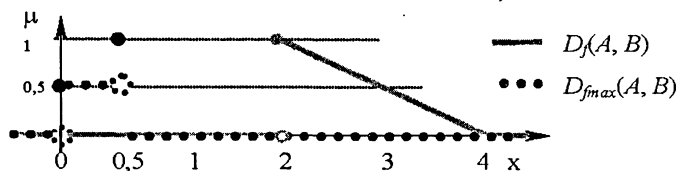


Fig. 8.3. Fuzzy distances $D_f(A, B)$ and $D_{f\max}(A, B)$.

have shown our own original definition of the distance of fuzzy sets and an original definition of a fuzzy metric space, whose properties we have formulated in several theorems and illustrated by examples. We have shown that, by fuzzifying a contractive mapping, we will obtain a fuzzy contractive mapping and, in a similar way, we have investigated in some detail fuzzy metric spaces defined on \mathbb{R}^n . We will refer to the outcomes achieved when further studying linear normed spaces and fuzzy convergences.

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