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THE TECHNIQUE OF SPLITTING OPERATORS IN PERTURBATION CONTROL THEORY

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Dedicated to the memory of our colleague and friend S. P. Patarinski.

The paper presents the technique of splitting operators, intended for perturbation analysis of control problems involving unitary matrices. Combined with the technique of Lyapunov majorants and the application of the Banach or Schauder fixed point principles, it allows to obtain rigorous non-local perturbation bounds for a set of sensitivity analysis problems. Among them are the reduction of linear systems into orthogonal canonical forms, the general feedback synthesis problem, and the pole assignment problem in particular, as well as other basic problems in control theory and linear algebra.

Keywords: perturbation analysis, canonical forms, feedback synthesis

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1. INTRODUCTION

The aim of perturbation analysis of a given problem is to provide bounds for the perturbations in the solution as functions of the perturbations in the data. There are at least three sound reasons to study the sensitivity of various problems relative to perturbations in the data.

First, this may give an independent and deep insight at the very nature of the problem, being therefore of independent theoretical interest.

Second, perturbation bounds provide a more realistic modelling framework for most problems. Indeed, there are inevitable measurement and other parametric and structural uncertainties, which means that we have to deal with a family of models rather than with a single model. In this case the perturbation bounds give us a tube in the space of models, to which the particular model actually belongs.

And third, when a numerically stable algorithm is applied to solve the problem then the solution, computed in finite arithmetics, will be close to the solution of a close problem. Having tight perturbation bounds and a knowledge of the equivalent perturbations for the computed solution, we may produce condition and accuracy estimates. Without such estimates a computational algorithm cannot be recognized as reliable from the viewpoint of modern computing standards.

In this paper we first present the technique, proposed in [13], which splits the equivalent operator of a perturbation problem involving unitary matrices, thus allowing an efficient application of Lyapunov majorants [1] and various fixed point principles [4, 12]. Then, we use this technique to obtain non-local perturbation bounds for the problem of computing orthogonal canonical forms of linear control systems and for the general feedback synthesis problem and the pole assignment problem in particular.

The following notations are used: $\mathcal{F}$ is the field of real $\mathcal{R}$ or complex $\mathcal{C}$ numbers; $\mathcal{F}^{m\times n}$ is the space of $m \times n$ matrices over $\mathcal{F}$; Z^T, Z^H and $Z^\dagger$ are the transpose, the complex conjugate transpose, and Moore–Penrose pseudoinverse of $Z = [z_{ij}]$; $\mathrm{spect}(Z)$ is the spectrum of $Z \in \mathcal{F}^{n\times n}$, i.e. the set of eigenvalues of Z counted with their algebraic multiplicity; I_n is the unit $n \times n$ matrix; $\mathcal{GL}_n$ is the group of nonsingular $n \times n$ matrices over $\mathcal{F}$ and $\mathcal{O}_n$, $\mathcal{U}_n \subset \mathcal{GL}_n$ are the groups of orthogonal and unitary matrices; $\|\cdot\|_2$ and $\|\cdot\|_F$ are the spectral and Frobenius norms in $\mathcal{F}^{m\times n}$, while $\|\cdot\|$ is the induced operator norm or an unspecified matrix norm. The Kronecker product of the matrices A, B is denoted by $A \otimes B$ and the symbol $:=$ stands for "equal by definition".

2. STATEMENT OF THE PROBLEM

Suppose that we have a matrix problem with data $D = (A, B, \dots, \Lambda)$, where $A, B, \dots$, are real or complex matrices and Λ is a collection (a set with repeated elements) of complex numbers. We consider problems, in which the resulting matrix (or the solution) $S = \mathsf{F}(D, U)$ is upper triangular, and is obtained from the data by multiplicative transformations with a unitary matrix U. We recall that the implementation of numerically stable unitary (or real orthogonal) transformations is highly desirable in order to improve the performance and reliability of the corresponding matrix numerical algorithm.

Let D be subject to a perturbation $D \mapsto D + \Delta D$. We consider D as an element of a normed linear space $\mathcal{D}$ with summation $(D, E) \mapsto D + E$, multiplication by scalars $(\alpha, D) \mapsto \alpha D$ and norm $D \mapsto \|D\|$, defined in some of the standard ways. Thus F is a map from a subset of $\mathcal{D}\times\mathcal{U}$ to the space $\mathcal{S}$ of upper triangular matrices, where $\mathcal{U}$ is the group of unitary matrices of corresponding size.

Suppose that the perturbed problem with data $D + \Delta D$ has a solution $S + \Delta S = \mathsf{F}(D + \Delta D, U + \Delta U)$, where $U + \Delta U$ is the perturbed unitary transformation matrix. Then the *perturbation problem* is to estimate the norm of the perturbation

$$\Delta S = \mathsf{F}(D + \Delta D, U + \Delta U) - \mathsf{F}(D, U)$$

in the solution S and ΔU in the transformation matrix U as functions of the perturbation ΔD in the data, e.g.

$$\|\Delta S\| \le f(\|\Delta D\|), \quad \|\Delta U\| \le g(\|\Delta D\|)$$

where f and g are non-decreasing functions with $f(0) = g(0) = 0$. When a more detailed information about the perturbations in the data is available, the perturbation bounds are in the form

$$\|\Delta S\| \le f(\|\Delta A\|, \|\Delta B\|, \ldots), \quad \|\Delta U\| \le g(\|\Delta A\|, \|\Delta B\|, \ldots). \tag{1}$$

Two types of perturbation bounds are usually used. First, these are the asymptotic bounds, which are linear or homogeneous first order expressions in the perturbation vector $\Delta = [\,\|\Delta A\|, \|\Delta B\|, \ldots]^T$. The linear perturbation bounds have the form

$$\|\Delta S\| \le K_{S,A}\|\Delta A\| + K_{S,B}\|\Delta B\| + \ldots + \mathrm{O}(\|\Delta\|^2) \tag{2}$$

$$\|\Delta U\| \le K_{U,A}\|\Delta A\| + K_{U,B}\|\Delta B\| + \ldots + \mathrm{O}(\|\Delta\|^2),\ \Delta \to 0 \tag{3}$$

where $K_{M,N}$ are the absolute condition numbers of the problem. There also exist improved first order perturbation bounds, which are not based on condition numbers and are generally better than (2), (3), see [9].

The asymptotic perturbation analysis usually does not give estimates for the $\mathrm{O}(\|\Delta\|^2)$ terms (this is actually the goal of non-local perturbation analysis) and in practice the asymptotic bounds are used simply neglecting the second and higher order terms. The resulting chopped bounds often produce acceptable results. However, they are not rigorous and may severely be violated in some cases. Without warning for the user, this may be a serious misleading.

These disadvantages of asymptotic perturbation bounds may be overcome using the techniques of non-local perturbation analysis. As a result we get non-local (and usually non-linear) rigorous perturbation bounds of type (1). They are valid for perturbations ΔD from certain domain $\mathcal{D}$, which may be small but is nevertheless finite. Moreover, the inclusion $\Delta D \in \mathcal{D}$ guarantees that the solution $\mathsf{F}(D + \Delta D, U + \Delta U)$ of the perturbed problem exists. This is an important issue from both theoretical and practical point of view. Note that chopped asymptotic bounds do not guarantee such existence and they "work" when the perturbed solution is either too large or is even non-existent.

To derive non-local perturbation bounds it is necessary to transform the initial perturbation problem into an equivalent operator equation. Then an application of a fixed point principle would produce the desired bound. The first phase – constituting the equivalent operator equation for problems involving unitary transformations, is done by the technique of splitting operators [7, 10, 13]. The second phase – application of fixed point principles [4, 12], is done by the technique of Lyapunov majorants [1, 10].

Writing $U + \Delta U$ as $U(I + X)$, where $X = U^H \Delta U$, we see that the matrix $I + X$ is unitary, i.e. $X^H + X + X^H X = 0$. An additional equation for X is obtained by the technique of splitting operators, see e.g. Sections 4 and 5 for two such examples. As a result we get an operator equation

$$X = \Pi(X, \Delta D), \quad \Pi(0, 0) = 0. \tag{4}$$

Let $|X| = [\xi_1, \ldots, \xi_k]^T \in \mathcal{R}_+^k$, $\mathcal{R}_+ = [0, \infty)$, be a generalized norm of X, i.e. $|X| \succeq 0$, $|\alpha X| = |\alpha||X|$ and $|X + Y| \preceq |X| + |Y|$. Here $\preceq$ is the partial order relation in $\mathcal{R}^k$, such that $X \preceq Y$ means $Y - X \in \mathcal{R}^k_+$. Let for instance $P_1, \dots, P_k$ be projectors in $\mathcal{C}^{n\times n}$ such that $P_1 + \dots + P_k = I$. Then we may choose $\xi_i = \|P_i X\|_F$.

Suppose that we can find a Lyapunov majorant function [1, 10] for equation (4). This is a differentiable function $(\xi, \Delta) \mapsto h(\xi, \Delta)$, $h = [h_1, \dots, h_k]^T$, $\xi = [\xi_1, \dots, \xi_k]^T$ such that

$$|\Pi(X, \Delta D)| \preceq h(|X|, \Delta).$$

In addition $h(0,0) = 0$ and the components h_i of h are non-decreasing functions in all their arguments. Under these conditions there exists $\Delta^0 \succ 0$, such that for $\Delta \preceq \Delta^0$ the vector equation $\xi = h(\xi, \Delta)$ has a solution $\xi = \varphi(\Delta)$, tending to zero together with Δ. Hence for $\Delta \preceq \Delta^0$ the operator $\Pi(\cdot, \Delta)$ maps the closed convex set

$$\mathcal{B}_\Delta = \{X : |X| \preceq \varphi(\Delta)\} \subset \mathcal{C}^{n\times n}$$

into itself. Then, according to the Schauder fixed point principle, there exists a solution $X \in \mathcal{B}_\Delta$ to the operator equation (4), such that

$$|X| \leq \varphi(\Delta), \quad 0 \preceq \Delta \preceq \Delta^0. \tag{5}$$

This is the desired non-local perturbation bound for the generalized norm of X. Now the estimates for $\|\Delta U\| = \|X\|$ and $\|\Delta S\|$ are straightforward.

In problems with unique solution it is possible to show that for $\Delta \preceq \Delta^0$ the operator $\Pi(\cdot, \Delta)$ is a contraction on the set $\mathcal{B}_\Delta$ Then applying the Banach fixed point principle we see that the solution X of equation (4), for which the estimate (5) holds, is unique.

3. SPLITTING OF THE EQUIVALENT OPERATOR

Denote by Low, Diag and Up the projectors from $\mathcal{C}^{m\times n}$ on the subspaces of strictly lower, diagonal and strictly upper $n \times m$ matrices, respectively.

In the considered perturbation problems, the perturbed resulting $n \times m$ matrix

$$\begin{aligned} \mathsf{F}(D + \Delta D, U + \Delta U) &= \mathsf{F}(D, U) + \mathsf{F}_D(D, U)[\Delta D] + \mathsf{F}_U(D, U)[\Delta U] + \mathrm{O}(\rho^2) \\ &= \mathsf{F}(D, U) + \mathsf{G}(D, U, \Delta D, \Delta U), \ \rho = \sqrt{\|\Delta D\|^2 + \|\Delta U\|^2} \end{aligned}$$

where $\mathsf{F}_Z(D, U)[\cdot]$ is the Fréchet derivative of F in the argument Z at the point (D, U), is again upper triangular, and we may write

$$\mathrm{Low}(\mathsf{F}(D + \Delta D, U + \Delta U)) = \mathrm{Low}(\mathsf{G}(D, U, \Delta D, \Delta U)) = 0. \tag{6}$$

The perturbed $n \times n$ transformation unitary matrix $U + \Delta U$ may be written as $U(I + X)$, where the matrix $I + X$ is unitary and the norm of X is small (of the order of the perturbations in the data). We may represent the matrix X in a splitted form as

$$X = X_1 + X_2 + X_3 = \mathrm{Low}(X) + \mathrm{Diag}(X) + \mathrm{Up}(X).$$

The crucial fact in application of the splitting operator technique is that the main (linear) part in (6), usually in the form $\mathrm{Low}(SX - XS)$, depends only on the strictly lower part X_1 of X rather on the whole matrix X. Then equation (6) yields

$$\mathsf{L}(X_1) = \Theta(X, \Delta D) \tag{7}$$

where L is a linear operator and $\|\Theta(X, \Delta D)\| = \mathrm{O}(\rho^2)$, $\rho \to 0$.

Under certain natural assumptions, the restriction L_1 of the operator L on the $n(n-1)/2$-dimensional subspace of strictly lower triangular matrices is invertible and hence we may write

$$X_1 = \Pi_1(X, \Delta D) := \mathsf{L}_1^{-1}(\Theta(X, \Delta D)). \tag{8}$$

We need two more equations for the diagonal X_2 and strictly upper X_3 parts of X. Due to the unitarity of $I + X$ we have

$$X^H + X + X^H X = X + X^H + XX^H = 0. \tag{9}$$

Applying the Diag and Up operators to (9) we get

$$X_2 = \Pi_2(X) = -\mathrm{Diag}(X^H X)/2 \tag{10}$$

and

$$X_3 = \Pi_3(X) = -\mathrm{Up}(X^H) - \mathrm{Up}(X^H X). \tag{11}$$

Equations (8), (10) and (11) constitute an operator equation

$$X = \Pi(X, \Delta D), \quad \Pi = (\Pi_1, \Pi_2, \Pi_3) \tag{12}$$

for X. In view of (10) and (11) we have

$$\|\Pi_2(X)\|_F \leq 0.5\|X\|_F^2$$
$$\|\Pi_3(X)\|_F \leq \|X_1\|_F + \sqrt{(n-1)/(2n)}\|X\|_F^2.$$

These inequalities together with (8) show that, for $\|\Delta D\|$ sufficiently small, the operator Π transforms a set of diameter $\mathrm{O}(\|\Delta D\|)$ into itself and we may apply the method of Lyapunov majorants, see [1, 10] and Section 2. For this purpose we use the generalized norm

$$|X| = [\xi_1, \xi_2, \xi_3]^T = [\|X_1\|_F, \|X_2\|_F, \|X_3\|_F]^T \in \mathcal{R}_+^3$$

in $\mathcal{C}^{n \times n}$. In certain cases (e. g. problems with unique solution) we can even show that the operator Π is a contraction, thus claiming the existence of an unique solution X of the operator equation (12).

In the next two sections we demonstrate the technique of splitting operators for solving two basic perturbation problems in control theory: the perturbation analysis of orthogonal canonical forms [11, 15, 17] and of the feedback synthesis problem [8, 16].

4. ORTHOGONAL CANONICAL FORMS

Consider first the controllable single-input system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{13}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^1$ and $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^n$. Further on system (13) is identified with the matrix pair $S = (A, B)$.

As it is well known [6], the canonical form of S relative to the orthogonal transformations group $\mathcal{O}_n$ is

$$S_c := (A_c, B_c) = (U^T AU, U^T B),\ U \in \mathcal{O}_n,$$

$$A_c = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix},\quad B_c = \begin{bmatrix} a_{1,0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{14}$$

where $a_{i,i-1} > 0$; $i = 1, \ldots, n$.

Let a, b be positive constants such that the pair $S + \Delta S := (A + \Delta A, B + \Delta B)$ remains controllable provided $\|\Delta A\|_F < a$, $\|\Delta B\| < b$. Denote by $(S + \Delta S)_c = ((A+\Delta A)_c, (B+\Delta B)_c)$ the orthogonal canonical form of the perturbed pair $S+\Delta S$ and let $U + \Delta U \in \mathcal{O}_n$ be the corresponding transformation matrix.

Our purpose is to estimate the perturbations in the canonical form S_c,

$$\Delta_{A_c} := \|(A + \Delta A)_c - A_c\|_F,\quad \Delta_{B_c} := \|(B + \Delta B)_c - B_c\|$$

as functions of the perturbations $\Delta_A := \|\Delta A\|_F$, $\Delta_B := \|\Delta B\|$ or $\Delta_S := \sqrt{\Delta_A^2 + \Delta_B^2}$.

The estimate for Δ_{B_c} is immediate:

$$\Delta_{B_c} = |\|(B + \Delta B)_c\| - \|B_c\|| \leq \|\Delta B\| = \Delta_B \tag{15}$$

with equality if ΔB is proportional to B.

Denoting $X := \Delta U^T U$, $E := (U+\Delta U)^T \Delta A(U+\Delta U)$, $F := (U+\Delta U)^T \Delta B$ we get

$$(A + \Delta A)_c - A_c = E + (XA_c - A_c X)(I_n + X^T) \tag{16}$$

$$(B + \Delta B)_c - B_c \doteq F + XB_c. \tag{17}$$

It follows from (16) that

$$\Delta_{A_c} \leq \|E\|_F + \|XA_c - A_c X\|_F \leq \Delta_A + \omega\Delta_U \tag{18}$$

where $\Delta_U := \|\Delta U\|_F = \|X\|_F$ and

$$\omega = \omega(A) := \max\{\ \|YA - AY\|_F : \|Y\|_F = 1\} = \|I_n \otimes A - A^T \otimes I_n\|_2.$$

Below we derive bounds for Δ_U which are nonlinear functions in Δ_S.

Let L be the linear operator mapping the subspace of strictly lower matrices $\mathrm{Low}(\mathcal{R}^{n\times n})$ into $\mathrm{Low}(\mathcal{R}^{n\times(n+1)})$ and defined from

$$\mathsf{L}(Y) := \mathrm{Low}[YB_c, YA_c - A_cY],\ Y \in \mathrm{Low}(\mathcal{R}^{n\times n}).$$

Then

$$\mathsf{L}(\mathrm{Low}(X)) = -\mathrm{Low}[F, E] - \mathrm{Low}[0, (XA_c - A_cX)X^T]. \tag{19}$$

The restriction L_1 of L into $\mathrm{Low}(\mathcal{R}^{n\times(n+1)})$ is invertible if and only if the dimension of the controllable subspace of S is not less than $n-1$. In this case

$$\mathrm{Low}(X) = \Pi_1(X) := -\mathsf{L}_1^{-1}\left(\mathrm{lw}(\mathrm{Low}[F, E] + \mathrm{Low}[0, (XA_c - A_cX)X^T])\right) \tag{20}$$

where $\mathrm{lw}(Z) := [z_{2,1}, \ldots, z_{n,1}, z_{3,2}, \ldots, z_{n,2}, \ldots, z_{n,n-1}]^T \in \mathcal{R}^{n(n-1)/2}$.

The general form of the block lower triangular matrix $L_1 \in \mathcal{R}^{s\times s}$, $s := n(n-1)/2$, of the operator L_1 is given in [15]. We have $L_1 = [L_{1_{i,j}}]$; $i, j = 1, \ldots, n-1$, where

$$L_{1_{i,j}} = [0_{(n-i)\times(i-j)}, a_{j,i-1}I_{n-i}] - \Delta_{i,j+1}A_c(i+1\!:n, j+1\!:n) \in \mathcal{R}^{(n-i)\times(n-j)}$$

if $i \geq j$ and $L_{1_{i,j}} = 0_{(n-i)\times(n-j)}$ if $i < j$. In particular $L_{1_{i,i}} = a_{i,i-1}I_{n-i}$.

Equation (19) together with

$$X + X^T + XX^T = 0 \tag{21}$$

constitute a system of matrix equations for determining X. We shall rewrite this system as an operator equation $X = \Pi(X)$, where $\Pi : \mathcal{R}^{n\times n} \to \mathcal{R}^{n\times n}$ is a nonlinear operator [7].

Let $X_1 = \mathrm{Low}(X)$, $X_2 = \mathrm{Diag}(X)$, $X_3 = \mathrm{Up}(X)$. Determine X_2, X_3 via (21) and X_1 – via (19). Then we have

$$X = X_1 + X_2 + X_3 = \Pi(X) := \Pi_1(X) + \Pi_2(X) + \Pi_3(X) \tag{22}$$

where the operator Π is defined as follows: $\Pi_1(X)$ is the right-hand side of (20) and

$$\Pi_2(X) := -\mathrm{Diag}(X^TX)/2$$
$$\Pi_3(X) := -X_1^T - \mathrm{Up}(X^TX).$$

Set $\xi := [\xi_1, \xi_2, \xi_3]^T$, $\xi_i := \|X_i\|_F$, $r := \|\xi\|^2$ and

$$\nu = \nu(S) := \max\{\mathrm{Low}[0, (YA_c - A_cY)Y^T]\|_F : \|Y\|_F = 1\}.$$

The maximization for determining ν may be done by the direct optimization technique proposed in [3].

It may be shown that

$$\|\Pi_1(X)\|_F \leq f_1(\xi, \Delta_S) = \phi_1(r, \Delta_S) := \mu\sqrt{(1-\alpha^2)\Delta_S^2 + (\alpha\Delta_S + \nu r)^2}$$

$$\|\Pi_2(X)\|_F = f_2(\xi) := \frac{r}{2}$$

$$\|\Pi_3(X)\|_F \leq f_3(\xi) := \xi_1 + \lambda_n r$$

where $\alpha := \Delta_A/\Delta_S \leq 1$ and $\lambda_n^2 := (n-1)/(2n)$. Note that in the above estimates ν may be replaced by the greater quantity

$$\nu_0 = \nu_0(S) := \left\| (I_n \otimes A_c^T - A_c \otimes I_n)(2n+1{:}n^2, 1{:}n^2) \right\|_2 .$$

Consider now the vector equation

$$c = f(c, \Delta_S) \tag{23}$$

where $c := [c_1, c_2, c_3]^T$, $f(c, \Delta_S) := [f_1(c, \Delta_S), f_2(c), f_3(c)]^T$. As follows from the analysis below, equation (23) has a positive solution $c = c(\Delta_S)$ for $\Delta_S > 0$ sufficiently small. The equivalent equation in r:

$$r = \phi(r, \Delta_S) := \phi_1^2(r, \Delta_S) + \frac{1}{4}r^2 + (\phi_1(r, \Delta_S) + \lambda_n r)^2 \tag{24}$$

may be written as an algebraic equation of fourth order.

Since $\phi(r, \Delta)$ is increasing in both r and Δ then applying the method of majorant Lyapunov functions [1] it may be shown that there exists a positive constant Δ^* (depending on μ, α and n) such that:

— For $0 < \Delta_S < \Delta^*$ there exist two positive solutions $r_{\min} = r_{\min}(\Delta_S) < r_{\max} = r_{\max}(\Delta_S)$ of (24)) (for $\Delta_S = 0$ one has $r_{\min} = 0$, $r_{\max} = 2/(\alpha_n^2 + \beta_n^2)$, see (27) for notations);

— For $\Delta_S = \Delta^*$ there exist one (double) positive solution $r^* := r_{\min}(\Delta^*) = r_{\max}(\Delta^*)$ of (24);

— For $\Delta_S > \Delta^*$ there are no real solutions of (24).

The pair (r^*, Δ^*) is obtained as a solution of the nonlinear system of equations

$$r = \phi(r, \Delta), \quad 1 = \phi_r'(r, \Delta).$$

We note that in general the critical value Δ^* and the corresponding solution of (23) $c^* = c(\Delta^*)$ are obtained via the nonlinear system [1]

$$c = f(c, \Delta), \quad \det(I_3 - f_c'(c, \Delta)) = 1$$

where $f_c'(c, \Delta) = [\partial f_i(c, \Delta)/\partial c_j]$ is the Jacobi matrix of f in the argument c.

The square root of the smallest positive root $r_{\min}$ of (24)) has the following expansion in Δ_S:

$$\begin{aligned} \rho(\Delta_S) := \sqrt{r_{\min}(\Delta_S)} &= \sqrt{2}\,\mu\Delta_S(1 + (\lambda_n + 2\alpha\nu)\mu\Delta_S \\ &\quad + (\frac{3}{2} - \frac{5}{4n} + 2\mu^2\nu^2 + 8\alpha\mu\nu\lambda_n + 6\alpha^2\mu^2\nu^2)\mu^2\Delta_S^2) + O(\Delta_S^4). \end{aligned} \tag{25}$$

For $\Delta_S \leq \Delta^*$ denote by $c_{\min} = [c_{\min,1}, c_{\min,2}, c_{\min,3}]^T = c_{\min}(\Delta_S)$ the solution of (23) corresponding to the root $r_{\min}$ of (24), and consider the set

$$\mathcal{B} = \mathcal{B}(\Delta_S) := \left\{ X \in \mathcal{R}^{n\times n} : \|X_i\|_F \leq c_{\min,i}(\Delta_S);\ i = 1,2,3 \right\} \subset \mathcal{R}^{n\times n}.$$

Since

$$(\|\Pi_1(X)\|_F, \|\Pi_2(X)\|_F, \|\Pi_3(X)\|_F)^T \preceq f(\xi, \Delta_S) \preceq f(c_{\min}, \Delta_S) = c_{\min}$$

then the operator Π maps the bounded closed convex set $\mathcal{B}$ into itself and hence there is a fixed point $X \in \mathcal{B}$ of Π [4] for which the estimate

$$\|X\|_F = \|\Pi(X)\|_F \leq \|c_{\min}(\Delta_S)\| = \rho(\Delta_S)$$

holds. Thus

$$\Delta_{A_c} \leq \Delta_A + \omega\rho(\Delta_S). \tag{26}$$

An explicit, although less sharp estimate of the norm of X may be derived as follows. Indeed,

$$f_1(\xi, \Delta_S) \leq \overline{f}_1(\xi, \Delta_S) := \mu(\Delta_S + \nu r)$$

and solving the system

$$c = \overline{f}(c, \Delta_S) := [\overline{f}_1(c, \Delta_S), f_2(c), f_3(c)]^T$$

instead of (23), we obtain the equation

$$\alpha_n^2 \overline{r}^2 - 2(1 - 2\mu\beta_n\Delta_S)\overline{r} + 4\mu^2\Delta_S^2 = 0 \tag{27}$$

where $\alpha_n^2 := 1 - \frac{1}{2n} + \beta_n^2$, $\beta_n := 2\mu\nu + \lambda_n$. Hence if

$$\Delta_S \leq \frac{1}{2\mu(\alpha_n + \beta_n)}$$

then equation (27) has a positive root

$$\overline{r}_{\min} = \overline{r}_{\min}(\Delta_S) := \frac{1 - 2\mu\beta_n\Delta_S - \sqrt{D(\Delta_S)}}{\alpha_n^2} = \frac{4\mu^2\Delta_S^2}{1 - 2\mu\beta_n\Delta_S + \sqrt{D(\Delta_S)}}$$

where $D(\Delta_S) := (1 - 2\mu\beta_n\Delta_S)^2 - 4\mu^2\alpha_n^2\Delta_S^2$. Now the bound for $\|X\|_F$ is

$$\|X\|_F \leq \overline{\rho}(\Delta_S) := \sqrt{\overline{r}_{\min}(\Delta_S)}$$

and according to (18)

$$\Delta_{A_c} \leq \Delta_A + \omega\overline{\rho}(\Delta_S). \tag{28}$$

Note that the linear perturbation bound (the first order term in Δ_S) for $\|X\|_F$ is $\sqrt{2}\mu\Delta_S$. The quantity $\sqrt{2}\,\mu$ is the absolute condition number of the problem of computing the matrix U transforming the pair S into orthogonal canonical form S_c.

The extension of the above results to the multi-input case is straightforward.

Consider the controllable multi-input system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$ and $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$.

Denote by $(m_1, \ldots, m_p)$ the collection of conjugate Kronecker indices of $S = (A, B)$. Then the orthogonal canonical form of S is

$$S_c := (A_c, B_c) = (U^TAU, U^TB), \;\; U \in \mathcal{O}_n$$

$$A_c = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,p-1} & A_{1,p} \\ A_{2,1} & A_{2,2} & A_{2,3} & \cdots & A_{2,p-1} & A_{2,p} \\ 0 & A_{3,2} & A_{3,3} & \cdots & A_{3,p-1} & A_{3,p} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{p,p-1} & A_{p,p} \end{bmatrix}, \quad B_c = \begin{bmatrix} A_{1,0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{29}$$

where the matrices $A_{i,i-1} \in R^{m_i \times m_{i-1}}$; $i = 1, \ldots, p$; $m_0 = m$ are upper trapezoidal and of full row rank. The detailed structure of the matrices $A_{i,i-1}$ which corresponds to the precise definition of a canonical set in the set of controllable systems is given in [6].

Let a, b be positive constants such that perturbations $\Delta S := (\Delta A, \Delta B)$ in (A, B) preserve controllability provided $\|\Delta A\|_F < a$, $\|\Delta B\|_F < b$. Denote by $(S + \Delta S)_c$ the orthogonal canonical form of the perturbed pair $S + \Delta S$ and let $U + \Delta U$ be the corresponding (unique) orthogonal matrix transforming $S + \Delta S$ into orthogonal canonical form. Similarly to the single-input case, the perturbation analysis of multi-input orthogonal canonical forms is aimed at estimating the F-norms of the perturbations

$$(A + \Delta A)_c - A_c, \quad (B + \Delta B)_c - B_c$$

as functions of the F-norms of ΔA, ΔB or ΔS.

In studying the sensitivity of multi-input orthogonal canonical forms, only the generic case is considered, when the first n columns of the controllability matrix $Q(S) \in \mathcal{R}^{n \times pm}$ of S are linearly independent. This is not a restrictive assumption since the lack of genericity could make the perturbation analysis of S_c meaningless. Indeed, in the nongeneric case the orthogonal canonical form S_c may even be discontinuous as a function of $\Delta_B := \|\Delta B\|_F, \Delta_A := \|\Delta A\|_F$, see Example 1 below.

For the generic pair S let again a, b be positive constants such that $S + \Delta S$ remains generic when $\Delta_A < a$, $\Delta_B < b$. We shall study only perturbations ΔS satisfying the last two inequalities. Then all main relations for single-input orthogonal canonical forms are valid formally for the multi-input case with some minor changes.

For the input matrix perturbation Δ_{B_c}, instead of (15) we have the bound [11]

$$\Delta_{B_c} \le \Delta_B \left(1 + \frac{\sqrt{2}\|B\|_2}{\sigma_{\min}(B) - \Delta_B} \right) = (1 + \sqrt{2}\,\mathrm{cond}_2(B))\,\Delta_B + O(\Delta_B^2) \tag{30}$$

where $\mathrm{cond}_2(B) = \|B\|_2 \|B^\dagger\|_2 = \sigma_{\max}(B)/\sigma_{\min}(B)$ is the condition number of the matrix B in the 2-norm, and $\sigma_{\max}(B)$ and $\sigma_{\min}(B)$ are the maximal and minimal singular values of B.

For the state matrix perturbation Δ_{A_c}, the bounds (26), (28) may be used, noting that the operator $\mathsf{L}_1 : \mathrm{Low}(\mathcal{R}^{n \times n}) \to \mathrm{Low}(\mathcal{R}^{n \times (n+m)})$ is formally determined by the expression for L. We stress that in the nongeneric case the operator L_1 is usually not invertible. More precisely, L_1 is invertible if and only if the first $n - 1$ columns of the controllability matrix $Q(S)$ are linearly independent. However, the expressions for the blocks $L_{1_{i,j}}$ of the matrix L_1 of L_1 are different in the multi-input case. If

e. g. $n = 5, m = m_1 = m_2 = 2, m_3 = 1$ and

$$A_c = \begin{bmatrix} a_{1,1} & a_{1,2} & * & * & * \\ a_{2,1} & a_{2,2} & * & * & * \\ a_{3,1} & a_{3,2} & * & * & * \\ 0 & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ 0 & 0 & a_{5,3} & a_{5,4} & a_{5,5} \end{bmatrix}, \quad B_c = \begin{bmatrix} b_{1,1} & b_{1,2} \\ 0 & b_{2,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then the blocks $L_{1_{i,j}} \in \mathcal{R}^{(5-i)\times(5-j)}$ of the upper triangular matrix $L_1 = [L_{1_{i,j}}]$; $i, j = 1, \ldots, 4$ of the operator L_1 are determined from

$$L_{1_{1,1}} = b_{1,1}I_4,\ L_{1_{2,2}} = b_{2,2}I_3,\ L_{1_{3,3}} = a_{3,1}I_2,\ L_{1_{4,4}} = a_{4,2},\ L_{1_{2,1}} = [0_{3\times1}, b_{1,2}I_3]$$

$$L_{1_{3,1}} = [0_{2\times2}, a_{1,1}I_2] - A_c(4{:}\,5, 2{:}\,5),\ L_{1_{3,2}} = [0_{2\times1}, a_{2,1}I_2]$$

$$L_{1_{4,1}} = [0_{1\times3}, a_{1,2}],\ L_{1_{4,2}} = [0_{1\times2}, a_{2,2}] - A_c(5{:}\,5, 3{:}\,5),\ L_{1_{4,3}} = [0, a_{3,2}].$$

In the definition of the quantity ν one has to replace $[0, A_c] = [0_{n\times1}, A_c] \in \mathcal{R}^{n\times(n+1)}$ with $[0_{n\times m}, A_c] \in \mathcal{R}^{n\times(n+m)}$.

Example 1. Consider the nongeneric system with matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for which $p = 3$ and $m_1 = m_2 = m_3 = 1$. Since the system is already in orthogonal canonical form, we have $U = I_3$. Let ΔB be a matrix with a single nonzero element $\beta > 0$ in position $(2, 1)$. Then the orthogonal canonical form of the perturbed pair $(A, B + \Delta B)$ has matrices

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (B + \Delta B)_c = \begin{bmatrix} \beta & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$U + \Delta U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$A_c - A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad (B + \Delta B)_c - B = \begin{bmatrix} \beta & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\Delta U = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$\Delta_U = 2,\ \Delta_{A_c} = 2,\ \Delta_{B_c} = \sqrt{2 + \Delta_S^2} = \sqrt{2 + \beta^2}$$

and the orthogonal canonical form S_c is discontinuous with a jump $\Delta_{S_c} = \sqrt{6}$ since $\Delta_{A_c} = 0$, $\Delta_{B_c} = 0$ for $\Delta_S = 0$.

Example 2. Consider a fifth order system with $m = m_1 = m_2 = 2$, $m_3 = 1$ and matrices

$$A = \begin{bmatrix} -2.00 & 1.00 & 0.00 & -9.00 & 17.00 \\ -1.00 & 3.00 & 2.00 & 5.00 & 8.00 \\ 0.01 & 0.00 & -4.00 & -7.00 & -6.00 \\ 0.00 & 0.01 & -3.00 & -1.00 & 5.00 \\ 0.00 & 0.00 & 0.20 & 0.00 & 1.00 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let the perturbations in the data be

$$\Delta A = 10^{i-13} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta B = 10^{i-13} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}.$$

The quantity $\nu = 10.99$ was obtained by the procedure `mdsmax` from [3]. The estimate ν_0 of ν is 22.00.

The results for Δ_{A_c} and Δ_{B_c} are shown in Table 1 for different values of i. In the case denoted by $*$ the estimate (28) does not exist since the quadratic equation for $\bar{r}$ has no real roots.

Table 1.

i	$\|\|\Delta_{A_c}\|\|_F$	Est. (28) with ν	Est. (28) with ν_0	$\|\|\Delta_{B_c}\|\|_F$	Est. (30)
1	1.22×10^{-8}	6.16×10^{-8}	6.16×10^{-8}	4.47×10^{-13}	2.58×10^{-11}
2	1.22×10^{-7}	6.16×10^{-7}	6.16×10^{-7}	4.47×10^{-12}	2.58×10^{-10}
3	1.22×10^{-6}	6.16×10^{-6}	6.17×10^{-8}	4.47×10^{-10}	2.58×10^{-8}
4	1.22×10^{-5}	6.19×10^{-5}	6.23×10^{-5}	4.47×10^{-9}	2.58×10^{-7}
5	1.22×10^{-4}	6.53×10^{-4}	7.02×10^{-4}	4.47×10^{-8}	2.58×10^{-6}
6	1.22×10^{-3}	$*$	$*$	4.47×10^{-8}	2.58×10^{-6}

5. FEEDBACK SYNTHESIS

Consider the controllable and observable system

$$\dot{x}(t) = Ax(t) + Bu(t), \ y(t) = Cx(t) \tag{31}$$

where $x(t) \in \mathcal{F}^n, u(t) \in \mathcal{F}^m, y(t) \in \mathcal{F}^r$ and $A \in \mathcal{F}^{n\times n}, B \in \mathcal{F}^{n\times m}, C \in \mathcal{F}^{r\times n}$. We assume that $\operatorname{rank}(B) = m < n, \operatorname{rank}(C) = r \leq n$ and identify system (31) with the matrix triple $S = (C, A, B)$.

The *general feedback synthesis problem* for system (31) is formulated as follows [8]. Let Γ be a subgroup of $\mathcal{GL}_n$. The matrix $F \in \mathcal{F}^{n\times n}$ is said to be a Γ-attainable form for (31) if there exists a gain matrix $K \in \mathcal{F}^{m\times r}$ and a matrix $U \in \Gamma$ such that

$$\Phi(S;U,K) := U^{-1}(A+BKC)U = F \tag{32}$$

or, equivalently

$$\Psi(S,F;U,K) := (A+BKC)U - UF = 0. \tag{33}$$

For a given Γ-attainable form F, the problem is to find K (and eventually U) such that (32) or (33) is valid.

The most important particular case of the feedback synthesis problem is the *pole assignment problem*: For a given set $\{\lambda_1,\ldots,\lambda_n\}$ of complex numbers (for real systems this set must be symmetric about the real axis) find a gain matrix K which preassigns the spectrum of the closed-loop system matrix, i. e. $\mathrm{spect}(A+BKC) = \{\lambda_1,\ldots,\lambda_n\}$.

From computational point of view it is preferable to reformulate the pole assignment problem as a problem of synthesis of an $\mathcal{U}_n$-attainable upper triangular matrix F with $\mathrm{spect}(F) = \{\lambda_1,\ldots,\lambda_n\}$:

$$\Phi_0(S;U,K) := (\mathrm{Low}+\mathrm{Diag})(U^H(A+BKC)U) = \Lambda \tag{34}$$

where $U \in \mathcal{U}_n$ and $\Lambda := \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$.

Let the matrix F be $\mathcal{GL}_n$-attainable and (U,K) be a solution of (32). Suppose that $\Delta S = (\Delta C,\Delta A,\Delta B)$ and ΔF are perturbations in S and F, such that $\|\Delta C\|, \|\Delta A\|, \|\Delta B\|, \|\Delta F\| \le \rho$, $\rho > 0$. For sufficiently small ρ the perturbed matrix $F+\Delta F$ is also $\mathcal{GL}_n$-attainable and the perturbed equation of type (32)

$$\Phi(S+\Delta S;U^*,K^*) = F+\Delta F \tag{35}$$

has a solution $U^* = U+\Delta U$, $K^* = K+\Delta K$.

The perturbation analysis of the feedback synthesis problem consists in finding estimates for the perturbation $\Delta_K := \|\Delta K\|_F$ in the solution for the gain matrix (and eventually for the corresponding perturbation in the transformation matrix U) as function of the perturbations $\Delta_Z := \|\Delta Z\|_F$ in the data $Z = C,A,B,F$.

Consider the linear operator $\mathsf{L}(\cdot,\cdot) : \mathcal{F}^{n\times n} \times \mathcal{F}^{m\times r} \to \mathcal{F}^{n\times n}$, defined from

$$\mathsf{L}(X_1,X_2) := FX_1 - X_1F + GX_2H \tag{36}$$

where $X_1 := U^{-1}\Delta U$, $X_2 := \Delta K$, $G := U^{-1}B$, $H := CU$. Denote by $L \in \mathcal{F}^{n^2\times s}$, $s := mr+n^2$, the matrix representation of the operator $\mathsf{L}(\cdot,\cdot)$:

$$L = [L_1,L_2];\ \ L_1 := I_n\otimes F - F^T\otimes I_n \in \mathcal{F}^{n^2\times n^2},\ \ L_2 := H^T\otimes G \in \mathcal{F}^{n^2\times mr} \tag{37}$$

corresponding to the columnwise vector expansion of its arguments. In view of the controllability and observability of (31) and the assumptions made, the operator $\mathsf{L}(\cdot,\cdot)$ is surjective, i. e. $\mathrm{rank}(L) = n^2$ [2].

The perturbed version of (33)

$$\Psi(S+\Delta S,F+\Delta F;U+\Delta U,K+\Delta K) = 0$$

may be rewritten as

$$\mathsf{L}(X_1, X_2) = \Theta(X_1, X_2) \tag{38}$$

where

$$\Theta(X_1, X_2) := (I_n + X_1)\Delta F - E - (GX_2H + E)X_1$$

$$E := U^{-1}\Delta AU + G(K + X_2)\Delta CU + U^{-1}\Delta B(K + X_2)H + U^{-1}\Delta B(K + X_2)\Delta CU.$$

Denote $\xi_1 := \mathrm{vec}(X_1) \in \mathcal{F}^{n^2}$, $\xi_2 := \mathrm{vec}(X_2) \in \mathcal{F}^{mr}$, $\xi := [\xi_1^T, \xi_2^T]^T \in \mathcal{F}^s, s := n^2 + mr$ and let

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \in \mathcal{F}^{s \times n^2}; \ R_1 \in \mathcal{F}^{n^2 \times n^2}, \ R_2 \in \mathcal{F}^{rm \times n^2}$$

be a matrix such that $LR = I_{n^2}$. Then it follows from (38) that ξ satisfies the operator equation $\xi = \Pi(\xi)$:

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \Pi(\xi) = \begin{bmatrix} \Pi_1(\xi) \\ \Pi_2(\xi) \end{bmatrix} := \begin{bmatrix} R_1 \mathrm{vec}(\Theta(X_1, X_2)) \\ R_2 \mathrm{vec}(\Theta(X_1, X_2)) \end{bmatrix}. \tag{39}$$

For $\rho_1, \rho_2 > 0$ denote

$$\mathcal{B}(\rho_1, \rho_2) := \{ \ \xi : \|\xi_i\|_2 \le \rho_i; \ i = 1, 2 \ \} \subset \mathcal{F}^s.$$

We shall show that under some conditions there exist a domain $\mathcal{D} \subset \mathcal{R}_+^4$ and functions $f_1, f_2 : \mathcal{D} \to \mathcal{R}_+$ with the properties of the function f from Section 2 and such that for $\Delta := [\Delta_C, \Delta_A, \Delta_B, \Delta_F]^T \in \mathcal{D}$ the operator Π maps the set $\mathcal{B}_\Delta := \mathcal{B}(f_1(\Delta), f_2(\Delta))$ into itself. Since $\mathcal{B}_\Delta$ is convex and compact then according to the Schauder fixed point principle there exists a solution $\xi \in \mathcal{B}_\Delta$ of the operator equation $\xi = \Pi(\xi)$ for which the estimate

$$\|\xi_2\|_2 = \Delta_K := \|\Delta K\|_F \le f_2(\Delta), \ \Delta \in \mathcal{D} \tag{40}$$

holds.

Let $\xi \in \mathcal{B}(\rho_1, \rho_2)$. Then (38), (39) yield

$$\|\Pi_i(\xi)\|_2 \le \|R_i\|_2 \|\Theta(X_1, X_2)\|_F \le r_i(a + a\rho_1 + b\rho_2 + c\rho_1\rho_2) \tag{41}$$

where $r_i := \|R_i\|_2$ and

$$a = a(\Delta) := \Delta_F + \mathrm{cond}_2(U)\Delta_A + b(\Delta)\|K\|_2$$

$$b = b(\Delta) := \|G\|_2\|U\|_2\Delta_C + \|H\|_2\|U^{-1}\|_2\Delta_B + \mathrm{cond}_2(U)\Delta_C\Delta_B$$

$$c = c(\Delta) := b(\Delta) + \|G\|_2\|H\|_2.$$

If the quantities ρ_i satisfy the system of algebraic equations

$$\rho_i = r_i(a + a\rho_1 + b\rho_2 + c\rho_1\rho_2); \ i = 1, 2$$

then it follows from (41) that $\|\Pi_i(\xi)\|_2 \le \rho_i$, i. e. $\Pi(\mathcal{B}(\rho_1, \rho_2)) \subset \mathcal{B}(\rho_1, \rho_2)$.

The system of equations for ρ_i yields

$$r_1 c\rho_2^2 - (1 - r_1 a - r_2 b)\rho_2 + r_2 a = 0. \tag{42}$$

Denote by $\mathcal{D}$ the set of all Δ satisfying the inequality

$$r_1 a(\Delta) + r_2 b(\Delta) + 2\sqrt{r_1 r_2 a(\Delta) c(\Delta)} \leq 1 \tag{43}$$

and define $f_2(\Delta)$ as the smaller root of (42):

$$f_2(\Delta) := \frac{1 - r_1 a(\Delta) - r_2 b(\Delta) - \sqrt{d(\Delta)}}{2 r_1 a(\Delta)} \tag{44}$$

where

$$d(\Delta) := (1 - r_1 a(\Delta) - r_2 b(\Delta))^2 - 4 r_1 r_2 a(\Delta) c(\Delta). \tag{45}$$

Since $\rho_2 := f_2(\Delta)$ satisfies (42), we see that if the vector Δ of perturbation norms satisfies (43) then inequality (40) in view of (44), (45) gives a nonlinear nonlocal perturbation bound for the solution of the general feedback synthesis problem. If only state feedback synthesis problem is considered, one must set $\Delta_C = 0$ and $C = I_n$ in the corresponding expressions.

Consider now the sensitivity analysis of the pole assignment problem for system (31).

As shown in the beginning of this section, the pole assignment problem may be stated as a feedback synthesis problem for an $\mathcal{U}_n$-reachable upper triangular form $F = [f_{i,j}]$ of the closed-loop system matrix $A + BKC$ with spect(F) equal to the set of desired poles: $f_{i,i} = \lambda_i$. Let $\Delta C, \Delta A, \Delta B$ and $\Delta\lambda_1, \ldots, \Delta\lambda_n$ be perturbations in the system matrices and in the preassigned poles of the closed-loop system. Similarly to (35), we obtain from (34)

$$\Phi_0(S + \Delta S; U^*, K^*) = \Lambda + \Delta\Lambda \tag{46}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $\Delta\Lambda = \mathrm{diag}(\Delta\lambda_1, \ldots, \Delta\lambda_n)$.

We define the linear pole assignment operator $\mathsf{L}_0(\cdot, \cdot)$ from

$$\mathsf{L}_0(X_1, X_2) := (\mathrm{Low} + \mathrm{Diag})(F\,\mathrm{Low}(X_1) - \mathrm{Low}(X_1)F + GX_2H)$$

where $F := U^H(A + BKC)U$, $X_1 := U^H \Delta U$, $X_2 := \Delta K$, $G := U^H B$, $H := CU$ and $U \in \mathcal{U}_n$.

Let $\xi_{01} := P\mathrm{vec}(X_1) = [x_{21}, \ldots, x_{n1}, x_{32}, \ldots, x_{n2}, \ldots, x_{n,n-1}]^T \in \mathcal{F}^{n(n-1)/2}$ be the columnwise vector representation of the lower part of $X_1 = [x_{i,j}]$, where

$$P := \begin{bmatrix} J_2 & 0 & \cdots & 0 & 0 \\ 0 & J_3 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & J_n & 0 \end{bmatrix} \in \mathcal{F}^{n(n-1)/2 \times n^2}$$

and $J_k := [0, I_{n-k+1}] \in \mathcal{F}^{(n-k+1)\times n}$; $k = 2, \ldots, n$.

Denote by L_0 the matrix representation of the pole assignment operator $\mathsf{L}_0(\cdot,\cdot)$:

$$L_0 := \mathrm{diag}(I_n, J_2, \ldots, J_n)[L_1 P^T, L_2] \in \mathcal{F}^{n(n+1)/2 \times \ell}$$

where $\ell := n(n-1)/2 + mr$ and the matrices L_1, L_2 are determined as in (37) with U^{-1} replaced by U^H. As in the general feedback synthesis problem, the assumptions made guarantee that the operator $\mathsf{L}_0(\cdot,\cdot)$ is surjective, i. e. $\mathrm{rank}(L_0) = n(n+1)/2$.

Let

$$R_0 = \begin{bmatrix} R_{01} \\ R_{02} \end{bmatrix} \in \mathcal{F}^{\ell \times n(n+1)/2};\; R_{01} \in \mathcal{F}^{n(n-1)/2 \times n(n+1)/2},\; R_{02} \in \mathcal{F}^{mr \times n(n+1)/2}$$

be a matrix such that $L_0 R_0 = I_{n(n+1)/2}$. Then the nonlocal perturbation bound for the pole assignment problem is obtained via (40) replacing r_i by $\|R_{0i}\|_2$ and setting to 1 the 2-norms of U and U^{-1}.

In particular for pole assignment by state feedback ($C = I_n, \Delta_C = 0, \Delta_\Lambda = 0, U \in \mathcal{U}_n$) the expressions for a, b, c are simplified as $a(\Delta) = \Delta_A + \|K\|_2 \Delta_B$, $b(\Delta) = \Delta_B$, $c(\Delta) = \|B\|_2 + \Delta_B$. Here the domain $\mathcal{D} \subset R^2_+$ is bounded by the nonnegative semi-axes and a parabola (or a straight line).

Example 3. Consider the state pole assignment problem for the real pair (A, B),

$$A = \begin{bmatrix} -6.9 & 5.8 & -7.9 & -16.4 & -5.1 & 30.0 \\ 10.9 & -10.8 & 9.9 & 26.4 & 6.1 & -50.0 \\ -4.9 & 5.9 & -3.9 & -13.6 & -3.1 & 24.3 \\ -4.9 & 4.8 & -1.9 & -14.4 & 0.9 & 25.0 \\ 4.9 & -3.8 & 4.9 & 10.4 & 3.1 & -20.0 \\ -6.9 & 6.8 & -4.9 & -18.4 & -2.1 & 33.0 \end{bmatrix},\quad B = \begin{bmatrix} -6.2 & -10.5 \\ 12.4 & 12.6 \\ -6.2 & -6.3 \\ -3.1 & -2.1 \\ 6.2 & 6.3 \\ -6.2 & -6.3 \end{bmatrix}.$$

For the set of desired poles $\{-0.1 + 0.2i, -0.1 - 0.2i, -0.3, -0.4, -0.5, -0.6\}$ the pole assignment algorithm from [14] produces the gain matrix

$$K = \begin{bmatrix} 0.3845 & -14.80 & -7.226 & 23.25 & 12.17 & -20.11 \\ 9.668 & -32.81 & -9.619 & 65.31 & 9.660 & -85.96 \end{bmatrix}.$$

The perturbations are taken as $\Delta\Lambda = 0$ and $\Delta A = 10^{-j} A_0$, $\Delta B = 10^{-j} B_0$,

$$A_0 = \begin{bmatrix} 0.1 & -0.8 & 0.4 & -0.7 & 0.1 & 0.2 \\ -0.2 & 1.0 & -0.6 & 1.0 & -0.2 & 0.3 \\ 0.3 & -0.4 & 0.2 & 0.0 & -0.2 & 0.2 \\ 0.5 & -0.3 & 2.0 & -3.0 & 0.5 & -0.9 \\ 2.0 & -3.0 & 0.9 & 0.9 & -2.0 & 2.0 \\ 1.0 & 3.0 & -2.0 & 5.0 & -2.0 & 3.0 \end{bmatrix},\quad B = \begin{bmatrix} 1.0 & -0.6 \\ -2.0 & 0.9 \\ 0.1 & 0.0 \\ 5.0 & -2.0 \\ -0.3 & 0.1 \\ -7.0 & 4.0 \end{bmatrix}$$

where j is an integer. The results are given in Table 2, where $\Delta_K / \|K\|_F$ is the exact relative perturbation in K and $f_2(\Delta)/\|K\|_F$ is its non-local estimate. For $j \leq 9$ the non-local estimate does not exist since (43) is violated, which is denoted by $*$.

Table 2.

i	$\Delta_K/\|K\|_F$	$f_2(\Delta)/\|K\|_F$
13	7.599×10^{-12}	115.1×10^{-12}
12	7.599×10^{-11}	115.2×10^{-11}
11	7.599×10^{-10}	115.9×10^{-10}
10	7.599×10^{-9}	123.8×10^{-9}
9	7.599×10^{-8}	$*$

6. CONCLUSION

The technique of splitting operators is essential in the perturbation analysis of matrix problems involving unitary transformations. For such problems, a preliminary step is the splitting of certain linear operator and its arguments into strictly lower, diagonal and strictly upper parts. This gives a majorant system of algebraic equations. The desired perturbation bounds follow directly from the solution of this majorant system. In order to obtain easily computable bounds, a modified majorant system may be constructed, whose solution produces less sharp but computationally efficient explicit estimates.

The technique of splitting operators makes possible to obtain perturbation bounds for basic problems in control theory and linear algebra. In this paper we have used the splitting operator technique to derive non-local perturbation bounds for the problem of computing orthogonal canonical forms of linear control systems and for the general feedback synthesis problem and the pole assignment problem in particular.

In the last decade, the technique of splitting operators has been also applied in the perturbation analysis of a number of linear algebra problems: Schur system of a matrix, QR decomposition of a matrix, generalized Schur form of a pair of matrices, polar decomposition of a matrix, and Hamiltonian Schur form of a Hamiltonian matrix [5, 7, 18, 19].

Other important and still unsolved problems in control theory and matrix analysis that can be addressed via the technique of splitting operators are the synthesis of state observers and dynamic compensators, block-Schur and Jordan-like forms of a matrix, and Hamiltonian matrix pencils.

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REFERENCES

[1] E. A. Grebenikov and Yu. A. Ryabov: Constructive Methods for Analysis of Nonlinear Systems (in Russian). Nauka, Moscow 1979.

[2] R. Hermann and C. L. Martin: Application of algebraic geometry to systems theory. Part I. IEEE Trans. Automatic Control *22* (1977), 19–25.

[3] N. J. Higham: Optimization by direct search in matrix computations. SIAM J. Matrix Anal. Appl. *14* (1993), 317–333.
[4] L. V. Kantorovich and G. P. Akilov: Functional Analysis in Normed Spaces. Pergamon, New York 1964.
[5] M. Konstantinov, V. Mehrmann, and P. Petkov: Perturbation Analysis for the Hamiltonian Schur Form. Technical Report 98-17, Fakultät für Mathematik, TU-Chemnitz, Chemnitz 1998.
[6] M. M. Konstantinov, P. Hr. Petkov, and N. D. Christov: Invariants and canonical forms for linear multivariable systems under the action of orthogonal transformation groups. Kybernetika *17* (1981), 413–424.
[7] M. M. Konstantinov, P. Hr. Petkov, and N. D. Christov: Nonlocal perturbation analysis of the Schur system of a matrix. SIAM J. Matrix Anal. Appl. *15* (1994), 383–392.
[8] M. M. Konstantinov, P. Hr. Petkov, and N. D. Christov: Sensitivity analysis of the feedback synthesis problem. IEEE Trans. Automatic Control *42* (1997), 568–573.
[9] M. M. Konstantinov, P. Hr. Petkov, N. D. Christov, D. W. Gu, and V. Mehrmann: Sensitivity of Lyapunov equations. In: Advances in Intelligent Systems and Computer Science (N. E. Mastorakis, ed.). WSES Press, 1999, pp. 289–292.
[10] M. M. Konstantinov, P. Hr. Petkov, D. W. Gu, and I. Postlethwaite: Perturbation Analysis in Finite Dimensional Spaces. Tech. Rep. 96–18, Engineering Department, Leicester University, Leicester 1996.
[11] M. M. Konstantinov, P. Hr. Petkov, D. W. Gu, and I. Postlethwaite: Perturbation analysis of orthogonal canonical forms. Linear Algebra Appl. *251* (1997), 267–291.
[12] J. Ortega and W. Rheinboldt: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York 1970.
[13] P. Hr. Petkov, N. D. Christov, and M. M. Konstantinov: A new approach to the perturbation analysis of linear control problems. Preprints 11th IFAC World Congr., Tallin 1990, pp. 311–316.
[14] P. Hr. Petkov, N. D. Christov, M. M. Konstantinov: Computational Methods for Linear Control Systems. Prentice–Hall, New York 1991.
[15] P. Hr. Petkov, N. D. Christov, and M. M. Konstantinov: Sensitivity of orthogonal canonical forms for single-input systems. In: Proc. 22nd Spring Conference of UBM, Sofia 1992, pp. 66–73.
[16] P. Hr. Petkov, N. D. Christov, and M. M. Konstantinov: Perturbation analysis of orthogonal canonical forms and pole assignment for single-input systems. In: Proc. 2nd European Control Conference, Groningen 1993, pp. 1397–1400.
[17] P. Hr. Petkov, N. D. Christov, and M. M. Konstantinov: Perturbation controllability analysis of linear multivariable systems. Preprints 12th IFAC World Congress, Sydney 1993, pp. 491–494.
[18] J.-G. Sun: On perturbation bounds for the QR factorization. Linear Algebra Appl. *215* (1995), 95–111.
[19] J.-G. Sun: Perturbation bounds for the generalized Schur decomposition. SIAM J. Matrix Anal. Appl. *16* (1995), 1328–1340.

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