Kybernetika

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Kybernetika, Vol. 41 (2005), No. 2, [177]--204

Persistent URL: http://dml.cz/dmlcz/135649

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KYBERNETIKA — VOLUME 41 (2005), NUMBER 2, PAGES 177–204

APPROXIMATIONS OF LATTICE–VALUED POSSIBILISTIC MEASURES

IVAN KRAMOSIL

Lattice-valued possibilistic measures, conceived and developed in more detail by G. De Cooman in 1997 [3], enabled to apply the main ideas on which the real-valued possibilistic measures are founded also to the situations often occurring in the real world around, when the degrees of possibility, ascribed to various events charged by uncertainty, are comparable only quantitatively by the relations like "greater than" or "not smaller than", including the particular cases when such degrees are not comparable at all. The aim of this work is to weaken the demands imposed on possibilistic measures in other direction: the condition that the value ascribed to the union of two or more events (taken as subsets of a universe of discourse) is identical with the supremum of the values ascribed to particular events is weakened in the sense that these two values should not differ "too much" from each other, in other words, that their (appropriately defined) difference should be below a given "small" threshold value. This idea is developed, in more detail, for the lattice-valued possibility degrees, resulting in the notion of lattice-valued quasi-possibilistic measures. Some properties of these measures are investigated and relevant mathematically formalized assertions are stated and proved.

Keywords: possibilistic measure, almost-maxitive approximation, fuzzy measure, complete lattice, lattice-valued measure

AMS Subject Classification: 28E10, 28E99

1. INTRODUCTION. A BRIEF SKETCH OF SOME GENERALIZATIONS OF POSSIBILISTIC MEASURES

Possibilistic (or possibility) measures were conceived by L. A. Zadeh in 1978 [11] as an auxiliary tool for numerical characterization and processing. They have emancipated, since, as an interesting mathematical model of the phenomenon of uncertainty, alternative to those offered by probability theory and mathematical statistics. The original very simple notion of real-valued and completely defined possibilistic measure has been subjected to various modifications, generalizations and weakenings with the times going. Some of them were motivated, or even forced, by the demands resulting from the nature of uncertainty charging the data coming from the real world around, to which possibilistic measures should be applied, other modifications were necessitated by the methodological demands of the mathematical tools used when building the mathematical theories under consideration.

Let us consider a nonempty set Ω (the universe of discourse). Each mapping $\pi : \Omega \to [0,1]$ such that $\sup_{\omega\in\Omega} \pi(\omega) = 1$ is called a *(normalized real-valued) possibilistic distribution on* Ω; in what follows, we write $\bigvee$ instead of sup. This distribution induces the *(normalized real-valued) possibilistic measure* Π on the power-set $\mathcal{P}(\Omega)$ of all subsets of Ω, setting $\Pi(A) = \bigvee_{\omega\in A} \pi(\omega)$ for every $\emptyset \neq A \subset \Omega$, so that $\Pi(\Omega) = 1$, and applying the convention $\Pi(\emptyset) = 0$ for the empty subset $\emptyset$ of Ω. Obviously, $\Pi(\bigcup \mathcal{A}) = \bigvee\{\Pi(A) : A \in \mathcal{A}\}$ follows for each nonempty system $\mathcal{A}$ of subsets of Ω, where $\bigcup \mathcal{A} = \bigcup\{A : A \in \mathcal{A}\}$, in particular, $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ holds for each $A, B \subset \Omega$. The same notion can be defined also axiomatically: (complete real-valued) possibilistic measure on $\mathcal{P}(\Omega)$ is a mapping $\Pi : \mathcal{P}(\Omega) \to [0,1]$ such that $\Pi(\emptyset) = 0, \Pi(\Omega) = 1$, and $\Pi(\bigcup \mathcal{A}) = \bigvee\{\Pi(A) : A \in \mathcal{A}\}$ for each $\emptyset \neq \mathcal{A} \subset \mathcal{P}(\Omega)$. Replacing the last condition by a weaker one, according to which $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ holds for each $A, B \subset \Omega$, we arrive at the notion of (not necessarily complete) possibilistic measure, weaker than that defined constructively above using a possiblistic distribution. Indeed, if Ω is infinite, $\Pi(A) = 1$ for infinite subsets of Ω, and $\Pi(A) = 0$ for finite $A \subset \Omega$, then Π is a (not complete) possibilistic measure on $\mathcal{P}(\Omega)$, which cannot be defined by a possibilistic distribution on Ω.

As a matter of fact, the axiomatic approach to possibilistic measures will be very useful in what follows, as the considered modifications and weakening of the notions of possibilistic measures can be explicitly defined by the appropriate changes of the axioms imposed on the mapping Π. One such weakening results when abandoning the idea that Π is defined on the whole $\mathcal{P}(\Omega)$, supposing that the definition domain of Π is a nonempty system $\mathcal{R} \subset \mathcal{P}(\Omega)$ and that the demands imposed on Π are those as above but relativized to $\mathcal{R}$. Hence, a mapping $\Pi : \mathcal{R} \to [0,1]$ is a *partial (normalized real-valued) possibilistic measure* on $\mathcal{R}$, if $\Pi(\emptyset) = 0$ and/or $\Pi(\Omega) = 1$ supposing that $\emptyset \in \mathcal{R}$ and/or $\Omega \in \mathcal{R}$, and $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for each $A, B \in \mathcal{R}$ such that $A \cup B \in \mathcal{R}$. The modification to the case of partial *complete* possibilistic measure is obvious. Let us recall that this notion is, indeed, very general, e. g., if $\mathcal{R}$ is a disjoint covering of Ω not containing $\emptyset$ and Ω, then *each* mapping $\Pi : \mathcal{R} \to [0,1]$ defines a partial complete possiblistic measure on $\mathcal{R}$, as no of the restricting conditions imposed on Π applies.

The shift from completely defined to partial possibilistic measures is inspired by the fact that very often the degree of possibility (the value of a possibilistic measure) is not known, or even not defined, for each event from the system of events charged by uncertainty under consideration. As a rule, some more conditions on the domain $\mathcal{R}$ are imposed in order to obtain interesting and nontrivial results concerning the partial possibilistic measure in question (e. g., $\mathcal{R}$ is supposed to be a field, a σ-field, an ample field, a nested system, ...).

However, our knowledge of the degrees of possibility ascribed to various events (subsets of the universe Ω, under our formalization) may be incomplete also in the sense that not the very numerical values of possibilistic measures, but only their qualitative relations to other values are at our disposal. Hence, what is known is that the value $\Pi(A)$ is greater than or equal to the value $\Pi(B)$, that the value $\Pi(C)$

is strictly smaller than that of $\Pi(D)$, or that the values $\Pi(E)$ and $\Pi(F)$ cannot be compared with respect to their sizes, A, B, C, D, E, and F being events (subsets of Ω) for which Π is defined.

Formalizing mathematically this intuition behind, we arrive at the idea of possibilistic measures, perhaps partial ones, with non-numerical values. The weakest condition to be imposed on the structure of the possibility degrees (values taken by non-numerical possibilistic measures) in order to obtain some non-trivial results seems to read that such possibilistic measures should take their values in a partially ordered set. Our reasoning will be much more easy and simple if we limit our considerations to *complete lattices,* i. e., to partially ordered sets, in which the supremum and the infimum of all nonempty subsets of elements are (obviously uniquely) defined, so that our results need not be conditioned to the cases when all the suprema and infima in question are defined. It is perhaps worth being re-called explicitly, that the notion of complete lattice seems to be the most specific (i. e., the least general one) still covering the two most often used structures for quantifying the uncertainty and possibility degrees: the unit interval of real numbers with respect to their standard linear ordering, and the complete Boolean algebra, in particular, the power-set of all subsets of a fixed space, partially ordered by the relation of set-theoretic inclusion.

Perhaps for the first time the idea of non-numerical uncertainty quantification and processing was applied by J. A. Goguen in 1967 [7], who introduced and investigated fuzzy sets with membership degrees (values of the membership functions) in a complete lattice. Partial possibilistic measures defined on an ample field of subsets of the basic space Ω and taking their values in a complete lattice were introduced by G. DeCooman in 1997 [3], who defined and proved numerous interesting and far going formal analogies between probabilistic and possibilistic measures resulting when interchanging mutually the operations of addition or series taking with that of supremum.

However, still another weakening of the original idea and notion of possibilistic measures comes almost immediately into one's mind, namely, to weaken the relation binding the values ascribed by the possibilistic measure in question to the sets A, B, and this one ascribed to their union $A \cup B$. In the most simple case of normalized real-valued possibilistic measure Π defined on $\mathcal{P}(\Omega)$ we replace, given a threshold value $0 \leq \varepsilon \leq 1$, the condition $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$, by a weaker one, according to which the inequality $|\Pi(A \cup B) - (\Pi(A) \vee \Pi(B))| \leq \varepsilon$ (or $< \varepsilon$) holds for each $A, B \subset \Omega$. If, moreover, $\Pi(\emptyset) = 0$ and $\Pi(\Omega) = 1$ hold, the mapping Π will be called an *ε-quasi-possibilistic measure on* $\mathcal{P}(\Omega)$. Obviously, for $\varepsilon = 0$ we arrive at the original notion of possibilistic measure.

In the rest of this paper we will present and analyze this idea at a more general and abstract level, introducing the notion of *t-quasi-possibilistic partial lattice-valued possibilistic measure* defined on a system $\mathcal{A} \subset \mathcal{P}(\Omega)$ (very often $\mathcal{A}$ will be an ample field). Here t is a value from the lattice in question playing the same role as ε in the real-valued case. It is why the next chapter will re-call and introduce some notions and properties related to complete lattices and necessary as preliminaries for our further reasoning and constructions.

An elementary explanation of possibility theory can be found in [4], the surveyal work [5] discusses the relations among various mathematical models and theories of uncertainty quantification and processing.

2. PRELIMINARIES ON COMPLETE LATTICES AND RELATED NOTIONS

Let T be a nonempty set. A binary relation $\leq$ on T (i. e., a subset of the Cartesian product $T \times T$) is called *pre-ordering* on T, if (i) $t \leq t$ (reflexivity) and (ii) if $t_1 \leq t_2$, and $t_2 \leq t_3$, then $t_1 \leq t_3$ (transitivity) holds for each $t, t_1, t_2, t_3 \in T$. A pre-ordering $\leq$ on T is a *partial ordering* if $t_1 = t_2$ holds for each $t_1, t_2 \in T$ such that $t_1 \leq t_2$ and $t_2 \leq t_1$ hold simultaneously (antisymmetry). If $\leq$ is a partial ordering on T, the pair $\mathcal{T} = \langle T, \leq \rangle$ is called a *partially ordered (p. o.) set* with the support T.

Let $\mathcal{T} = \langle T, \leq \rangle$ be a p. o. set, let S be a nonempty subset of T. An element $\bigvee S$ of T is called the *supremum* of S (w.r.to $\mathcal{T}$), if (i) $s \leq \bigvee S$ holds for each $s \in S$ and (ii) if there is $s_0 \in T$ such that $s \leq s_0$ holds for each $s \in S$, then $\bigvee S \leq s_0$ holds as well. An element $\bigwedge S$ of T is called the infimum of S (w.r.to $\mathcal{T}$), if (iii) $\wedge S \leq s$ holds for each $s \in S$ and (iv) if there is $s_1 \in T$ such that $s_1 \leq s$ holds for each $s \in S$, then $s_1 \leq \bigwedge S$ holds as well.

In general, neither $\bigvee S$ not $\bigwedge S$ need be defined for every $S \subset T$, but if they are defined, they are defined uniquely. If $\bigvee T$ and/or $\bigwedge T$ are defined, we write $\mathbf{1}_{\mathcal{T}}$ for $\bigvee T$ (the *maximum* or the *unit* element of $\mathcal{T}$), and we write $\oslash_{\mathcal{T}}$ for $\bigwedge T$ (the *minimum* or the *zero* element of $\mathcal{T}$). Supposing that $\mathbf{1}_{\mathcal{T}}$ and/or $\oslash_{\mathcal{T}}$ are defined, we define $\bigvee \emptyset = \oslash_{\mathcal{T}}$ and $\bigwedge \emptyset = \mathbf{1}_{\mathcal{T}}$ for the empty subset of T. If $S = \{a, b\}$, $S = \{a_1, a_2, \ldots, a_n\}$, or $S = \{a_1, a_2, \ldots\}$, we write often $\bigvee S = a \vee b$, $\bigvee S = a_1 \vee a_2 \vee \ldots \vee a_n$ or $\bigvee S = \bigvee_{i=1}^{n} a_i$, and $\bigvee S = \bigvee_{i=1}^{\infty} a_i$, and similarly for $\bigwedge S$ supposing that $\bigvee S$ and/or $\bigwedge S$ are defined. P. o. set $\mathcal{T} = \langle T, \leq \rangle$ is an *upper semilattice,* if $s \vee t$ is defined for each $s, t \in T$ and this upper semilattice is *complete,* if $\bigvee S$ is defined for each $\emptyset \neq S \subset T$. Dually, $\mathcal{T}$ is a *lower semilattice,* if $s \wedge t$ is defined for each $s, t \in T$ and this lower semilattice is *complete,* if $\bigwedge S$ is defined for each $\emptyset \neq S \subset T$. $\mathcal{T}$ is a *lattice,* if it is an upper and a lower semilattice, and $\mathcal{T}$ is a *complete lattice,* if it is a complete upper semilattice and a complete lower semilattice. Hence, if $\mathcal{T} = \langle T, \leq \rangle$ is a complete lattice, $\bigvee S$ and $\bigwedge S$ are defined for any $S \subset T$ (for $S = \emptyset$ the conventions apply). As already noted above, the unit interval of reals equipped by their standard linear ordering, as well as the power-set $\mathcal{P}(X)$ of all subsets of a nonempty set X with the set inclusions as the partial ordering relation, are obviously complete lattices.

Like as in the case of real-valued possibilistic measures, also when introducing their non-numerical variants the supremum operation plays the key role. So, when supposing that the structure over the set of values taken by non-numerical possibilistic measures in just that of p. o. sets, we have to make relative, explicitly and repeatedly, all our statements to the cases when all the suprema and infima in question exist. In order to simplify substantially our position we will suppose that the non-numerical possibilistic measures under consideration take their values in a complete lattice $\mathcal{T} = \langle T, \leq \rangle$, so that the existence of all suprema and infima values follows immediately as it is the case for real-valued possibilistic measures.

As complete lattices do not contain an operation of complement as their primary notion, at least the two ways how to proceed can be considered. Either, we may introduce more specific structures with complement as one of their primary operations axiomatically bounded together with the other operations (this is the way leading to Boolean algebras, e. g.). Or, we can introduce a pseudo-complement operation weaker than that of complement in the unit interval ($1 - x$ for $x \in [0,1]$) or in Boolean algebras (e. g., set-complement operation), but definable in every complete lattice. Preferring this last approach and considering a complete lattice $\mathcal{T} = \langle T, \leq \rangle$, we define for every $t \in T$ its *(pseudo-) complement* t^C by

$$t^C = \bigvee \{s \in T : s \wedge t = \oslash_{\mathcal{T}}\}, \tag{1}$$

let us recall that $\oslash_{\mathcal{T}} = \bigwedge T$. The element t^C is obviously always defined and this definition is close to that of complement in Boolean algebras. Indeed, if $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$ for some $X \neq \emptyset$, then for each $A \subset X$ we obtain that $A^C = X - A$, as a matter of fact, for each complete Boolean algebra $\mathcal{B} = \langle B, \vee, \wedge, \neg \rangle$ we obtain that $t^C = \neg t$ for every $t \in B$. Contrary to this intuitive fact, if $\mathcal{T} = \langle [0,1], \leq \rangle$, then for every $0 < x \leq 1$ we obtain that $x^C = \bigvee \{y \in [0,1] : \inf(x,y) = 0\} = 0$.

Obviously, also the dual definition of (pseudo-) complement is possible and perhaps worth being considered, i. e., we could set

$$t^d = \bigwedge \{s \in T : s \vee t = \mathbf{1}_{\mathcal{T}}\} \tag{2}$$

for every $t \in T$. E. g., when $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$, both the definitions agree so that $A^C = A^d = X - A$ of every $A \subset X$. However, let us focus our attention to the case (2.1), postponing a more detailed investigation of the alternative approach (2.2) till another occasion.

As can be easily seen (cf. examples in [8], e. g.), neither $t^C \wedge t = \oslash_{\mathcal{T}}$ nor $t^C \vee t = \mathbf{1}_{\mathcal{T}}$ holds in general in each complete lattice. A complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is called *semi-Boolean,* if $t^C \wedge t = \oslash_{\mathcal{T}}$ is valid for each $t \in T$, and it is called *Boolean-like,* if also $t^C \vee t = \mathbf{1}_{\mathcal{T}}$ holds for each $t \in T$. The complete lattice $\mathcal{T} = \langle [0,1], \leq \rangle$ can be introduced as an example of a lattice which is semi-Boolean but not Boolean-like. A complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is *completely distributive (c. d.),* if the identities

$$t \wedge \left(\bigvee S\right) = \bigvee_{s \in S} (t \wedge s), t \vee \left(\bigwedge S\right) = \bigwedge_{s \in S} (t \vee s) \tag{3}$$

are valid for every $t \in T$ and $\emptyset \neq S \subset T$. A completely distributive complete lattice is called *Brouwerian lattice* ([1]).

When introducing the basic idea of quasi-possibilistic measures in the case of normalized real-valued measures (the end of Chapter 1), we took the value $|\Pi(A \cup B) - (\Pi(A) \vee \Pi(B))|$ as a reasonable quantitative distance between the real numbers $\Pi(A \cup B)$ and $\Pi(A) \vee \Pi(B)$, hence, we used the absolute value of difference of two real numbers as a metric on the real line and, in particular, on the unit interval of reals. Aiming to apply this idea to the case of lattice-valued possibilistic measures, we need to define a lattice-valued metric on the support set T of the complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is question. Taking inspiration from the notion of symmetric difference in the elementary set theory, considering a complete lattice $\mathcal{T} = \langle T, \leq \rangle$, and taking profit of the notion of (pseudo-) complement defined by (2.1), we set for each $t_1, t_2 \in T$

$$\rho(t_1, t_2) = (t_1 \wedge t_2^C) \vee (t_2 \wedge t_1^C). \tag{4}$$

Indeed, if $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$, we obtain easily that, for each $A_1, A_2 \subset X$, $\rho(A_1, A_2) = A_1 \div A_2 = (A_1 - A_2) \cup (A_2 - A_1)$.

The following assertion shows (cf. Theorem 2.1 in [8] and its proof) that the mapping ρ possesses certain properties due to which it can be taken as a lattice-valued metric.

Fact 2.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian and Boolean-like lattice, let $\rho : T \times T \to T$ be defined by (2.4). Then this mapping is

(i) *reflexive,* i. e., $\rho(t_1, t_1) = \emptyset_{\mathcal{T}}$ for each $t_1 \in T$,

(ii) *symmetric,* i. e., $\rho(t_1, t_2) = \rho(t_2, t_1)$ for each $t_1, t_2 \in T$,

(iii) *triangular inequality* holds, i. e., $\rho(t_1, t_3) \leq \rho(t_1, t_2) \vee \rho(t_2, t_3)$ for each $t_1, t_2, t_3 \in T$.

As a matter of fact, the assertion (i) concerning the reflexivity of the relation ρ can be strengthened as follows.

Lemma 2.1. Let $\mathcal{T}$ and ρ be as in Fact 2.1. Then $\rho(t_1, t_2) = \emptyset_{\mathcal{T}}$ holds iff $t_1 = t_2$.

Proof. First of all, let us prove that under the conditions imposed on $\mathcal{T}$ the identity $(t^C)^C = t$ holds for every $t \in T$. As $t \wedge t^C = \emptyset_{\mathcal{T}}$, the inequality $t \leq (t^C)^C$ follows from the definition of $(t^C)^C$. As $t \vee t^C = \mathbf{1}_{\mathcal{T}}$, we obtain that, due to the distributivity of $\mathcal{T}$,

$$(t^C)^C = (t^C)^C \wedge \mathbf{1}_{\mathcal{T}} = (t^C)^C \wedge (t^C \vee t) = ((t^C)^C \wedge t^C) \vee ((t^C)^C \wedge t) = (t^C)^C \wedge t, \tag{5}$$

as $(t^C)^C \wedge t^C = \emptyset_{\mathcal{T}}$. Consequently, $(t^C)^C \leq t$ and also $(t^C)^C = t$ follows.

Now, we have to prove that $\rho(t_1, t_2) > \emptyset_{\mathcal{T}}$ holds for each $t_1 \neq t_2$ from $\mathcal{T}$. The distributivity property imposed on $\mathcal{T}$ yields that

$$\begin{aligned} \rho(t_1, t_2) &= (t_1 \wedge t_2^C) \vee (t_2 \wedge t_1^C) = [(t_1 \wedge t_2^C) \vee t_2] \wedge [(t_1 \wedge t_2^C) \vee t_1^C] \\ &= [(t_1 \vee t_2) \wedge (t_2^C \vee t_2)] \wedge [(t_1 \vee t_1^C) \wedge (t_2^C \vee t_1^C)] \\ &= [(t_1 \vee t_2) \wedge \mathbf{1}_{\mathcal{T}}] \wedge [\mathbf{1}_{\mathcal{T}} \wedge (t_2^C \vee t_1^C)] \\ &= (t_1 \vee t_2) \wedge (t_2^C \vee t_1^C). \end{aligned} \tag{6}$$

Let $t_1, t_2 \in T$ be such that $t_1 \leq t_2$ holds. Then

$$\begin{aligned} \rho(t_1, t_2) &= (t_1 \wedge t_2^C) \vee (t_2 \wedge t_1^C) \leq (t_2 \wedge t_2^C) \vee (t_2 \wedge t_1^C) \\ &= \emptyset_{\mathcal{T}} \vee (t_2 \wedge t_1^C) = t_2 \wedge t_1^C \end{aligned} \tag{7}$$

follows. Suppose, in order to arrive at a contradiction, that $t_1 < t_2$ and $t_2 \wedge t_1^C = \oslash_\mathcal{T}$ hold simultaneously. Then $t_2 \leq (t_1^C)^C = t_1$ holds due to the definition of $(t_1^C)^C$ and the contradiction follows. Hence, if $t_1 < t_2$, then $t_2 \wedge t_1^C > \oslash_\mathcal{T}$ holds.

For each $t_1, t_2 \in T$,if $t_1 \neq t_2$, then $t_1 \wedge t_2 < t_1 \vee t_2$ and, consequently, either $t_1 < t_1 \vee t_2$ or $t_2 < t_1 \vee t_2$ holds, without any loss of generality let us consider only the case $t_1 < t_1 \vee t_2$. Applying what we have just proved to this case, we obtain that

$$\oslash_\mathcal{T} < \rho(t_1 \vee t_2, t_1) = (t_1 \vee t_2) \wedge t_1^C \leq (t_1 \vee t_2) \wedge (t_1^C \vee t_2^C) = \rho(t_1, t_2) \tag{8}$$

holds due to (2.6). The lemma is proved. □

For more details on lattices and related notions, e. g. [2, 6] and [10] can be recommended.

3. PARTIAL LATTICE–VALUED POSSIBILISTIC AND QUASI–POSSIBILISTIC MEASURES

The following definition tries to copy, within the framework of complete lattices, that one of normalized real-valued partial possibilistic measures, conserving also the same high degree of generality. The only new more specifying and simplifying condition will be that the domain $\mathcal{R}$ of partial possibilistic measures under investigation contains the empty subset of Ω and Ω itself.

Definition 3.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let Ω be a nonempty set, let $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$ be a system of subsets of Ω.

(i) A mapping $\Pi : \mathcal{R} \to T$ is called a *partial $\mathcal{T}$-(-valued) (normalized) monotone measure* on $\mathcal{R}$, if $\Pi(\emptyset) = \oslash_\mathcal{T}, \Pi(\Omega) = \mathbf{1}_\mathcal{T}$, and $\Pi(A) \leq \Pi(B)$ holds for each $A, B \in \mathcal{R}$ such that $A \subset B$ (instead of "monotone measure" sometimes also the term "fuzzy measure" is used).

(ii) A mapping $\Pi : \mathcal{R} \to T$ is called a *partial $\mathcal{T}$-(valued) (normalized) possibilistic measure* on $\mathcal{R}$, if $\Pi(\emptyset) = \oslash_\mathcal{T}$, $\Pi(\Omega) = \mathbf{1}_\mathcal{T}$, and $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ for every $A, B, A \cup B \in \mathcal{R}$ (it follows easily that each partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$ is also partial $\mathcal{T}$-monotone measure on $\mathcal{R}$).

(iii) A partial $\mathcal{T}$-possibilistic measure Π and $\mathcal{R}$ is called *complete,* if $\Pi(\bigcup \mathcal{R}_0) = \bigvee\{\Pi(A) : A \in \mathcal{R}_0\}$ holds for each $\emptyset \neq \mathcal{R}_0 \subset \mathcal{R}$ such that $\bigcup \mathcal{R}_0 (= \bigcup_{A \in \mathcal{R}_0} A)$ is in $\mathcal{R}$.

(iv) Let $t \in T$ be fixed. A mapping $\Pi : \mathcal{R} \to T$ is called a *partial $\mathcal{T}$-(valued) (normalized) t-quasi-possibilistic measure* on $\mathcal{R}$, if $\Pi(\emptyset) = \oslash_\mathcal{T}, \Pi(\Omega) = \mathbf{1}_\mathcal{T}$, and for every $A, B, A \cup B \in \mathcal{R}$ the relation $\rho(\Pi(A \cup B), \Pi(A) \vee \Pi(B)) \leq t$ holds, where ρ is defined by (2.4).

(v) A partial $\mathcal{T}$-t-quasi-possibilistic measure on $\mathcal{R}$ is called *complete,* if the relation $\rho(\Pi(\bigcup \mathcal{R}_0), \bigvee\{\Pi(A) : A \in \mathcal{R}_0\}) \leq t$ holds for each $\emptyset \neq \mathcal{R}_0 \subset \mathcal{R}$ such that $\bigcup \mathcal{R}_0 \in \mathcal{R}$.

If $\mathcal{T}$ satisfies the conditions of Fact 2.1, then the identity $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$ implies that

$$\rho(\Pi(A \cup B), \Pi(A) \vee \Pi(B)) = \oslash_{\mathcal{T}} \leq t \tag{9}$$

for each $t \in T$, so that the notion of t-quasi-possibilistic measure is a straightforward weakening of that of partial $\mathcal{T}$-possibilistic measure (the modification for the case of complete measures is obvious).

A class of lattice-valued partial quasi-possibilistic measures can be obtained when applying the following construction.

Theorem 3.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like lattice, let $\mathcal{A}$ be a nonempty ample filed of subsets of a nonempty space Ω (so that, for every $A \in \mathcal{A}$ and $\emptyset \neq \mathcal{A}_0 \subset \mathcal{A}$, the sets $\Omega - A$, $\bigcup \mathcal{A}_0$ and $\bigcap \mathcal{A}_0$ are in $\mathcal{A}$), let $t_0 \in T$ be fixed, let $\Pi_1 : \mathcal{A} \to T$ be a partial complete $\mathcal{T}$-possibilistic measure on $\mathcal{A}$, let $\Pi_2 : \mathcal{A} \to T$ be any mapping such that $\Pi_2(\emptyset) = \oslash_{\mathcal{T}}$ and $\Pi_2(\Omega) = \mathbf{1}_{\mathcal{T}}$. Set, for each $A \in \mathcal{A}$,

$$\Pi(A) = (\Pi_1(A) \wedge t_0^C) \vee (\Pi_2(A) \wedge t_0) \tag{10}$$

Then Π is a partial $\mathcal{T}$-t_0-quasi-possibilistic measure on $\mathcal{A}$.

Proof. Let us prove, first of all, that for each $s, t_1, t_2 \in T$ the inequality

$$\rho(s \vee t_1, s \vee t_2) \leq \rho(t_1, t_2) \tag{11}$$

is valid. Indeed, under the conditions imposed on $\mathcal{T}$, for each $s_1, s_2 \in T$ such that $s_1 \geq s_2$ holds, the inequality $s_1^C \leq s_2^C$ follows. In particular, $s \vee t_1 \geq s$ and $s \vee t_1 \geq t_1$ yields that $(s \vee t_1)^C \leq s^C$ and $(s \vee t_1)^C \leq t_1^C$, hence, $(s \vee t_1)^C \leq s^C \wedge t_1^C$ and, analogously, $(s \vee t_2)^C \leq s^C \wedge t_2^C$ hold. Using these relations we obtain that

$$\begin{aligned}
\rho(s \vee t_1, s \vee t_2) &= [(s \vee t_1) \wedge (s \vee t_2)^C] \vee [(s \vee t_2) \wedge (s \vee t_1)^C] \\
&\leq [(s \vee t_1) \wedge s^C \wedge t_2^C] \vee [(s \vee t_2) \wedge s^C \wedge t_1^C] \\
&= (s \wedge s^C \wedge t_2^C) \vee (t_1 \wedge s^C \wedge t_2^C) \\
&\vee \ (s \wedge s^C \wedge t_1^C) \vee (t_2 \wedge s^C \wedge t_1^C) \\
&= (t_1 \wedge t_2^C \wedge s^C) \vee (t_2 \wedge t_1^C \wedge s^C) \\
&\leq (t_1 \wedge t_2^C) \vee (t_2 \wedge t_1^C) = \rho(t_1, t_2),
\end{aligned} \tag{12}$$

as each Brouwerian lattice is distributive and $s \wedge s^C = \oslash_{\mathcal{T}}$ holds in every semi-Boolean (i.e., also in every Boolean-like) lattice, so that (2.3) is proved.

For the empty set $\emptyset$ and the space Ω, which obviously belong to each nonempty ample field $\mathcal{A} \subset \mathcal{P}(\Omega)$, we obtain that

$$\Pi(\emptyset) = (\Pi_1(\emptyset) \wedge t_0^C) \vee (\Pi_2(\emptyset) \wedge t_0) = (\oslash_{\mathcal{T}} \wedge t_0^C) \vee (\oslash_{\mathcal{T}} \wedge t_0) = \oslash_{\mathcal{T}}, \tag{13}$$

and

$$\Pi(\Omega) = (\Pi_1(\Omega) \wedge t_0^C) \vee (\Pi_2(\Omega) \wedge t_0) = (\mathbf{1}_{\mathcal{T}} \wedge t_0^C) \vee (\mathbf{1}_{\mathcal{T}} \wedge t_0) = t_0^C \vee t_0 = \mathbf{1}_{\mathcal{T}}, \tag{14}$$

as the lattice $\mathcal{T}$ is Boolean-like. Given $A, B \in \mathcal{R}$, we obtain that

$$\begin{aligned}\Pi(A \cup B) &= (\Pi_1(A \cup B) \wedge t_0^C) \vee (\Pi_2(A \cup B) \wedge t_0) \\ &= ((\Pi_1(A) \vee \Pi_1(B)) \wedge t_0^C) \vee (\Pi_2(A \cup B) \wedge t_0) \\ &= (\Pi_1(A) \wedge t_0^C) \vee (\Pi_1(B) \wedge t_0^C) \vee (\Pi_2(A \cup B) \wedge t_0), \end{aligned} \tag{15}$$

and

$$\begin{aligned}\Pi(A) \vee \Pi(B) &= [(\Pi_1(A) \wedge t_0^C) \vee (\Pi_2(A) \wedge t_0)] \\ &\vee\ [(\Pi_1(B) \wedge t_0^C) \vee (\Pi_2(B) \wedge t_0)] \\ &= [(\Pi_1(A) \vee \Pi_1(B)) \wedge t_0^C] \vee [(\Pi_2(A) \vee \Pi_2(B)) \wedge t_0]. \end{aligned} \tag{16}$$

Hence, applying (3.3) to $s = (\Pi_1(A) \vee \Pi_1(B)) \wedge t_0^C = (\Pi_1(A) \wedge t_0^C) \vee (\Pi_1(B) \wedge t_0^C)$, $t_1 = \Pi_2(A \cup B) \wedge t_0$, and $t_2 = (\Pi_2(A) \vee \Pi_2(B)) \wedge t_0$, we obtain that

$$\begin{aligned} & \rho(\Pi(A \cup B), \Pi(A) \vee \Pi(B)) \\ = \ & \rho[(\Pi(A) \wedge t_0^C) \vee (\Pi_1(B) \wedge t_0^C) \vee (\Pi_2(A \cup B) \wedge t_0), \\ & (\Pi_1(A) \wedge t_0^C) \vee (\Pi_1(B) \wedge t_0^C) \vee (\Pi_2(A) \wedge t_0) \vee (\Pi_2(B) \wedge t_0)] \\ \leq \ & \rho(\Pi_2(A \cup B) \wedge t_0, (\Pi_2(A) \vee \Pi_2(B)) \wedge t_0) \\ = \ & [(\Pi_2(A \cup B) \wedge t_0 \wedge ((\Pi_2(A) \vee \Pi_2(B)) \wedge t_0)^C] \\ \vee \ & [(\Pi_2(A) \vee \Pi_2(B)) \wedge t_0 \wedge (\Pi_2(A \cup B) \wedge t_0)^C] \leq t_0 \vee t_0 = t_0. \end{aligned} \tag{17}$$

The assertion is proved. □

The following lemma will be of use in our further considerations.

Lemma 3.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like lattice. Then, for every $t_1, t_2 \in T$, the identity

$$(t_1 \wedge t_2)^C = t_1^C \vee t_2^C \tag{18}$$

holds.

Proof. Under the conditions imposed on $\mathcal{T}$ we obtain that

$$\begin{aligned}(t_1 \wedge t_2)^C &= (t_1 \wedge t_2)^C \wedge \mathbf{1}_{\mathcal{T}} = (t_1 \wedge t_2)^C \wedge [(t_1^C \vee t_2^C) \vee (t_1^C \vee t_2^C)^C] \\ &= [(t_1 \wedge t_2)^C \wedge (t_1^C \vee t_2^C)] \vee [(t_1 \wedge t_2)^C \wedge (t_1^C \vee t_2^C)^C]. \end{aligned} \tag{19}$$

As $t_1 \wedge t_2 \leq t_1$ and $t_1 \wedge t_2 \leq t_2$ hold, the inverse inequalities $(t_1 \wedge t_2)^C \geq t_1^C, (t_1 \wedge t_2)^C \geq t_2^C$ for the corresponding pseudo-complements are obvious and the inequality $(t_1 \wedge t_2)^C \geq t_1^C \vee t_2^C$ follows. Hence, setting

$$s = (t_1 \wedge t_2)^C \wedge (t_1^C \vee t_1^C)^C, \tag{20}$$

(3.11) can be rewritten as

$$(t_1 \wedge t_2)^C = (t_1^C \vee t_2^C) \vee s., \tag{21}$$

Suppose, in order to arrive at a contradiction, that the strict inequality $(t_1 \wedge t_2)^C > t_1^C \vee t_2^C$ is valid, consequently, that $s > \oslash_{\mathcal{T}}$ holds. From (3.12) it follows easily, that

$$s \wedge (t_1^C \vee t_2^C) = (t_1 \wedge t_2)^C \wedge (t_1^C \vee t_2^C)^C \wedge (t_1^C \vee t_2^C) = \oslash_{\mathcal{T}}, \tag{22}$$

so that $s \wedge t_1^C = s \wedge t_2^C = \oslash_{\mathcal{T}}$, hence, $s \leq (t_1^C)^C = t_1, s \leq (t_2^C)^C = t_2$, and $s \leq t_1 \wedge t_2$ follow, so that $s \wedge t_1 \wedge t_2 = s > \oslash_{\mathcal{T}}$. However, (3.13) yields that $s \leq (t_1 \wedge t_2)^C$, so that $s \wedge (t_1 \wedge t_2) \leq (t_1 \wedge t_2)^C \wedge (t_1 \wedge t_2) = \oslash_{\mathcal{T}}$, and we have arrived at a contradiction. The lemma is proved. □

Remark 1. As a matter of fact, Lemma 3.1. is valid in each Brouwerian lattice (cf. Remark (2.72) in [1]), however, for our purposes the more specific case introduced in Lemma 3.1. seems to be worth being stated and proved explicitly.

Lemma 3.2. Under the notations and conditions of Lemma 3.1, for every $s_1, s_2, t \in T$ the inequality

$$\rho(s_1 \wedge t, s_2 \wedge t) \leq \rho(s_1, s_2) \tag{23}$$

is valid.

Proof. Applying Lemma 3.1 we obtain that

$$\begin{aligned}
\rho(s_1 \wedge t, s_2 \wedge t) &= [(s_1 \wedge t) \wedge (s_2 \wedge t)^C] \vee [(s_2 \wedge t) \wedge (s_1 \wedge t)^C] \\
&= [(s_1 \wedge t) \wedge (s_2^C \vee t^C)] \vee [(s_2 \wedge t) \wedge (s_1^C \vee t^C)] \\
&= (s_1 \wedge t \wedge s_2^C) \vee (s_1 \wedge t \wedge t^C) \vee (s_2 \wedge t \wedge s_1^C) \vee (s_2 \wedge t \wedge t^C) \\
&= (s_1 \wedge t \wedge s_2^C) \vee (s_2 \wedge t \wedge s_1^C) \leq (s_1 \wedge s_2^C) \vee (s_2 \wedge s_1^C) \\
&= \rho(s_1, s_2),
\end{aligned} \tag{24}$$

as $t \wedge t^C = \oslash_{\mathcal{T}}$. The lemma is proved. □

Let $\mathcal{T} = \langle T, \leq \rangle$ be a partially ordered set, let $t_0 \in T$, let $T(t_0) = \{s \wedge t_0 : s \in T\}$, let $\leq_{t_0}$, be the partial ordering on $T(t_0)$ defined by

$$s_1 \wedge t_0 \leq_{t_0} s_2 \wedge t_0 \Leftrightarrow s_1 \leq s_2 \tag{25}$$

Then $\mathcal{T}(t_0) = \langle T(t_0), \leq_{t_0} \rangle$ is a p.o. set. If $\mathcal{T}$ is a Brouwerian Boolean-like lattice, also $\mathcal{T}(t_0)$ is such lattice. Obviously, $\oslash_{\mathcal{T}(t_0)} = \oslash_{\mathcal{T}}$ and $\mathbf{1}_{\mathcal{T}(t_0)} = t_0$ for every $t_0 \in T$.

The next theorem can be taken, in a sense, as an assertion inverse to that of Theorem 3.1.

Theorem 3.2. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like lattice, let $\mathcal{A}$ be a nonempty ample-field of subsets of a nonempty basic space Ω, let $t_0 \in T$, let $\Pi : \mathcal{A} \to T$ be a partial $\mathcal{T}$-t_0-quasi-possibilistic measure on $\mathcal{A}$. Then there exist a partial $\mathcal{T}(t_0^C)$-possibilistic measure Π_1 on $\mathcal{A}$ and a mapping $\Pi_2 : \mathcal{A} \to T(t_0)$ such that $\Pi_2(\emptyset) = \oslash_{\mathcal{T}(t_0)}, \Pi_2(\Omega) = \mathbf{1}_{\mathcal{T}(t_0)}$ and the identity

$$\Pi(A) = \Pi_1(A) \vee \Pi_2(A) \tag{26}$$

holds for each $A \in \mathcal{A}$.

Proof. Under the conditions imposed on $\mathcal{T}$, $t_0 \vee t_0^C = \mathbf{1}_{\mathcal{T}}, t_0 \wedge t_0^C = \oslash_{\mathcal{T}}$, so that, for every $A \in \mathcal{A}$,

$$\Pi(A) = \Pi(A) \wedge \mathbf{1}_{\mathcal{T}} = (\Pi(A) \wedge t_0^C) \vee (\Pi(A) \wedge t_0). \tag{27}$$

Setting $\Pi_1(A) = \Pi(A) \wedge t_0^C$, $\Pi_2(A) = \Pi(A) \wedge t_0$, let us prove that Π_1 and Π_2 possess the properties declared in the assertion. The constraints for $\emptyset$ and Ω are obvious. Indeed,

$$\begin{aligned}
\Pi_1(\emptyset) &= \Pi(\emptyset) \wedge t_0^C = \oslash_{\mathcal{T}} \wedge t_0^C = \oslash_{\mathcal{T}} = \oslash_{\mathcal{T}(t_0^C)}, \\
\Pi_1(\Omega) &= \Pi(\Omega) \wedge t_0^C = \mathbf{1}_{\mathcal{T}} \wedge t_0^C = t_0^C = \mathbf{1}_{\mathcal{T}(t_0^C)}, \\
\Pi_2 \emptyset) &= \Pi(\emptyset) \wedge t_0 = \oslash_{\mathcal{T}} \wedge t_0 = \oslash_{\mathcal{T}} = \oslash_{\mathcal{T}(t_0)}, \\
\Pi_2(\Omega) &= \Pi(\Omega) \wedge t_0 = \mathbf{1}_{\mathcal{T}} \wedge t_0 = t_0 = \mathbf{1}_{\mathcal{T}(t_0)},
\end{aligned} \tag{28}$$

Let $A, B \in \mathcal{A}$. As Π is a partial $\mathcal{T}$-t_0-quasi-possibilistic measure on $\mathcal{A}$, the inequality

$$\rho(\Pi(A \cup B), \Pi(A) \vee \Pi(B)) \leq t_0 \tag{29}$$

holds. However,

$$\begin{aligned}
\Pi_1(A \cup B) &= \Pi(A \cup B) \wedge t_0^C, \\
\Pi_1(A) \vee \Pi_1(B) &= (\Pi(A) \wedge t_0^C) \vee (\Pi(B) \wedge t_0^C) \\
&= (\Pi(A) \vee \Pi(B)) \wedge t_0^C,
\end{aligned} \tag{30}$$

so that

$$\begin{aligned}
&\rho(\Pi_1(A \cup B), \Pi_1(A) \vee \Pi_1(B)) \\
=\ &\rho(\Pi(A \cup B) \wedge t_0^C, (\Pi(A) \vee \Pi(B)) \wedge t_0^C) \\
\leq\ &\rho(\Pi(A \cup B), \Pi(A) \vee \Pi(B)) \leq t_0
\end{aligned} \tag{31}$$

due to Lemma 3.2 and (3.21). At the same time, however,

$$\begin{aligned}
&(\Pi_1(A \cup B)) \wedge (\Pi_1(A) \vee \Pi_1(B))^C \\
=\ &\Pi(A \cup B) \wedge t_0^C \wedge (\Pi_1(A) \vee \Pi_1(B)^C \leq t_0^C
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&(\Pi_1(A) \vee \Pi_1(B)) \wedge (\Pi_1(A \cup B))^C \\
=\ &(\Pi(A) \vee \Pi(B)) \wedge t_0^C \wedge (\Pi_1(A \cup B))^C \leq t_0^C
\end{aligned} \tag{33}$$

hold, se that the inequality

$$\rho(\Pi_1(A \cup B), \Pi_1(A) \vee \Pi_1(B)) \leq t_0^C \tag{34}$$

follows. Combining (3.23) and (3.26) together, we obtain that

$$\rho(\Pi_1(A \cup B), \Pi_1(A) \vee \Pi_1(B)) \leq t_0 \wedge t_0^C = \oslash_{\mathcal{T}} \tag{35}$$

is valid, Hence, Lemma 2.1 yields that

$$\Pi_1(A \cup B) = \Pi_1(A) \vee \Pi_1(B), \tag{36}$$

so that Π_1 is a partial $\mathcal{T}(t_0^C)$-possibilistic measure on $\mathcal{A}$ and the assertion is proved. □

4. OPERATIONS OVER LATTICE–VALUED MONOTONE, POSSIBILISTIC AND QUASI–POSSIBILISTIC MEASURES

Given a lattice $\mathcal{T} = \langle T, \leq \rangle$, a nonempty system $\mathcal{R}$ of subsets of a basic space Ω and mappings Π_1, Π_2, both of them taking $\mathcal{R}$ into T, we may apply the operations of supremum and infimum defined w.r.to $\leq$ in $\mathcal{T}$ to the values of these mappings, so arriving at the mappings $\Pi_1 \wedge \Pi_2, \Pi \vee \Pi_2 : \mathcal{R} \to T$. Hence, for each $A \in \mathcal{R}$,

$$(\Pi_1 \wedge \Pi_2)(A) = \Pi_1(A) \wedge \Pi_2(A), (\Pi_1 \vee \Pi_2)(A) = \Pi_1(A) \vee \Pi_2(A). \tag{37}$$

The generalization to the case of complete lattice $\mathcal{T}$ and any non-empty set $\mathcal{S}$ of mappings, each of them taking $\mathcal{R}$ into T, is straightforward. Also the operation of (pseudo-) complement can be applied to produce new $\mathcal{T}$-valued mappings, setting, for each $A \in \mathcal{R}$ and each $\Pi : \mathcal{R} \to T$, $\Pi^C(A) = (\Pi(A))^C$. However, a more detailed analysis of mappings resulting when (pseudo-) complements are applied will be postponed till the next chapter.

Lemma 4.1. Let $\mathcal{R} = \langle T, \leq \rangle$ be a partially ordered set, let $\Omega \neq \emptyset$, let $\emptyset \neq \mathcal{R} \subset \mathcal{P}(\Omega)$, let $\Pi_1, \Pi_2 : \mathcal{R} \to T$ be $\mathcal{T}$-monotone measures on $\mathcal{R}$. If $\mathcal{T}$ is an upper semi-lattice (a lower semi-lattice, resp.), then $\Pi_1 \vee \Pi_2 (\Pi_1 \wedge \Pi_2$, resp.) is also a $\mathcal{T}$-monotone measure on $\mathcal{R}$.

Proof. Let Π_1, Π_2 be $\mathcal{T}$-monotone measures on $\mathcal{R}$, let $A, B \in \mathcal{R}$ be such that $A \subset B$, so that $\Pi_i(A) \leq \Pi_i(B)$ holds for both $i = 1, 2$. If $\mathcal{T}$ is a lower semi-lattice, the inequality

$$(\Pi_1 \wedge \Pi_2)(A) = \Pi_1(A) \wedge \Pi_2(B) \leq \Pi_1(B) \wedge \Pi_2(B) = (\Pi_1 \wedge \Pi_2)(B) \tag{38}$$

follows immediately. If $\mathcal{T}$ is an upper semilattice, the inequality

$$\Pi_i(A) \leq \Pi_1(B) \vee \Pi_2(B) \tag{39}$$

is valid for both $i = 1, 2$, so that the relation

$$(\Pi_1 \vee \Pi_2)(A) = \Pi_1(A) \vee \Pi_2(A) \leq \Pi_1(B) \vee \Pi_2(B) = (\Pi_1 \vee \Pi_2)(B) \tag{40}$$

follows and the lemma is proved. □

The following generalization is obvious.

Corollary 4.1. Let Ω and $\mathcal{R}$ be as in Lemma 4.1, let $\mathcal{T} = \langle T, \leq \rangle$ be a p. o. set, let $\mathcal{S}$ be a nonempty of $\mathcal{T}$-monotone measures, each of them taking $\mathcal{R}$ into T. If $\mathcal{T}$ is a complete upper semi-lattice, then the mapping $\bigvee \mathcal{S} : \mathcal{R} \to T$, defined by

$$\left(\bigvee \mathcal{S}\right)(A) = \bigvee \{\Pi(A) : \Pi \in \mathcal{S}\} \tag{41}$$

for every $A \in \mathcal{R}$, is a $\mathcal{T}$-monotone measure on $\mathcal{R}$. If $\mathcal{T}$ is a complete lower semilattice, then the mapping $\bigwedge \mathcal{S} : \mathcal{R} \to T$, defined by

$$\left(\bigwedge \mathcal{S}\right)(A) = \bigwedge \{\Pi(A) : \Pi \in \mathcal{S}\} \tag{42}$$

for every $A \in \mathcal{R}$, is a $\mathcal{T}$-monotone measure on $\mathcal{R}$.

For $\mathcal{T}$-possibilistic measures only the operation of supremum conserves the conditions imposed on such measures.

Lemma 4.2. Let Ω and $\mathcal{R}$ be as in Lemma 4.1, let $\mathcal{T} = \langle T, \leq \rangle$ be an upper semi-lattice, let $\Pi_1, \Pi_2 : \mathcal{R} \to T$ be partial $\mathcal{T}$-possibilistic measures on $\mathcal{R}$. Then $\Pi_1 \vee \Pi_2$ is also a partial $\mathcal{T}$-possibilistic measure on $\mathcal{R}$.

Proof. The constraints according to which $(\Pi_1 \vee \Pi_2)(\emptyset) = \oslash_{\mathcal{T}}$ and/or $(\Pi_1 \vee \Pi_2)(\Omega) = \mathbf{1}_{\mathcal{T}}$ supposing that $\emptyset \in \mathcal{R}$ and/or $\Omega \in \mathcal{R}$ are obviously satisfied. let $A, B, A \cup B \in \mathcal{R}$, then

$$\begin{aligned} & (\Pi_1 \vee \Pi_2)(A \cup B) = \Pi_1(A \cup B) \vee \Pi_2(A \cup B) = (\Pi_1(A) \vee \Pi_1(B)) \\ \vee \quad & (\Pi_2(A) \vee \Pi_2(B)) = (\Pi_1(A) \vee \Pi_2(A)) \vee (\Pi_1(B) \vee \Pi_2(B)) \\ = \quad & (\Pi_1 \vee \Pi_2)(A) \vee (\Pi_1 \vee \Pi_2)(B) \end{aligned} \tag{43}$$

and the assertion is proved. □

The generalization to the case of the mapping $\bigvee \mathcal{S}$ defined by (4.5) supposing that $\mathcal{T}$ is a complete upper semi-lattice, is obvious.

On the other side, if $\Omega, \mathcal{R}, \Pi_1, \Pi_2$ are as in Lemma 4.2 and if $\mathcal{T}$ is a complete lower semi-lattice, the mapping $\Pi_1 \wedge \Pi_2$ need not be, in general, a $\mathcal{T}$-possibilistic measure on $\mathcal{R}$, as the following simple example demonstrates. Let $\mathcal{R} = \{\emptyset, A, \Omega - A, \Omega\}$, where $\emptyset \neq A \neq \Omega$, let $\mathcal{T} = \langle T, \leq \rangle = \langle \{\oslash_{\mathcal{T}}, \mathbf{1}_{\mathcal{T}}\}, \leq \rangle$ be the most trivial two-element lattice with $\oslash_{\mathcal{T}} \leq \mathbf{1}_{\mathcal{T}}, \oslash_{\mathcal{T}} \neq \mathbf{1}_{\mathcal{T}}$. Let $\Pi_1, \Pi_2 : \mathcal{R} \to T$ be such that $\Pi_1(\emptyset) = \Pi_2(\emptyset) = \oslash_{\mathcal{T}}$, $\Pi_1(\Omega) = \Pi_2(\Omega) = \mathbf{1}_{\mathcal{T}}, \Pi_1(A) = \Pi_2(\Omega - A) = \oslash_{\mathcal{T}}$, $\Pi_1(\Omega - A) = \Pi_2(A) = \mathbf{1}_{\mathcal{T}}$. Both Π_1 and Π_2 are obviously $\mathcal{T}$-possibilistic measures on $\mathcal{R}$, as for both $i = 1, 2$,

$$\Pi_i(A \cup (\Omega - A)) = \Pi_i(\Omega) = \mathbf{1}_{\mathcal{T}} = \mathbf{1}_{\mathcal{T}} \vee \oslash_{\mathcal{T}} = \Pi(A) \vee \Pi_i(\Omega - A), \tag{44}$$

for the trivial unions containing $\emptyset$ and/or Ω as their components the condition (4.8) is also trivially satisfied. But

$$\begin{aligned} (\Pi_1 \wedge \Pi_2)(A) &= \Pi_1(A) \wedge \Pi_2(A) = \oslash_{\mathcal{T}} \wedge \mathbf{1}_{\mathcal{T}} = \oslash_{\mathcal{T}}, \\ (\Pi_1 \wedge \Pi_2)(\Omega - A) &= \Pi_1(\Omega - A) \wedge \Pi_2(\Omega - A) = 1 \wedge \oslash_{\mathcal{T}} = \oslash_{\mathcal{T}}, \end{aligned} \tag{45}$$

so that

$$
\begin{aligned}
& (\Pi_1 \wedge \Pi_2)(A \cup (\Omega - A)) = (\Pi_1 \wedge \Pi_2)(\Omega) = \Pi_1(\Omega) \wedge \Pi_2(\Omega) \\
= \;& \mathbf{1}_{\mathcal{T}} \wedge \mathbf{1}_{\mathcal{T}} = \mathbf{1}_{\mathcal{T}} \neq (\Pi_1 \wedge \Pi_2)(A) \vee (\Pi_1 \wedge \Pi_2)(\Omega - A) \\
= \;& \oslash_{\mathcal{T}} \vee \oslash_{\mathcal{T}} = \oslash_{\mathcal{T}},
\end{aligned} \tag{46}
$$

so that $\Pi_1 \wedge \Pi_2$ is not a $\mathcal{T}$-possibilistic measure on $\mathcal{R}$.

Using the lattice-valued metric function ρ, the relation (4.10) can be written as

$$\rho((\Pi_1 \wedge \Pi_2)(A \cup (\Omega - A)), (\Pi_1 \wedge \Pi_2)(A) \vee (\Pi_1 \wedge \Pi_2)(\Omega - A)) = \mathbf{1}_{\mathcal{T}} \tag{47}$$

supposing that $\mathcal{T}$ fulfills the conditions under which ρ was defined. This fact will be of use when considering the case of quasi-possibilistic measures (below).

The following lemma will be of use when generalizing the assertion of Lemma 4.2 to quasi-possibilistic measures.

Lemma 4.3. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like complete lattice. Then, for every $s_1, s_2, t_0, t_1, t_2 \in T$, if $\rho(s_1, t_1) \leq t_0$ and $\rho(s_2, t_2) \leq t_0$ hold, the inequality $\rho(s_1 \vee s_2, t_1 \vee t_2) \leq t_0$ holds as well.

Proof. Due to the conditions imposed on $\mathcal{T}$, following inequalities are valid.

$$
\begin{aligned}
& \rho(s_1 \vee s_2, t_1 \vee t_2) = [(s_1 \vee s_2) \wedge (t_1 \vee t_2)^C] \vee [(t_1 \vee t_2) \wedge (s_1 \vee s_2)^C] \\
\leq \;& [(s_1 \vee s_2) \wedge (t_1^C \wedge t_2^C)] \vee [(t_1 \vee t_2) \wedge (s_1^C \wedge s_2^C)] \\
= \;& (s_1 \wedge t_1^C \wedge t_2^C) \vee (s_2 \wedge t_1^C \wedge t_2^C) \vee (t_1 \wedge s_1^C \wedge s_2^C) \vee (t_2 \wedge s_1^C \wedge s_2^C) \\
\leq \;& (s_1 \wedge t_1^C) \vee (s_2 \wedge t_2^C) \vee (t_1 \wedge s_1^C) \vee (t_2 \wedge s_2^C) \\
= \;& \rho(s_1, t_2) \vee \rho(s_2, t_2) \leq t_0 \vee t_0 = t_0.
\end{aligned} \tag{48}
$$

The lemma is proved. □

Theorem 4.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like lattice, let $\Omega \neq \emptyset$, let $\{\emptyset, \Omega\} \subset \mathcal{R} \subset \mathcal{P}(\Omega)$ be a system of subsets of Ω, let $t_0 \in T$, let $\Pi_1, \Pi_2 : \mathcal{R} \to T$ be$\mathcal{T}$-t_0-quasi-possibilistic measures on $\mathcal{R}$. Then $\Pi_1 \vee \Pi_2$ is also a $\mathcal{T}$-t_0-quasi-possibilistic measure on $\mathcal{R}$.

Proof. The constraints for $\emptyset$ and Ω obviously hold. Let $A, B, A \cup B \in \mathcal{R}$, so that the inequality

$$\rho(\Pi_i(A \cup B), \Pi_i(A) \vee \Pi_i(B)) \leq t_0 \tag{49}$$

is valid for both $i = 1, 2$. Setting $s_i = \Pi_i(A \cup B)$ and $t_i = \Pi_i(A) \vee \Pi_i(B)$ for both $i = 1, 2$, and applying Lemma 4.3, we obtain that

$$
\begin{aligned}
& \rho((\Pi_1 \vee \Pi_2)(A \cup B), (\Pi_1 \cup \Pi_2)(A) \vee (\Pi_1 \vee \Pi_2)(B)) \\
= \;& \rho(\Pi_1(A \cup B) \vee \Pi_2(A \cup B), (\Pi_1(A) \vee \Pi_2(A)) \vee (\Pi_1(B) \vee \Pi_2(B))) \\
\leq \;& \rho(\Pi_1(A \cup B), \Pi_1(A) \vee \Pi_1(B)) \vee \rho(\Pi_2(A \cup B), \Pi_2(A) \vee \Pi_2(B)) \\
\leq \;& t_0 \vee t_0 = t_0.
\end{aligned} \tag{50}
$$

The assertion is proved. □

As in the case of partial $\mathcal{T}$-possibilistic measures, the conjunction (infimum) of $\mathcal{T}$-t_0-quasi-possibilistic measures need not be, in general, a $\mathcal{T}$-t_0-quasi-possibilistic measure over the same domain. Indeed, re-considering the same counter-example as above, we obtain that the relation $\rho(\Pi_i(A \cup B, \Pi_i(A) \vee \Pi_i(B)) = \emptyset_{\mathcal{T}} \leq t_0$ holds for each $A, B, A \cup B \in \mathcal{R}$, each $t_0 \in T$, and both $i = 1, 2$, but (4.11) yields that

$$\rho((\Pi_1 \wedge \Pi_2)(A \cup (\Omega - A)), (\Pi_1 \wedge \Pi_2)(A) \vee (\Pi_1 \wedge \Pi_2)(\Omega - A)) \leq t_0 \qquad (51)$$

does not hold, if $t_0 < \mathbf{1}_{\mathcal{T}}$.

5. FROM REAL–VALUED TO LATTICE–VALUED NECESSITY MEASURES

In the case of a real-valued possibilistic measure $\Pi : \mathcal{P}(\Omega) \rightarrow [0,1]$, the necessity measure Σ (or Σ_Π, to make explicit the original possibilistic measure Π) is the mapping, defined for every $A \subset \Omega$, by

$$\Sigma(A) = 1 - \Pi(\Omega - A). \qquad (52)$$

Its properties are, in a sense, dual to those of Π. Obviously, $\Sigma(\emptyset) = 0$ and $\Sigma(\Omega) = 1$, moreover, for every $A, B \subset \Omega$ we obtain that

$$\begin{aligned} & \Sigma(A \cap B) = 1 - \Pi(\Omega - (A \cap B)) = 1 - \Pi((\Omega - A) \cup (\Omega - B)) \\ = \; & 1 - [\Pi(\Omega - A) \vee \Pi(\Omega - B)] = (1 - \Pi(\Omega - A)) \wedge (1 - \Pi(\Omega - B)) \\ = \; & \Sigma(A) \wedge \Sigma(B), \end{aligned} \qquad (53)$$

denoting, for a while, by $\vee$ and $\wedge$ the supremum and infimum in $\langle [0,1], \leq \rangle$. Moreover, as can be easily checked, for each $A \subset \Omega$ the inequality $\Sigma(A) \leq \Pi(A)$ is valid.

As a matter of fact, necessity measure can be defined also axiomatically as a primitive notion. In the most simple case just investigated, real-valued necessity measure, defined on the power-set of all subsets of a fixed space Ω, is a mapping $\Sigma : \mathcal{P}(\Omega) \rightarrow [0,1]$ such that $\Sigma(\emptyset) = 0, \Sigma(\Omega) = 1$, and $\Sigma(A \cap B) = \Sigma(A) \wedge \Sigma(B)$ holds for every $A, B \subset \Omega$. If the relation $\Sigma(\bigcap \mathcal{A}_0) = \bigwedge \{\Sigma(A) : A \in \mathcal{A}_0\}$ is valid for every nonempty system $\mathcal{A}_0 \subset \mathcal{P}(\Omega)$, where $\bigcap \mathcal{A}_0 = \bigcap_{A \in \mathcal{A}_0} A$, the necessity measure Σ is called *complete*. The duality between possibilistic and necessity measures goes so far that, setting for a necessity measure $\Sigma, \Pi_\Sigma(A) = 1 - \Sigma(\Omega - A)$ for every $A \subset \Omega$, the mapping Π_Σ can be easily proved to define a possibilistic measure on $\mathcal{P}(\Omega)$. Moreover, if $\Sigma = \Sigma_\Pi$ for a possibilistic measure Π on $\mathcal{P}(\Omega)$, the identity $\Pi(A) = \Pi_{(\Sigma_\Pi)}(A)$ holds for every $A \subset \Omega$.

Let us consider an alternative way of definition of necessity measure induced by a given real-valued possibilistic measure on $\mathcal{P}(\Omega)$. Namely, instead of the operation of substraction $1-$. we apply the operation of pseudo-complement defined in $\langle [0,1], \leq \rangle$, setting for every $A \subset \Omega$

$$\hat{\Sigma}(A) = (\Pi(\Omega - A))^C = \bigvee \{x \in [0,1] : x \wedge \Pi(\Omega - A) = 0\}. \qquad (54)$$

As a matter of fact, $\hat{\Sigma}$ takes $\mathcal{P}(\Omega)$ into the binary set $\{0,1\}$. Indeed, if $\Pi(\Omega-A)>0$, then $\hat{\Sigma}(A) = 0$, if $\Pi(\Omega - A) = 0$, then $\hat{\Sigma}(A) = 1 = \Sigma(A)$, so that the inequality $\hat{\Sigma}(A) \leq \Sigma(A)$ obviously holds for every $A \subset \Omega$ (In what follows, when using Σ and Π together, they are supposed to be related to each other by (5.1)). Consequently, if Π takes only the values 0 or $1, \Sigma$ and $\hat{\Sigma}$ are identical set functions. However, this is not the case if there exists $B \subset \Omega$ such that $0 < \Pi(A) < 1$ (e. g., Π is defined by a possibilistic distribution $\pi : \Omega \to [0,1]$ such that $0 < \pi(\omega_0) < 1$ holds for some $\omega_0 \in \Omega$, then $0 < \Pi(B) < 1$ holds for $B = \{\omega_0\}$). Setting $A = \Omega - B$ we obtain that

$$\Sigma(A) = 1 - \Pi(\Omega - A) = 1 - \Pi(\Omega - (\Omega - B)) = 1 - \Pi(B) > 0, \tag{55}$$

but

$$\begin{aligned}\hat{\Sigma}(A) &= \bigvee\{x \in [0,1] : x \wedge \Pi(\Omega - A) = 0\} \\ &= \bigvee\{x \in [0,1] : x \wedge \Pi(B) = 0\} = 0\end{aligned} \tag{56}$$

Nevertheless, (5.2) holds even when replacing Σ by $\hat{\Sigma}$, i. e., the relation

$$\hat{\Sigma}(A \cap B) = \hat{\Sigma}(A) \wedge \hat{\Sigma}(B) \tag{57}$$

is valid for every $A, B \subset \Omega$. Indeed,

$$\begin{aligned} & \hat{\Sigma}(A \cap B) = (\Pi(\Omega - (A \cap B)))^C = [(\Pi(\Omega - A)) \vee (\Pi(\Omega - B))]^C \\ = \; & \bigvee\{x \in [0,1] : x \wedge (\Pi(\Omega - A) \vee \Pi(\Omega - B)) = 0 \\ \leq \; & \left(\bigvee\{x \in [0,1] : x \wedge \Pi(\Omega - A) = 0\}\right) \\ \wedge \; & \left(\bigvee\{x \in [0,1] : x \wedge \Pi(\Omega - B) = 0\}\right) \\ = \; & (\Pi(\Omega - A))^C \wedge (\Pi(\Omega - B))^C = \hat{\Sigma}(A) \wedge \hat{\Sigma}(B), \end{aligned} \tag{58}$$

hence, if $\hat{\Sigma}(A) = 0$ or $\hat{\Sigma}(B) = 0$, also $\hat{\Sigma}(A \cap B) = 0$ and (5.6) holds. If $\hat{\Sigma}(A) = \hat{\Sigma}(B) = 1$, then $\Pi(\Omega - A) = \Pi(\Omega - B) = 0 = \Pi(\Omega - A) \vee \Pi(\Omega - B)$, hence, for every $x \in [0,1]$ the relation $x \wedge [\Pi(\Omega - A) \vee \Pi(\Omega - B)] = x \wedge \Pi(\Omega - (A \cap B)) = 0$ holds, so that $\hat{\Sigma}(A \cap B) = 1$ and (5.6) is valid also in this case. Combining the inequalities $\Sigma(A) \leq \Pi(A)$ and $\hat{\Sigma}(A) \leq \Sigma(A)$, proved above to be valid for every $A \subset \Omega$, we obtain that $\hat{\Sigma}(A) \leq \Pi(A)$ also holds for every $A \subset \Omega$.

Given a necessity measure Σ on $\mathcal{P}(\Omega)$, let us define the mapping $\hat{\Pi}$ (or $\hat{\Pi}_\Sigma$) : $\mathcal{P}(\Omega) \to [0,1]$ such that, for every $A \subset \Omega$,

$$\hat{\Pi}(A) = (\Sigma(\Omega - A))^C. \tag{59}$$

Applying the way of reasoning dual to that used when proving (5.6) above we obtain that $\hat{\Pi}$ takes only the values 0 or 1 and that it is a possibilistic measure on $\mathcal{P}(\Omega)$. However, if Π is a possibilistic measure on $\mathcal{P}(\Omega)$ such that $0 < \Pi(A) < 1$ holds for at least one $A \subset \Omega$, if $\hat{\Sigma}(= \hat{\Sigma}_\Pi)$ is defined by (5.3) and $\hat{\Pi} = \hat{\Pi}_{\hat{\Sigma}} = \hat{\Pi}_{(\hat{\Sigma}_\Pi)}$ is defined by (5.8) then, contrary to the case when the substraction $1 - .$ is applied, $\hat{\Pi}$ is not identical with the original possibilistic measure Π, as $\hat{\Pi}(A)$ can be only 0 or 1.

Relation (5.3) can be easily modified to the case of lattice-valued possibilistic measures.

Definition 5.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like lattice, let $\Omega \neq \emptyset$, let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be a nonempty ample field, let $\Pi : \mathcal{A} \to T$ be a $\mathcal{T}$-possibilistic measure on $\mathcal{A}$. Then the mapping Σ (or Σ_Π) : $\mathcal{A} \to T$, defined by

$$\Sigma_\Pi(A) = (\Pi(\Omega - A))^C \tag{60}$$

for every $A \in \mathcal{A}$ is called the *$\mathcal{T}$-(valued) necessity measure* induced by Π on $\mathcal{A}$.

Theorem 5.1. Let the notations and conditions of Definition 5.1 hold. Then

(i) $\Sigma_\Pi(\emptyset) = \oslash_\mathcal{T}, \Sigma_\Pi(\Omega) = \mathbf{1}_\mathcal{T}$,

(ii) $\Sigma_\Pi(A) \leq \Pi(A)$ for every $A \in \mathcal{A}$,

(iii) $\Sigma_\Pi(A \cap B) = \Sigma_\Pi(A) \wedge \Sigma_\Pi(B)$ for every $A, B \in \mathcal{A}$.

Proof.

$$\begin{aligned} \Sigma_\Pi(\emptyset) &= (\Pi(\Omega - \emptyset))^C = \mathbf{1}_\mathcal{T}^C = \oslash_\mathcal{T}, \\ \Sigma_\Pi(\Omega) &= (\Pi(\Omega - \Omega))^C = \oslash_\mathcal{T}^C = \mathbf{1}_\mathcal{T}, \end{aligned} \tag{61}$$

so that (i) is proved. Given $A, B \in \mathcal{A}$,

$$\begin{aligned} & \Sigma_\Pi(A \cap B) = (\Pi(\Omega - (A \cap B)))^C = \Pi((\Omega - A) \cup (\Omega - B)))^C \\ = \; & (\Pi(\Omega - A) \vee \Pi(\Omega - B))^C = (\Pi(\Omega - A))^C \wedge (\Pi(\Omega - B))^C \\ = \; & \Sigma_\Pi(A) \wedge \Sigma_\Pi(B) \end{aligned} \tag{62}$$

due to the assumptions imposed on the complete lattice $\mathcal{T} = \langle T, \leq \rangle$, so that (iii) is also proved. Finally, for each $A \in \mathcal{A}$ and each $t \in T$,

$$t = t \wedge \mathbf{1}_\mathcal{T} = t \wedge (\Pi(A) \vee \Pi(\Omega - A)) = (t \wedge \Pi(A)) \vee (t \wedge \Pi(\Omega - A)). \tag{63}$$

Hence, for each $t \in T$ such that $t \wedge \Pi(\Omega - A) = \oslash_\mathcal{T}$ the identity $t = t \wedge \Pi(A)$ and, consequently, also the inequality $t \leq \Pi(A)$ follow. So, also the inequality

$$\Sigma(A) = (\Pi(\Omega - A))^C = \bigvee \{t \in T : t \wedge \Pi(\Omega - A) = \oslash_\mathcal{T}\} \leq \Pi(A) \tag{64}$$

is valid and the proof of Theorem 5.1 is completed. □

6. QUASI–NECESSITY MEASURES

The operation replacing $\Pi(A)$ by $\Sigma(A) = (\Pi(\Omega - A))^C$ can be applied to any mapping $\Pi : \mathcal{A} \to T$ such that $\mathcal{T} = \langle T, \leq \rangle$ is a complete lattice, so that the (pseudo-) complement t^C is defined for every $t \in T$, and supposing that $\mathcal{A}$ is closed with respect to complements. When Π is a $\mathcal{T}$-t_0-quasi-possibilistic measure on $\mathcal{A}$, the following assertion can be proved.

Lemma 6.1. Let $\Omega \neq \emptyset$, let $\emptyset \neq \mathcal{A} \subset \mathcal{P}(\Omega)$ be an ample field of subsets of Ω, let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like complete lattice, let $\Pi : \mathcal{A} \to T$ be a $\mathcal{T}$-t_0-quasi-possibilistic measure on $\mathcal{A}$, where $t_0 \in T$ is fixed. Then for every $A, B \in \mathcal{A}$ the inequality

$$\rho(\Sigma(A \cap B), \Sigma(A) \wedge \Sigma(B)) \leq t_0 \tag{65}$$

is valid, where Σ is induced by Π due to (5.1).

Proof. Under the conditions imposed on $\mathcal{T}$ the identities $s = (s^C)^C$ and $t = (t^C)^C$ hold for every $s, t \in T$, so that

$$\rho(s,t) = (s \wedge t^C) \vee (t \wedge s^C) = ((s^C)^C \wedge t^C) \vee ((t^C)^C \wedge s^C) = \rho(s^C, t^C). \tag{66}$$

Hence, for every $A, B \in \mathcal{A}$ also $\Omega - A, \Omega - B \in \mathcal{A}$ and

$$\begin{aligned} & \rho(\Sigma(A \cap B), \Sigma(A) \wedge \Sigma(B)) \\ = \ & \rho((\Pi(\Omega - (A \cap B)))^C, (\Pi(\Omega - A))^C \wedge (\Pi(\Omega - B))^C) \\ = \ & \rho((\Pi((\Omega - A) \cup (\Omega - B)))^C, (\Pi(\Omega - A) \vee \Pi(\Omega - B))^C) \\ = \ & \rho(\Pi((\Omega - A) \cup (\Omega - B)), \Pi(\Omega - A) \vee \Pi(\Omega - B)) \leq t_0, \end{aligned} \tag{67}$$

as Π is a t_0-quasi-possibilistic measure on $\mathcal{A}$. The assertion is proved. □

Evidently, the mapping Σ defined in Lemma 6.1 could be called the $\mathcal{T}$-t_0-quasi-necessity measure on $\mathcal{A}$, induced by the $\mathcal{T}$-t_0-quasi-possibilistic measure Π, as it weakens the condition $\Sigma(A \cap B) = \Sigma(A) \wedge \Sigma(B)$ in the same way and degree in which the definition of $\mathcal{T}$-t_0-quasi-possibilistic measure Π weakens the condition $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$. Hence, the notion of $\mathcal{T}$-t_0-quasi-necessity measure can be introduced also axiomatically, with (6.1) in the role of the key axiom. The following definition is purposely conceived at a rather general level.

Definition 6.1. Let Ω be a nonempty set, let $\mathcal{R}$ be a non-empty system of subsets of Ω, let $\mathcal{T} = \langle T, \leq \rangle$ be a lower semi-lattice (i. e., for every $s, t \in T$, $s \wedge t$ is defined in T), let $t_0 \in T$. A mapping $\Sigma : \mathcal{R} \to T$ is called a *$\mathcal{T}$-(valued)-t_0-quasi-necessity measure* on $\mathcal{R}$, if $\oslash_{\mathcal{T}} = \bigwedge T$ is defined and $\Sigma(\emptyset) = \oslash_{\mathcal{T}}$ supposing that $\emptyset \in \mathcal{R}$, if $\mathbf{1}_{\mathcal{T}} = \bigvee T$ is defined and $\Sigma(\Omega) = \mathbf{1}_{\mathcal{T}}$ supposing that $\Omega \in \mathcal{R}$, and if the relation

$$\rho(\Sigma(A \cap B), \Sigma(A) \wedge \Sigma(B)) \leq t_0 \tag{68}$$

is valid for every $A, B, A \cap B \in \mathcal{R}$. If $\mathcal{T}$ is a complete lower semi-lattice (i. e., $\bigwedge A = \bigwedge_{t \in A} t$ is defined for every $\emptyset \neq A \subset T$) and if the relation

$$\rho\left(\Sigma\left(\bigcap \mathcal{A}\right), \bigwedge \{\Sigma(A) : A \in \mathcal{A}\}\right) \leq t_0 \tag{69}$$

is valid for every $\emptyset \neq \mathcal{A} \subset \mathcal{R}$ such that $\bigcap \mathcal{A} (= \bigcap_{A \in \mathcal{A}} A)$ is in $\mathcal{R}$, the $\mathcal{T}$-t_0-quasi-necessity measure Σ is called *complete.*

The following assertion is analogous to Theorem 3.1.

Theorem 6.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like lattice, let $\mathcal{A}$ be a nonempty ample field of subsets of a non-empty space Ω, let Σ_1 be a $\mathcal{T}$-necessity measure on $\mathcal{A}$, let $\Sigma_2 : \mathcal{A} \to T$ be any mapping such that $\Sigma_2(\emptyset) = \oslash_{\mathcal{T}}$ and $\Sigma_2(\Omega) = \mathbf{1}_{\mathcal{T}}$, let $t_0 \in T$ be fixed. let

$$\Sigma(A) = (\Sigma_1(A) \wedge t_0^C) \vee (\Sigma_2(A) \wedge t_0) \tag{70}$$

for every $A \in \mathcal{A}$. Then Σ is a $\mathcal{T}$-t_0-quasi-necessity measure on $\mathcal{A}$.

Proof. The relations $\Sigma(\emptyset) = \oslash_{\mathcal{T}}$ and $\Sigma(\Omega) = \mathbf{1}_{\mathcal{T}}$ are obvious. Let $A, B \in \mathcal{A}$, then

$$\begin{aligned}
& \rho(\Sigma(A \cap B), \Sigma(A) \wedge \Sigma(B)) \\
= \ & \rho((\Sigma_1(A \cap B) \wedge t_0^C) \vee (\Sigma_2(A \cap B) \wedge t_0), \\
& [(\Sigma_1(A) \wedge t_0^C) \vee (\Sigma_2(A) \wedge t_0)] \wedge [(\Sigma_1(B) \wedge t_0^C) \vee (\Sigma_2(B) \wedge t_0)]) \\
= \ & \rho((\Sigma_1(A) \wedge \Sigma_1(B) \wedge t_0^C) \vee (\Sigma_2(A \cap B) \wedge t_0), \\
& (\Sigma_1(A) \wedge \Sigma_1(B) \wedge t_0^C) \vee (\Sigma_2(A) \wedge t_0 \wedge \Sigma_1(B) \wedge t_0^C) \vee \\
& (\Sigma_1(A) \wedge t_0^C \wedge \Sigma_2(B) \wedge t_0) \vee (\Sigma_2(A) \wedge \Sigma_2(B) \wedge t_0)) \\
= \ & \rho((\Sigma_1(A) \wedge \Sigma_1(B) \wedge t_0^C) \vee (\Sigma_2(A \cap B) \wedge t_0), \\
& (\Sigma_1(A) \wedge \Sigma_1(B) \wedge t_0^C) \vee (\Sigma_2(A) \wedge \Sigma_2(B) \wedge t_0)),
\end{aligned} \tag{71}$$

as $t_0 \wedge t_0^C = \oslash_{\mathcal{T}}$ due to the conditions imposed on $\mathcal{T}$. Applying (3.3) with $s = \Sigma_1(A) \wedge \Sigma_1(B) \wedge t_0^C$, $t_1 = \Sigma_2(A \cap B) \wedge t_0$ and $t_2 = \Sigma_2(A) \wedge \Sigma_2(B) \wedge t_0$, we obtain the inequality

$$\begin{aligned}
& \rho(\Sigma(A \cap B), \Sigma(A) \wedge \Sigma(B)) \leq \rho(\Sigma_2(A \cap B) \wedge t_0, \Sigma_2(A) \wedge \Sigma_2(B) \wedge t_0) \\
= \ & [\Sigma_2(A \cap B) \wedge t_0 \wedge (\Sigma_2(A) \wedge \Sigma_2(B) \wedge t_0)^C] \\
\vee \ & [\Sigma_2(A) \wedge \Sigma_2(B) \wedge t_0 \wedge (\Sigma_2(A \cap B) \wedge t_0)^C] \leq t_0 \vee t_0 = t_0.
\end{aligned} \tag{72}$$

Hence, Σ is a $\mathcal{T}$-t_0-quasi-necessity measure on $\mathcal{A}$. □

Also an assertion for t_0-quasi-necessity measures analogous to that of Theorem 3.2 for t_0-quasi-possibilistic measures can be stated and proved.

Theorem 6.2. Let $\mathcal{T} = \langle T, \leq \rangle, t_0, \mathcal{T}(t_0), \mathcal{T}(t_0^C), \Omega$ and $\mathcal{A}$ be as in Theorem 3.2, let $\Sigma : \mathcal{A} \to T$ be a partial $\mathcal{T}$-t_0-quasi-necessity measure on $\mathcal{A}$. Then there exist a partial $\mathcal{T}(t_0^C)$-valued necessity measure Σ_1 on $\mathcal{A}$ and a mapping $\Sigma_2 : \mathcal{A} \to T(t_0)$ such that $\Sigma_2(\emptyset) = \oslash_{\mathcal{T}(t_0)}, \Sigma_2(\Omega) = \mathbf{1}_{\mathcal{T}(t_0)}$, and that the identity $\Sigma(A) = \Sigma_1(A) \vee \Sigma_2(A)$ holds for each $A \in \mathcal{A}$.

Proof. The proof more or less copies that of Theorem 3.2 above. Under the conditions imposed on $\mathcal{T}$, the relation

$$\Sigma(A) = (\Sigma(A) \wedge t_0^C) \vee (\Sigma(A) \wedge t_0) \tag{73}$$

holds for every $A \in \mathcal{A}$. Setting $\Sigma_1(A) = \Sigma(A) \wedge t_0^C$ and $\Sigma_2(A) = \Sigma(A) \wedge t_0$, the equalities $\Sigma_1(\emptyset) = \Sigma_2(\emptyset) = \oslash_{\mathcal{T}(t_0^C)} = \oslash_{\mathcal{T}(t_0)} = \oslash_{\mathcal{T}}$, $\Sigma_1(\Omega) = \mathbf{1}_{\mathcal{T}(t_0^C)}$, and $\Sigma_2(\Omega) = \mathbf{1}_{\mathcal{T}(t_0)}$ are obviously valid, cf. (3.20) for more detail.

Let $A, B \in \mathcal{A}$. As Σ is a partial $\mathcal{T}$-t_0-quasi-necessity measure on $\mathcal{A}$, the inequality

$$\rho(\Sigma(A \cap B), \Sigma(A) \wedge \Sigma(B)) \leq t_0 \tag{74}$$

is valid. Due to the definition of Σ_1,

$$\begin{aligned} & \rho(\Sigma_1(A \cap B), \Sigma_1(A) \wedge \Sigma_1(B)) \\ = \quad & \rho(\Sigma(A \cap B) \wedge t_0^C, \Sigma(A) \wedge \Sigma(B) \wedge t_0^C) \\ \leq \quad & \rho(\Sigma(A \cap B), \Sigma(A) \wedge \Sigma(B)) \leq t_0, \end{aligned} \tag{75}$$

applying Lemma 3.2 and (6.10). On the other side, however,

$$(\Sigma_1(A \cap B)) \wedge (\Sigma_1(A) \wedge \Sigma_1(B))^C \leq \Sigma(A \cap B) \wedge t_0^C \leq t_0^C, \tag{76}$$

and

$$(\Sigma_1(A) \wedge \Sigma_1(B)) \wedge (\Sigma_1(A \cap B))^C \leq \Sigma(A) \wedge \Sigma(B) \wedge t_0^C \leq t_0^C, \tag{77}$$

so that the inequality

$$\rho(\Sigma_1(A \cap B), \Sigma_1(A) \wedge \Sigma_1(B) \leq t_0^C \tag{78}$$

follows. Combining (6.11) and (6.14) together, we obtain that

$$\rho(\Sigma_1(A \cap B), \Sigma_1(A) \wedge \Sigma_1(B)) \leq t_0 \wedge t_0^C = \oslash_{\mathcal{T}} \tag{79}$$

holds. Lemma 2.1 then yields that

$$\Sigma_1(A \cap B) = \Sigma_1(A) \wedge \Sigma_1(B), \tag{80}$$

so that Σ_1 is a partial $\mathcal{T}(t_0^C)$-necessity measure on $\mathcal{A}$ and the assertion is proved. □

7. QUASI-MONOTONE MEASURES

The same idea on which the weakening of possibilistic and necessity measures to quasi-possibilistic and quasi-necessity measures is based can be applied to perhaps the most general and still nontrivial set functions reflecting only the most general common property of all set functions measuring the sizes of sets from their domain – the property of monotonicity with respect to set inclusion. Very often such set functions are called *fuzzy measures,* but this term seems to be rather misleading, as these mappings have nothing in common neither with the phenomenon of fuzziness itself, nor with its mathematical processing within the framework of the theory of fuzzy sets. Hence, in what follows, we use the perhaps more adequate term *monotone measures.* Let us introduce the appropriate notation and recall the basic definition.

Definition 7.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a partially ordered set, let $\mathcal{R}$ be a nonempty system of subsets of a nonempty set Ω. A mapping $\Pi : \mathcal{A} \to T$ is called a $\mathcal{T}$-*(valued) monotone measure* on $\mathcal{R}$, if for each $A, B \in \mathcal{R}$ such that $A \subset B$ holds the relation $\Pi(A) \leq \Pi(B)$ is valid. If $\mathcal{T} = \langle T, \leq \rangle$ is such that the *minimum element* $\oslash_{\mathcal{T}} = \bigwedge T = \bigwedge_{t \in T} t$ and the *maximum element* $\mathbf{1}_{\mathcal{T}} = \bigvee_{t \in T} t = \bigvee T$ are defined, if $\emptyset$ and Ω are in $\mathcal{R}$, and if the relations $\Pi(\emptyset) = \oslash_{\mathcal{T}}$ and $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ hold, then the $\mathcal{T}$-monotone measure Π is *normalized*. Here $\bigwedge$ and $\bigvee$ denote the (in general, partial) operations of infimum and supremum induced by the partial ordering relation $\leq$. Obviously, the inequalities $\Pi(\emptyset) \leq \Pi(A) \leq \Pi(\Omega)$ hold for each monotone measure Π and each $A \in \mathcal{R}$ supposing that $\emptyset \in \mathcal{R}$ and/or $\Omega \in \mathcal{R}$.

The inspiration for our further reasoning will be taken from the particular case $\mathcal{T} = \langle T, \leq \rangle = \langle \mathcal{P}(X), \subset \rangle$, when the values of the monotone measure Π in question are subsets of a fixed nonempty set X partially ordered by the relation of set inclusion. The condition of monotonicity obviously reads, in this case, that $\Pi(A) \subset \Pi(B)$ or, what turns to be the same, that $\Pi(A) - \Pi(B) = \emptyset$ holds for each $A, B \in \mathcal{R}$ such that $A \subset B$. As in $\langle \mathcal{P}(X), \subset \rangle$ the minimum element $\emptyset$ and the maximum element X are obviously defined, a $\langle \mathcal{P}(X), \subset \rangle$-valued monotone measure Π on $\mathcal{R}$ is normalized, iff $\emptyset$ and Ω are in $\mathcal{R}, \Pi(\emptyset) = \emptyset$, and $\Pi(\Omega) = X$. The weakened modification is almost evident. Let $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$, let X_0 be a fixed subset of X A mapping $\Pi : \mathcal{R} \to T$ is a $\langle \mathcal{P}(X), \subset \rangle$-*valued-$X_0$-quasi-monotone measure* on $\mathcal{R}$, if for each $A, B \in \mathcal{R}$ such that $A \subset B$ the inclusion $\Pi(A) - \Pi(B) \subset X_0$ is valid. If $X_0 = \emptyset$, we are back at the $\langle \mathcal{P}(X), \subset \rangle$-monotone measure on $\mathcal{R}$.

When trying to generalize and re-phrase this idea to the case of lattice-valued monotone measures, let us limit ourselves, for the sake of simplicity, to the case when the set-theoretic equivalence $Y_1 \subset Y_2$ iff $Y_1 - Y_2 (= Y_1 \cap (X - Y_2)) = \emptyset$, trivially valid for each subsets Y_1, Y_2 of a nonempty set X, can be replaced by the equivalence

$$t_1 \leq t_2 \text{ iff } t_1 \wedge t_2^C \left(= t_1 \wedge \bigvee \{ s \in T : s \wedge t_2 = \oslash_{\mathcal{T}} \} \right) = \oslash_{\mathcal{T}}, \tag{81}$$

valid for every t_1, t_2 from the lattice $\mathcal{T} = \langle T, \leq \rangle$ under consideration. So, we arrive at the following definition.

Definition 7.2. Let $\mathcal{T} = \langle T, \leq \rangle$ be a Brouwerian Boolean-like lattice, let $\mathcal{A}$ be a nonempty ample field of subsets of a nonempty set Ω, let $t_0 \in T$ be fixed. A mapping $\Pi : \mathcal{R} \to T$ is called a $\mathcal{T}$-(valued)-t_0-*quasi-monotone measure* on $\mathcal{A}$, if for each $A, B \in \mathcal{A}$ such that $A \subset B$ holds, the inequality

$$\Pi(A) \wedge (\Pi(B))^C \leq t_0 \tag{82}$$

is valid.

As can be easily seen, under the conditions imposed on $\mathcal{T}$, if $t_0 = \oslash_{\mathcal{T}}$, then $\Pi(A) \wedge (\Pi(B))^C = \oslash_{\mathcal{T}}$ implies that $\Pi(A) \leq \Pi(B)$ holds. Hence, as in the case with $\mathcal{T} = \langle \mathcal{P}(X), \subset \rangle$ and $X_0 = \emptyset$, we obtain the original definition of $\mathcal{T}$-monotone measures on $\mathcal{A}$. The notations and conditions of Definition 7.2 being valid, we obtain easily that for each $A, B \in \mathcal{A}$ such that $A \subset B$, the inequality

$$\Pi(A) \leq \Pi(B) \vee t_0 \tag{83}$$

is valid. Indeed,

$$\begin{aligned} \Pi(A) = \Pi(A) \wedge \mathbf{1}_{\mathcal{T}} &= \Pi(A) \wedge (\Pi(B) \vee (\Pi(B))^C) = \Pi(A) \wedge \Pi(B)) \\ &\vee (\Pi(A) \wedge (\Pi(B))^C) \leq \Pi(B) \vee t_0. \end{aligned} \tag{84}$$

$\mathcal{T}$-monotone measures can be induced by no matter which $\mathcal{T}$-valued set functions using the standard idea of inner and outer measures.

Definition 7.3. Let $\Omega \neq \emptyset$, let $\emptyset \neq \mathcal{R} \subset \mathcal{P}(\Omega)$, let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\Pi : \mathcal{R} \to T$ be any mapping. The *inner (or lower) measure* Π_*, and the *outer (or upper) measure* Π^* induced by Π are mappings both taking $\mathcal{P}(\Omega)$ into T and defined, for each $A \subset \Omega$, by

$$\Pi_*(A) = \bigvee \{\Pi(C) : C \in \mathcal{R}, C \subset A\}, \tag{85}$$

$$\Pi^*(A) = \bigwedge \{\Pi(C) : C \in \mathcal{R}, C \supset A\}, \tag{86}$$

applying the conventions $\bigvee \emptyset = \oslash_{\mathcal{T}}$ and $\bigwedge \emptyset = \mathbf{1}_{\mathcal{T}}$ for the empty subset of T, if necessary.

Definition 7.4. Let $\mathcal{R}$ and $\mathcal{T}$ be as in Definition 7.3. A mapping $\Pi : \mathcal{R} \to T$ is called *normalized,* if $\Pi(\emptyset) = \oslash_{\mathcal{T}}$ and/or $\Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ supposing that $\emptyset \in \mathcal{R}$ and/or $\Omega \in \mathcal{R}$.

Lemma 7.1. Let $\mathcal{R}$ and $\mathcal{T}$ be as in Definition 7.3, let $\Pi : \mathcal{R} \to T$ be a normalized mapping. Then both Π_* and Π^* are normalized monotone measures on $\mathcal{P}(\Omega)$. If, moreover, Π is a monotone measure on $\mathcal{R}$, then both Π_* and Π^* conservatively extend Π from $\mathcal{R}$ to $\mathcal{P}(\Omega)$ (i.e., $\Pi_*(A) = \Pi^*(A) = \Pi(A)$ for every $A \in \mathcal{R}$), and the inequality $\Pi_*(A) \leq \Pi^*(A)$ holds for every $A \subset \Omega$.

Proof. If $\emptyset \in \mathcal{R}$ and/or $\Omega \in \mathcal{R}$, the identities $\Pi_*(\emptyset) = \Pi^*(\emptyset) = \Pi(\emptyset) = \oslash_{\mathcal{T}}$ and $\Pi_*(\Omega) = \Pi^*(\Omega) = \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ are evident. For every $A \subset B \subset \Omega$, if $C \in \mathcal{R}$ is such that $C \subset A$ holds, then $C \subset B$ holds as well, hence, the set inclusion

$$\{\Pi(C) : C \in \mathcal{R}, C \subset A\} \subset \{\Pi(C) : C \in \mathcal{R}, C \subset B\} \tag{87}$$

follows and the inequality $\Pi_*(A) \leq \Pi_*(B)$ results. The proof of the same inequality for Π^* is completely dual. Hence, both Π_* and Π^* are $\mathcal{T}$-monotone measures on $\mathcal{P}(\Omega)$.

If Π is a $\mathcal{T}$-monotone measure on $\mathcal{R}$, then for each $C, D \in \mathcal{R}$ such that $C \subset A \subset D$ holds the inequality $\Pi(C) \leq \Pi(D)$ follows, hence,

$$\Pi_*(A) = \bigvee \{\Pi(C) : C \in \mathcal{R}, C \subset A\} \leq \bigwedge \{\Pi(D) : D \in \mathcal{R}, D \supset A\} = \Pi^*(A) \tag{88}$$

is valid for every $A \subset \Omega$. If $A \in \mathcal{R}$, the identities $\Pi_*(A) = \Pi(A) = \Pi^*(A)$ immediately follow from (3.8). The assertion is proved. □

As can be easily seen, if Π is not a $\mathcal{T}$-monotone measure on $\mathcal{A}$, neither Π_* nor Π^* are identical with Π on $\mathcal{A}$. Indeed, let $A, B \in \mathcal{A}$ be such that $A \subset B$ and $\Pi(A) > \Pi(B)$ hold, then the inequalities

$$\Pi_*(B) = \bigvee\{\Pi(C) : C \in \mathcal{A}, C \subset B\} \geq \Pi(A) > \Pi(B) \tag{89}$$

and

$$\Pi^*(A) = \bigwedge\{\Pi(D) : D \in \mathcal{A}, D \supset A\} \leq \Pi(B) < \Pi(A) \tag{90}$$

easily follow.

Let us note that under the notations and conditions of Lemma 7.1 and if $\mathcal{T}$ is Brouwerian the implication inverse to (7.3) is valid, i.e., if $\Pi : \mathcal{A} \to T$ is such that $\Pi(\emptyset) = \oslash_{\mathcal{T}}, \Pi(\Omega) = \mathbf{1}_{\mathcal{T}}$ and the inequality $\Pi(A) \leq \Pi(B) \vee t_0$ is valid for a fixed $t_0 \in T$ and for each $A, B \in \mathcal{A}$ such that $A \subset B$ holds, then Π is a $\mathcal{T}$-t_0-quasi-monotone measure on $\mathcal{A}$. Indeed, $\Pi(A) \leq \Pi(B) \vee t_0$ implies that the inequality

$$\begin{aligned} \Pi(A) \wedge (\Pi(B))^C &\leq (\Pi(B) \vee t_0) \wedge (\Pi(B))^C \\ = (\Pi(B) \wedge (\Pi(B))^C) &\vee (t_0 \wedge (\Pi(B))^C) \leq \oslash_{\mathcal{T}} \vee t_0 = t_0 \end{aligned} \tag{91}$$

is valid.

Under the same notations and conditions we can prove that if $\Pi : \mathcal{A} \to T$ is a $\mathcal{T}$-t_0-quasi-monotone measure on $\mathcal{A}$, then the mapping $\Pi_0 : \mathcal{A} \to T$, defined by $\Pi_0(\emptyset) = \oslash_{\mathcal{T}}, \Pi_0(A) = \Pi(A) \vee t_0$ for each $\emptyset \neq A \in \mathcal{A}$, is a $\mathcal{T}$-monotone measure on $\mathcal{A}$. The constraints for $\Pi_0(\emptyset)$ and $\Pi_0(\Omega)$ are obvious and for every $\emptyset \neq A \subset B, A, B \in \mathcal{A}$, we obtain that the inequality

$$\Pi_0(A) = \Pi(A) \vee t_0 \leq (\Pi(B) \vee t_0) \vee t_0 = \Pi(B) \vee t_0 = \Pi_0(B) \tag{92}$$

holds, applying (7.3).

The class of quasi-monotone measures is closed with respect to the operation of supremum in the following sense.

Theorem 7.1. Let $\mathcal{T} = \langle T, \leq \rangle$ be a complete lattice, let $\mathcal{A}$ be a nonempty ample field of subsets of a nonempty set Ω, let S be a nonempty subset of T. let $\mathcal{P}$ be a nonempty set of mappings such that each $\Pi \in \mathcal{P}$ takes $\mathcal{A}$ into T and, for some $s \in S$, Π is a $\mathcal{T}$-s-quasi-monotone measure on $\mathcal{A}$. Let $\bigvee \mathcal{P} : \mathcal{A} \to T$ and $\bigwedge \mathcal{P} : \mathcal{A} \to T$ be the mappings defined, for any $A \in \mathcal{A}$, by

$$\left(\bigvee \mathcal{P}\right)(A) = \bigvee\{\Pi(A) : \Pi \in \mathcal{P}\}, \tag{93}$$

$$\left(\bigwedge \mathcal{P}\right)(A) = \bigwedge\{\Pi(A) : \Pi \in \mathcal{P}\}. \tag{94}$$

Then $\bigvee \mathcal{P}$ is a $\mathcal{T}$-$\bigvee S$-quasi-monotone measure on $\mathcal{A}$, let us recall that $\bigvee S = \bigvee_{s \in S} s$. If $\mathcal{T}$ is, moreover, completely distributive, then $\bigwedge \mathcal{P}$ is also a $\mathcal{T}$-$\bigvee S$-quasi-monotone measure on $\mathcal{A}$.

In particular, if $\mathcal{P}$ is a nonempty set of $\mathcal{T}$-t_0-quasi-monotone measures on $\mathcal{A}$ and $\mathcal{T}$ is a complete lattice, then $\bigvee \mathcal{P}$ is also a $\mathcal{T}$-t_0-quasi-monotone measure on $\mathcal{A}$. If $\mathcal{T}$ is, moreover, completely distributive, the same holds true for $\bigwedge \mathcal{P}$.

Proof. Due to the conditions imposed on $\mathcal{T}$ and on the mappings from $\mathcal{P}$, for each $\Pi \in \mathcal{P}$ the inequality (7.3) holds, hence, there exists $s(= s_\Pi) \in S$ such that, for each $A \subset B, A, B \in \mathcal{A}$, the inequality

$$\Pi(A) \leq \Pi(B) \vee s_\Pi \leq \Pi(B) \vee \left(\bigvee S\right) \tag{95}$$

is valid. Hence, the relation

$$\begin{aligned} \left(\bigvee \mathcal{P}\right)(A) &= \bigvee\{\Pi(A) : \Pi \in \mathcal{P}\} \leq \bigvee\{\Pi(B) \vee \left(\bigvee S\right) : \Pi \in \mathcal{P}\} \\ &= \left(\bigvee\{\Pi(B) : \Pi \in \mathcal{P}\}\right) \vee \left(\bigvee S\right) \\ &= \left(\bigvee \mathcal{P})(B) \vee (\bigvee S\right) \end{aligned} \tag{96}$$

easily follows. If $\mathcal{T}$ is completely distributive, a similar relation holds also for $\bigwedge \mathcal{P}$, i.e.,

$$\begin{aligned} \left(\bigwedge \mathcal{P}\right)(A) &= \bigwedge\{\Pi(A) : \Pi \in \mathcal{P}\} \leq \bigwedge\{\Pi(B) \vee \left(\bigvee S\right) : \Pi \in \mathcal{P}\} \\ &= \left(\bigwedge\{\Pi(B) : \Pi \in \mathcal{P}\}\right) \vee \left(\bigvee S\right) = \left(\bigwedge \mathcal{P}\right)(B) \vee \left(\bigvee S\right). \end{aligned} \tag{97}$$

The relations $(\bigvee \mathcal{P})(\emptyset) = \oslash_{\mathcal{T}} = (\bigwedge \mathcal{P})(\emptyset)$ and $(\bigvee \mathcal{P})(\Omega) = \mathbf{1}_{\mathcal{T}} = (\bigwedge \mathcal{P})(\Omega)$ are obviously valid. Hence, under the conditions imposed on $\mathcal{T}$ in both the particular cases, $\bigvee \mathcal{P}$ and $\bigwedge \mathcal{P}$ are $\mathcal{T}$-$\bigvee S$-quasi-monotone measures on $\mathcal{A}$. □

As a matter of fact, in the case of real-valued monotone measures which take their values in the unit interval of real numbers equipped by the standard linear ordering, i.e., in the case when $\mathcal{T} = \langle [0,1], \leq \rangle$, the idea of quasi-monotone measure leads to trivial results beyond any interest, as the following assertion demonstrates.

Lemma 7.2. Let $\mathcal{T} = \langle [0,1], \leq \rangle$, let $\mathcal{R}$ be a nonempty system of subsets of a nonempty set Ω, let $\Pi : \mathcal{R} \to [0,1]$ be any mapping such that $\Pi(\emptyset) = 0$ and/or $\Pi(\Omega) = 1$ supposing that $\emptyset \in \mathcal{R}$ and/or $\Omega \in \mathcal{R}$, and such that Π is monotone on zero sets, i.e., if $A, B \in \mathcal{R}$, $A \subset B$ and $\Pi(B) = 0$, then $\Pi(A) = 0$. Then Π is a $\mathcal{T}$-ε-quasi-monotone measure on $\mathcal{R}$ for every $\varepsilon > 0$.

Remark 2. E.g., any mapping $\Pi : \mathcal{R} \to [0,1]$ such that $\Pi(A) = 0$ iff $A = \emptyset$ and $\emptyset \in \mathcal{R}$ and $\Pi(\Omega) = 1$ supposing that $\Omega \in \mathcal{R}$ meets the conditions of Lemma 7.2.

Proof. The definition of pseudo-complement in $\mathcal{T} = \langle [0,1], \leq \rangle$ yields that, for each $x \in [0,1], x^C = \bigvee\{y \in [0,1] : x \wedge y = 0\}$, hence, $x^C = 0$ for any $x > 0$ and $0^C = 1$. So, for any $A, B \in \mathcal{R}$ such that $A \subset B$ holds we obtain that $\Pi(A) \wedge (\Pi(B))^C = 0$.

Indeed, if $\Pi(B) > 0$, then $(\Pi(B))^C = 0$, if $\Pi(B) = 0$, then $\Pi(A) = 0$ due to the conditions imposed on Π. Hence, for any $A, B \in \mathcal{R}, A \subset B$, and for any $\varepsilon > 0$ the inequality $\Pi(A) \wedge (\Pi(B))^C < \varepsilon$ holds, so that Π is a $\langle [0,1], \leq \rangle$-$\varepsilon$-monotone measure on $\mathcal{R}$. □

On the other side, for non-numerical, in particular, lattice, Boolean or set-valued monotone measures the operation of pseudo-complement is far not so trivial as in the case with $\mathcal{R} = \langle [0,1], \leq \rangle$ in Lemma 7.2 above. An intuition behind lattice-valued metrics and monotone measures will be developed and analyzed in the next chapter.

8. AN INTUITION BEHIND LATTICE–VALUED METRICS AND RELATED QUASI–MEASURES

Let us introduce and analyze the following model of multicriterial evaluation and decision making. Consider a problem of no matter which nature; a more detailed specification will not play any role below. Let Ω be a nonempty set of possible or potential solutions to this problem (or candidates which can be taken into consideration when looking for the best, in a sense, solution). Each solution $\omega \in \Omega$ can be evaluated with respect to a number of various criteria, in other words, various aspects of each solution are evaluated. Let us consider just the most simple case when all these criteria are qualitative and binary-valued. Hence, given $\omega \in \Omega$ and a criterion x, the solution ω may be either good (adequate, appropriate, acceptable,...) with respect to the criterion $x(x(\omega) = 1$, is symbols), or ω may be wrong (bad, inadequate, inacceptable, inappropriate,...) w.r.to $x(x(\omega) = 0$, in symbols). Denoting by X the set of all criteria under consideration, we define the mapping π (or π_X) $: \Omega \to \mathcal{P}(X)$ which ascribes to each $\omega \in \Omega$ the subset $\pi(\omega) = \{x \in X : x(\omega) = 1\}$ of those criteria with respect to which ω is good (adequate,...). This subset of X can be taken as the value ascribed to the solution $\omega \in \Omega$ by the multidimensional set-valued evaluation (function) π.

The mapping $\pi : \Omega \to \mathcal{P}(X)$ defines the partial ordering $\leq$ on Ω, setting simply $\omega_1 \leq \omega_2$ iff $\pi(\omega_1) \subset \pi(\omega_2)$ holds. It is quite intuitive to say that ω_2 is at least as good solution (better, resp.) solution to the problem under consideration as ω_1 (than ω_1, resp.), if the relation $\omega_1 \leq \omega_2(\omega_1 < \omega_2$, i.e., $\omega_1 \leq \omega_2$ but not $\omega_2 \leq \omega_1$, resp.) is valid, as this relation means that ω_2 is good (adequate,...) w.r.to the same criteria as ω_1 (and perhaps w.r.to some more criteria). Still another construction, this time leading to a set-valued monotone measure, is possible. Define the equivalence relation $\approx$ on Ω in this way: $\omega_1 \approx \omega_2$ iff $\omega_1 \leq \omega_2$ and $\omega_2 \leq \omega_1$ hold together, i.e., iff $\pi(\omega_1) = \pi(\omega_2)$. Consider the factor-space $\Omega_0 = \Omega/\approx$ and identify each equivalence class $[\omega] \in \Omega_0$ with the value $\pi(\omega) \subset X$, this mapping is obviously one-to-one. The identity mapping $\Pi_{id} : \Omega_0 \to \mathcal{P}(X)$ is obviously a $\langle \mathcal{P}(X), \subset \rangle$-valued monotone measure on $\mathcal{A} = \Omega_0 \subset \mathcal{P}(\Omega)$. Instead of the p.o. set $\langle \mathcal{P}(X), \subset \rangle$ we can construct the p.o. set $\langle \Omega_0, \leq \rangle$ such that $[\omega_1] \leq [\omega_2]$ holds iff $[\omega_1] \subset [\omega_2]$ holds, i.e., iff $\omega_1 \leq \omega_2$ holds in $\langle \Omega, \leq \rangle$. The identity mapping Π_{id} is then an $\langle \Omega_0, \leq \rangle$-monotone measure on Ω_0.

The difficulties arise when searching, given a set $A \subset \Omega_0$ of solutions, for the best solution with respect to the monotone set function Π_{id} on $\mathcal{A} = \Omega_0$. If the supremum of A w.r.to $\leq$ is in A, i.e., if there is $\omega_A \in A$ such that $\omega_A \geq \omega$ holds for each $\omega \in A$, the decision is obvious; this ω_A is the best solution to the problem under consideration within the framework of A and this decision completely meets the intuition behind the partial ordering $\leq$ on Ω_0. If this is not the case, one of the following partial remedies can be taken into consideration.

(i) The solution $\omega_A = \bigcup A (= \bigcup_{\omega \in A} \omega)$ is joined with A, so that the best solution in $A \cup \{\omega_A\}$ will be just ω_A. However, at the level of applications this assumption implies that we are able to combine sophistically the solutions from A so that the positive features of each solution are conserved, i.e., for every criterion $x \in X$ and every $\omega \in A$, if ω is good w.r.to x, also the combined solution is good w.r.to x. It is a matter of the real situation, problems and solutions under consideration whether such an assumption is realistic or not.

(ii) We can pick up and restrict our considerations to the non-dominated solutions in A; a solution $\omega \in A$ is dominated, if there exists $\omega_1 \in A$ such that $\omega < \omega_1$ holds. However, if (i) is not the case, the subset $A_0 \subset A$ of non-dominated solutions contains at least two elements, so that some further and ontologically independent principle must be introduced, if we have to choose just one solution. Hence, every such decision rule involves some more conditions under which this rule is applicable and reasonable when applied within our model. Nevertheless, to limit ourselves to non-dominated solutions from $A_0 \subset A$ can be taken as a reasonable intermediate step before applying further measures.

(iii) Another remedy is to introduce a linear ordering $\leq_*$ on A conservatively extending the partial ordering $\leq$ on A defined by the set inclusion on $\mathcal{P}(X)$. Hence, for every $\omega_1, \omega_2 \in A$ either $\omega_1 \leq_* \omega_2$ or $\omega_2 \leq_* \omega_1$ holds, and if $\omega_1 \leq \omega_2$ holds, $\omega_1 \leq_* \omega_2$ holds as well. (this can be done for each partial ordering; in finite cases a constructive proof exists, in infinite cases Zorn lemma applies). In particular, any real-valued monotone measure λ on $\Omega_0 \subset \mathcal{P}(\Omega)$ defines such a linear ordering, setting $\omega_1 \leq_* \omega_2$ iff $\lambda(\omega_1) \leq \lambda(\omega_2)$, holds. Again, the choice of such a monotone measure λ is a new and ontologically independent step in our reasoning, i.e., up to the most trivial cases no such choice can be uniquely deduced from and completely justified by the model described above.

(iv) The same idea as in (i) can be applied, but only with respect to the criteria from a proper subset $Y_0 \subset X$, in other words, the criteria from the set $X_0 = X - Y_0$ are neglected when defining a partial ordering $\leq_{x_0}$ in Ω_0. In symbols, $\omega_1 \leq_{x_0} \omega_2$ holds for $\omega_1, \omega_2 \in \Omega_0$, iff the set inclusion $\omega_1 \cap (X - X_0) \subset \omega_2 \cap (X - X_0)$ is valid. The particular choice of the subset X_0 of neglected criteria is, again, an ontologically independent input in our decision procedure and can be motivated by various reasons. E.g., if the set X is infinite, and the value $x(\omega)$, given $\omega \in \Omega_0$, can be evaluated only individually, taking one $x \in X$ after another, then each demand of computational effectiveness requests to limit ourselves to a finite number of particular criteria, i.e., to a finite, hence, proper subset Y_0 of X. Other motivation for omitting some criteria may read that, for some reasons, they are not taken as much important, in the given context, or that the cost to be paid or the obstacles to be overcome when evaluating the solutions with respect to these criteria, are too high when relating them to their importance when searching for the optimal solution. The elimination of some criteria from consideration can proceed also gradually and sequentially, step by step, hoping that an acceptable compromise between the gradually "less and less partial" ordering $\leq_{x_0}$ on Ω_0 (with X_0 increasing), and the demands imposed on the chosen solution taken for the best one, will be reached sooner or later.

Throughout this paper, the standard demands imposed on lattice-valued possibilistic measures have been weakened uniformly in the sense that the threshold values $t \in T$, up to which the value $\Pi(A \cup B)$ and $\Pi(A) \vee \Pi(B)$ may differ, was taken the same for each A, B, $A \cup B \in \mathcal{R}$ (the definition domain of Π). A further generalization could read as follows: the still accepted difference between $\Pi(A \cup B)$ and $\Pi(A) \vee \Pi(B)$ could differ for different A, B, $A \cup B \in \mathcal{R}$. E.g., for "important" or often occurring cases only a very small or even none (i.e. $t = \oslash_{\mathcal{T}}$) difference is acceptable, on the other side, for some rarely occurring or "not too important" sets A, B, $A \cup B$ also a large value of t, or even $t = \mathbf{1}_{\mathcal{T}}$, are acceptable. Supposing we were able to define, in an appropriate way, the expected value of the difference between $\Pi(A \cup B)$ and $\Pi(A) \vee \Pi(B)$, and demanding only this expected value to be beyond a threshold value, we would arrive at a lattice-valued analogy of the Bayesian risk and Bayesian decision function well-known in statistical decision theory. However, let us postpone a more detailed analysis and formalization of this approach to another occasion.

ACKNOWLEDGEMENT

This work has been sponsored by the Cost Action 274 (TARSKI).

(Received February 27, 2004.)

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