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ON TYPES OF FUZZY NUMBERS UNDER ADDITION

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We consider the question whether, for given fuzzy numbers, there are different pairs of t-norm such that the resulting membership function within the extension principle under addition are identical. Some examples are given.

Keywords: fuzzy number, extension principles, t-norms

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1. INTRODUCTION (PRELIMINARIES)

In general, a functional relationship in a fuzzy environment (see [2]) is given by $f(x,y) = z$, where the input values are available as fuzzy numbers X and Y, and the corresponding response value Z must be computed via an extension principle.

The usual performance consists in choosing a triangular form for the two input values and the min-connection within the extension principle resulting

$$m_{F(X,Y)}(z) = \sup_{x,y:z=f(x,y)} \min(m_X(x), m_Y(y)).$$

Two problems can be considered. First, we have to specify the membership functions for X and Y, and secondly, we have to choose a connection within the extension principle. There are vast diversity of possibility for these two choices. Concerning with this question, can we reduce this diversity in a reasonable manner? If the functional relationship is an arithmetic one, the rules for a possibly approximate computing can be simple, even if the fuzzy numbers are not triangular.

Recently, fuzzy arithmetic has grown in importance during recent years as a tool of advance in fuzzy optimization and control theory. Fuzzy arithmetic based on the sup-(t-norm) convolution, with the controllability of the increase of fuzziness, enables us to construct more flexible and adaptable mathematical models in several intelligent technologies based on approximate reasoning and fuzzy logic. Recent results on this topic and applications can be found in [3–21].

Gebhardt [7] considered some class of combination functions under the restriction of fuzzy numbers to those of LR-type, where the specification assigns special reference function and the value of its parameters.

In this paper we restrict f to $+$ (algebraic addition) and study more on this topics and generalize a result of Gebhardt [7].

2. DEFINITIONS

A t-norm is a function $t : [0,1] \times [0,1] \to [0,1]$ defined by

$$t(1,x) = x, \quad t(0,x) = 0, \quad t(x, t(y,z)) = t(t(x,y),z),$$
$$t(x,y) = t(y,x), \quad x \le y \Rightarrow t(x,z) \le t(y,z).$$

A reference function is a mapping $M : [0,\infty) \to [0,1]$ defined by $M(0) = 1$ and M monotonically decreasing.

A fuzzy number Z is called of LR-type if there are two reference functions L and R, possibly equal, by which the membership function of z can be represented as

$$m_Z(z) = \begin{cases} L\left(\dfrac{z_0 - z}{l}\right) & \text{for} \quad z \le z_0, \\ R\left(\dfrac{z - z_0}{r}\right) & \text{for} \quad z \ge z_0. \end{cases}$$

Then Z can be represented by $Z = (z_0; l, r)_{LR}$ as usual. Moreover, we have supp $Z = \{x |\ m_Z(x) > 0\} = [z_0 - l, z_0 + r]$. The special cases $L(x) = R(x)$ will be indicated by $Z = (z_0; l, r)_{LL} =: (z_0; l, r)_L$ and considered throughout the following.

The given function

$$f : R \times R \to R$$

with the domain $D(f)$ will be denoted also by $x * y = f(x,y)$. When fuzzifying the function f to

$$F : \mathcal{F}(R) \times \mathcal{F}(R) \to \mathcal{F}(R),$$

the extension principle is used in its general form with a given t-norm t:

$$m_{F(x,y)}(z) = \sup_{x,y; z = f(x,y)} t(m_X(x), m_Y(y)), \quad z \in R,\ X, Y \in \mathcal{F}(R)$$

and abbreviated by $X *_t Y := F(X,Y)$.

If $f(x,y) = x + y$, it shall be denoted by $\oplus$, and hence the t-sum of fuzzy numbers is defined by :

$$m_{X \oplus Y}(z) = \sup_{x+y=z} t(m_X(x), m_Y(y)), \quad z \in R,\ X, Y \in \mathcal{F}(R)$$

and abbreviated by $X \oplus_t Y$.

A continuous t-norm t is said to be Archimedean if $T(x,x) < x$ for each $x \in (0,1)$. Every Archimedean t-norm T can be represented by means of an additive generator f (i. e., by a non-negative, continuous, strictly decreasing function defined on the interval $[0,1]$ and satisfying $f(1) = 0$).

Namely,

$$T(x,y) = f^{[-1]}(f(x) + f(y)) \quad \text{for all } x, y \in [0,1],$$

where $f^{[-1]}$ is pseudo-inverse of f given by $f^{[-1]}(y) = f^{-1}(\min(y, f(0)))$. It means that for each continuous Archimedean t-norm t, the t-sum of fuzzy number X, Y can be expressed in the form :

$$X \oplus_t Y(z) = m_{X \oplus_t Y}(z) = \sup_{x \in R} f^{[-1]}(f(m_X(x)) + f(m_Y(z - x))), \quad z \in R.$$

A function h defined on $[0,\infty)$ is subadditive if for all $s,t \geq 0$, $h(s+t) \leq h(s)+h(t)$.

3. EQUIVALENCE OF FUZZIFYING PRINCIPLES FOR ADDITION

In this section, we consider the problem whether, for given fuzzy numbers, there are different pairs of t-norm such that the resulting membership functions within the extension principle under addition are identical. Throughout this section, we restrict to fuzzy number of LL-type.

Definition 1. Let L_1, L_2 be reference functions, and t_1, t_2 be t-norms. (L_1, t_1) and (L_2, t_2) are identical if for any $m, n, l, r, s, t \in R$,

$$(m; l, r)_{L_1} \oplus_{t_1} (n; s, t)_{L_1} = (m; l, r)_{L_2} \oplus_{t_2} (n; s, t)_{L_2}.$$

The following theorems are due to Gebhardt [4].

Theorem 1. (Gebhardt [7]) If (L_1, t_1) and (L_2, t_2) are identical, then $L_1 = L_2$.

Theorem 2. (Gebhardt [7]) Let t_L be the Lukasiewicz t-norm, i.e., $t_L(x,y) = \max\{0, x+y-1\}$, t_λ be the Weber-family of t-norms, i.e., $t_\lambda(x,y) = \max\{0, \frac{x+y-1+\lambda xy}{1+\lambda}\}$, $-1 < \lambda < 0$, and L be a convex function. Then (L, t_L) and (L, t_λ) are identical.

We now generalize above result and study more on this problem.

The following lemmas are easy to check.

Lemma 1. Let t_1, t_2 be t-norms such that $t_1 \leq t_2$ and L be reference function. Then for any $m, n, l, s, t \in R$

$$(m; l, r)_L \oplus_{t_1} (n; s, t)_L \leq (m; l, r)_L \oplus_{t_2} (n; s, t)_L.$$

Lemma 2. Let t be a t-norm and L be a reference function. Then for any $m, n, l, r, s, t \in R$

$$(m; l, r)_L \oplus_t (n; s, t)_L(m+n+z) = (0; l, r)_L \oplus_t (0; s, t)_L(z).$$

Let t_W denote the weakest t-norm defined by

$$t_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

and let t_M denote the strongest t-norm defined by

$$t_M(x,y) = \min(x,y) \quad \text{for all } x, y \in [0,1].$$

Then for each t-norm t it holds $t_W \leq t \leq t_M$.

Theorem 3. Let t be an Archimedean t-norm with additive generator f and let L be a reference function such that $f \circ L$ is subadditive. If $t_i \leq t$, $i = 1, 2$, then (L, t_1) and (L, t_2) are identical.

Proof. By Lemma 2, it suffices to prove that for $l, r, s, t \in R$

$$(0; l, r)_L \oplus_{t_1} (0; s, t)_L = (0; l, r)_L \oplus_{t_2} (0; s, t)_L.$$

By Theorem 3 of Hong [12] and Lemma 1, we have for $i = 1, 2$

$$\begin{aligned}(0; l, r)_L \oplus_{t_i} (0; s, t)_L &\leq (0; l, r)_L \oplus_t (0; s, t)_L \\ &= (0; \max(l, s), \max(r, t))_L.\end{aligned}$$

On the other hand, by Theorem 1 of Hong [12] and Lemma 1, we have for $i = 1, 2$,

$$\begin{aligned}(0; l, r)_L \oplus_{t_i} (0; s, t)_L &\geq (0; l, r)_L \oplus_{t_W} (0; s, t)_L \\ &= (0; \max(l, s), \max(r, t))_L.\end{aligned}$$

Hence we have

$$(0; l, r)_L \oplus_{t_1} (0; s, t)_L = (0; l, r)_L \oplus_{t_2} (0; s, t)_L$$

which completes the proof. □

If $f \circ L$ is concave, then $f \circ L$ is subadditive. Hence the following is immediate.

Corollary 1. Let t be an Archimedean t-norm with additive generator f and let L be a reference function such that $f \circ L$ is a concave function. If $t_i \leq t$, $i = 1, 2$, then (L, t_1) and (L, t_2) are identical.

The additive generator g of the Lukasiewicz t-norm t_L is given by $f(x) = 1 - x$, $x \in [0, 1]$. If L is convex, then $f \circ L$ is concave. Hence we have the following result.

Corollary 2. Let L be a convex reference function, then for $t_i \leq t_L$, $i = 1, 2$, (L, t_1) and (L, t_2) are identical.

Note. Theorem 2 (Gebhardt [7]) is a special case of Corollary 2. For this, we show that $t_\lambda \leq t_L$. Since $0 \leq (x - 1)(y - 1) = xy - (x + y - 1)$, for $-1 < \lambda < 0$,

$$\begin{aligned}\frac{x + y - 1 + \lambda xy}{1 + \lambda} &\leq \frac{x + y - 1 + \lambda(x + y - 1)}{1 + \lambda} \\ &= \frac{(1 + \lambda)(x + y - 1)}{1 + \lambda} \\ &= x + y - 1.\end{aligned}$$

In Theorem 3, what if $f \circ L$ is convex? Consider the following theorem.

Theorem 4. Let t_i, $i = 1, 2$ be Archimedean t-norms with additive generator f_i, $i = 1, 2$. Suppose $f_i \circ L$, $i = 1, 2$ are convex and (L, t_1) and (L, t_2) are identical. Then $t_1 = t_2$.

We need the following lemma due to Hong and Hwang [11].

Lemma 3. (Hong and Hwang [11]) Let t be an Archimedean t-norm with additive generator f. Suppose $f \circ L$ is convex then

$$(0; 1, 1)_L \oplus_t (0; 1, 1)_L(z) = f^{[-1]}\left(2f \circ L\left(\frac{z}{2}\right)\right) \quad \text{on } [0, 2].$$

Proof of Theorem 4. Suppose f_i, $i = 1, 2$, are normed additive generator of t_i, $i = 1, 2$, respectively. Then it suffices to show that $f_1 = f_2$. Suppose $f_1 \neq f_2$ and let $x^* = \sup\{x \leq 1 | f_1(x) \neq f_2(x)\}$ and $x^{**} = \inf\{x < x^* | (x, x^*) \subset \{f_1 \neq f_2\}\}$. We note that $\{f_1 \neq f_2\}$ is open set since f_1 and f_2 are continuous function. Then $(x^{**}, x^*) \subset \{f_1 \neq f_2\}$. There are two possible cases:

Case 1) $x^* = 1$. Without loss of generality, we assume $f_1 < f_2$ on $(x^{**}, 1)$. By construction $f_1(x^{**}) = f_2(x^{**}) \equiv y^{**}$, say. Let $f_1^{[-1]}(\frac{y^{**}}{2}) = x_0$. Then $2f_2(x_0) > 2f_1(x_0) = y^{**}$ and hence

$$\begin{aligned} f_1^{[-1]}(2f_1(x_0)) &= f_1^{[-1]}(y^{**}) = x^{**} \\ &= f_2^{[-1]}(y^{**}) > f_2^{[-1]}(2f_2(x_0)), \end{aligned}$$

which is contradict.

Case 2) $x^* < 1$. Without loss of generality, we assume $f_1 < f_2$ on (x^{**}, x^*). Let $f_1(x^{**}) = f_2(x^{**}) = y^{**}$ and $f_1(x^*) = f_2(x^*) = y^*$. Choose y_0 such that $y^* < y_0 < y^{**}$ and $x_0 = f_2^{[-1]}(\frac{y_0}{2}) = f_1^{[-1]}(\frac{y_0}{2}) > x^*$. Then $2f_1(x_0) = 2f_2(x_0) = y_0$ but, since $f_1^{[-1]}(y_0)$, $f_2^{[-1]}(y_0) \in (x^{**}, x^*)$ and $f_1 < f_2$ on (x^{**}, x^*), $f_1^{[-1]}(2f_1(x_0)) = f_1^{[-1]}(y_0) < f_2^{[-1]}(y_0) = f_2^{[-1]}(2f_2(x_0))$ which is contradict. This completes the proof. □

For the case of $t = t_M$, we consider the following theorem.

Theorem 5. Let L be continuous and let (L, t) and (L, t_M) be identical, then $t = t_M$.

Proof. Suppose (L, t) and (L, t_M) are identical, then for any $z \in R$,

$$\sup_{x+y=z} t(L(x), L(y)) = \sup_{x+y=z} \min(L(x), L(y)).$$

Note that

$$\sup_{x+y=z} \min(L(x), L(y)) = L(\frac{z}{2}).$$

Then there exists $x_0 \in R$ such that $t(L(x_0), L(z-x_0)) = L(\frac{z}{2})$. If $L(x_0) \neq L(z-x_0)$, then

$$\begin{aligned} t(L(x_0), L(z-x_0)) &\leq \min(L(x_0), L(z-x_0)) \\ &< L(\frac{z}{2}). \end{aligned}$$

Hence $L(x_0) = L(z-x_0)$, which implies $t(L(\frac{z}{2}), L(\frac{z}{2})) = L(\frac{z}{2})$. From the continuity of L, we have for any $z \in [0,1]$, $t(z,z) = z$, and hence $t = t_M$. □

The continuity of L in Theorem 5 is essential. To construct an example we need some definitions about ordinal sums of t-norm and a result of Baets and Marková [1].

Definition 2. Consider $(a,b) \in R^2$, $a \neq b$, then $\phi_{(a,b)}$ is the linear transformation defined by

$$\phi_{(a,b)}(x) = \frac{x-a}{b-a}$$

Note that the inverse mapping $\phi^{-1}_{(a,b)}$ of $\phi_{(a,b)}$ is given by $\phi^{-1}_{(a,b)}(x) = a + (b-a)x$.

Definition 3. Consider a family $(t_k)_{k \in K}$ of t-norms and a family $((\alpha_k, \beta_k))_{k \in K}$ of pairwise disjoint open non-degenerate subintervals of $[0,1]$. The $[0,1]^2 \to [0,1]$ mapping t defined by

$$t(x,y) = \begin{cases} \phi_k^{-1}(t_k(\phi_k(x), \phi_k(y))), & \text{if } (x,y) \in [\alpha_k, \beta_k]^2, \\ t_M(x,y) & \text{elsewhere,} \end{cases}$$

where $\phi_k = \phi_{(\alpha_k, \beta_k)}$, is a t-norm. t is called the ordinal sum of the summands $\langle \alpha_k, \beta_k, t_k \rangle$, and is denoted by $t \equiv (\langle \alpha_k, \beta_k, t_k \rangle \mid k \in K)$.

The notion 'ordinal sum' has led to the following important characterization of continuous t-norms.

Theorem 6. A $[0,1]^2 \to [0,1]$ mapping t is a continuous t-norm if and only if it is an ordinal sum of continuous Archimedean t-norms.

Definition 4. Consider a fuzzy number A and $(a,b) \in [0,1]^2$, $a < b$.

(i) The fuzzy number $A^{[a,b]}$ is defined as

$$A^{[a,b]} = \mathrm{tr} \circ \phi_{(a,b)} \circ A \quad \text{i.e.,} \quad A^{[a,b]}(x) = \mathrm{tr}((A(x)-a)/(b-a)),$$

with tr the $R \to [0,1]$ mapping defined by

$$\mathrm{tr}(x) = \begin{cases} 0, & \text{if } \ x < 0, \\ x, & \text{if } \ 0 \leq x \leq 1, \\ 1, & \text{if } \ x > 1. \end{cases}$$

(ii) The fuzzy number $A_{[a,b]}$ is defined by

$$A_{[a,b]}(x) = \begin{cases} \phi^{-1}_{(a,b)}(A(x)), & \text{if } A(x) > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

The following theorem is due to Baets and Marková [1].

Theorem 7. (Baets and Marková [1]) Consider an ordinal sum $t \equiv (\langle a_i, b_i, t_i \rangle \mid i \in I)$ written in such a way that $\bigcup_{i \in I}[a_i, b_i] = [0,1]$, and fuzzy numbers A_1, A_2, then t-sum $A_1 \oplus_t A_2$ is given by

$$A_1 \oplus_t A_2(x) = \sup_{i \in I} \left(A_1^{[a_i,b_i]} \oplus_t A_2^{[a_i,b_i]} \right)_{[a_i,b_i]}(x).$$

The following example shows that the continuity of L in Theorem 5 is essential.

Example 1. Consider the ordinal sum $t \equiv \left(\langle 0, \frac{1}{2}, t_M \rangle, \langle \frac{1}{2}, 1, t_L \rangle\right)$, and the reference function L defined by

$$L(x) = \begin{cases} 1 - x, & \text{if } x \in [0, \frac{1}{2}], \\ 0, & \text{elsewhere.} \end{cases}$$

Then, by Theorem 7, (L, t) and (L, t_M) are identical but $t \neq t_M$.

We also have various types of reference function and continuous t-norm which are identical if we apply Theorem 3, 4 and 5 to Theorem 6 and 7. For instance, consider the following example.

Example 2. Consider the ordinal sums $t_1 \equiv \left(\langle 0, \frac{1}{2}, t_L \rangle, \langle \frac{1}{2}, 1, t_L \rangle\right)$, and $L(x) = 1 - x^2$ on $[0,1]$. Let $t_2 \equiv \left(\langle 0, \frac{1}{2}, t_{21} \rangle, \langle \frac{1}{2}, 1, t_{22} \rangle\right)$ and let $f_L(x) = 1 - x$ be the additive generator of t_L. Note that $f_L \circ L^{[\frac{1}{2},1]}(x) = 2x$ is concave and $f_L \circ L_{[0,\frac{1}{2}]}(x) = 0$ on $\left[0, \frac{1}{2}\right)$ and $2x - 1$ on $\left[\frac{1}{2}, 1\right]$ is convex on $[0,1]$. Hence by Theorem 3, 4 and 7, if $t_{22} \leq t_L$, and $t_{21} = t_L$ then (L, t_1) and (L, t_2) are identical.

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Dug Hun Hong, Department of Mathematics, Myongji University, Kyunggi 449-728. South Korea.
e-mail: dhhong@mju.ac.kr